

Module 23: The wave equation-I

Lecture 23: The wave equation-I

23.1 Longitudinal elastic waves

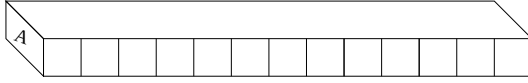


Figure 23.1: An elastic beam

Consider a beam made of elastic material with cross sectional area A as shown in Figure 23.1. Lines have been shown at an uniform spacing along the length of the beam .

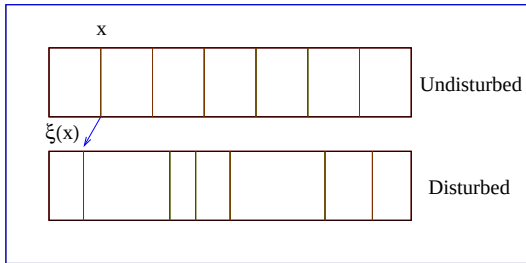


Figure 23.2: Compression and rarefaction in a disturbed beam

A disturbance is introduced in this beam as shown in Figure 23.2. This also shows the undisturbed beam. The disturbance causes the rod to be compressed at some places (where the lines have come closer) and to get rarefied at some other places (where the lines have moved apart). The beam is made of elastic material which tries to oppose the deformation i.e. the compressed region tries to expand again to its original size and same with the rarefied region. We would like to study the behaviour of these disturbances in an elastic beam.

Let us consider the material originally at the point x of the undisturbed beam (Figure 23.2). This material is displaced to $x + \xi(x)$ where $\xi(x)$ is the horizontal displacement of a the point x on the rod. What happen to an elastic solid when it is compressed or extended?

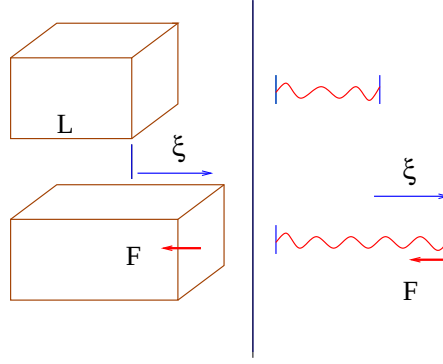


Figure 23.3: A rarefied section of the beam

$$\text{Stress} = F/A,$$

$$\text{Strain} = \frac{\xi}{L},$$

$$Y = \frac{\text{Stress}}{\text{Strain}} \quad (\text{Young's modulus}), \quad (23.1)$$

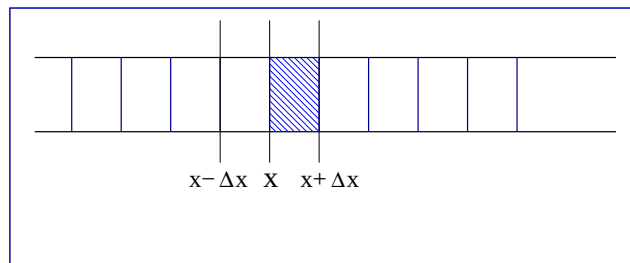
$$F = \left(\frac{YA}{L} \right) \xi, \quad (23.2)$$

$$F = k\xi \rightarrow \text{Spring}. \quad (23.3)$$

Coming back to our disturbed beam, let us divide it into small slabs of length Δx each. Each slab acts like a spring with spring constant

$$k = \frac{YA}{\Delta x}.$$

We focus our attention to one particular slab (shaded below).



Writing the equation of motion for this slab we have,

$$\Delta x \rho A \frac{\partial^2 \xi(x, t)}{\partial t^2} = F, \quad (23.4)$$

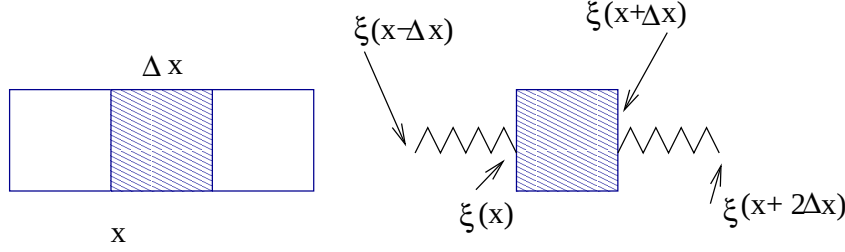


Figure 23.4: One particular slab in the beam

where ρ is the density of the rod, $\rho A \Delta x$ the mass of the slab, $\partial^2 \xi(x, t) / \partial t^2$ its acceleration. F denotes the total external forces acting on this slab.

The external forces arise from the adjacent slabs which are like springs.

This force from the spring on the left is

$$F_L = -k [\xi(x, t) - \xi(x - \Delta x, t)], \quad (23.5)$$

$$\approx -YA \frac{\partial \xi}{\partial x}(x, t). \quad (23.6)$$

The force from the spring on the right is

$$F_R = -k [\xi(x + \Delta x, t) - \xi(x + 2\Delta x, t)], \quad (23.7)$$

$$\approx YA \frac{\partial \xi}{\partial x}(x + \Delta x, t). \quad (23.8)$$

The total force acting on the shaded slab $F = F_L + F_R$ is

$$F = YA \frac{\partial}{\partial x} [\xi(x + \Delta x, t) - \xi(x, t)], \quad (23.9)$$

$$\approx YA \Delta x \frac{\partial^2}{\partial x^2} \xi(x, t). \quad (23.10)$$

Using this in the equation of motion of the slab (eq. 23.4) we have

$$\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = YA \Delta x \frac{\partial^2 \xi}{\partial x^2}, \quad (23.11)$$

which gives us

$$\frac{\partial^2 \xi}{\partial x^2} - \left(\frac{\rho}{Y} \right) \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (23.12)$$

This is a wave equation. Typically the wave equation is written as

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (23.13)$$

where c_s is the phase velocity of the wave. In this case

$$c_s = \sqrt{\frac{Y}{\rho}}.$$

In three dimensions the wave equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \xi - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0. \quad (23.14)$$

This is expressed in a compact notation as

$$\nabla^2 \xi - \frac{1}{c_s^2} \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (23.15)$$

where ∇^2 denotes the Laplacian operator defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

We next check that the familiar sinusoidal plane wave discussed earlier

$$\xi(x, t) = \tilde{a} e^{i(\omega t - kx)} \quad (23.16)$$

is a solution of the wave equation. Substituting this in the wave equation (23.15) gives us

$$k^2 = \frac{\omega^2}{c_s^2}. \quad (23.17)$$

Such a relation between the wave vector $\vec{k}$ and the angular frequency ω is called a dispersion relation. We have

$$\omega = \pm c_s k \quad (23.18)$$

which tells us that the constant c_s which appears in the wave equation is the phase velocity of the wave.

Problems

1. Consider the wave equations

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi(\vec{r}, t) = \frac{1}{L^2} \psi(\vec{r}, t)$$

- a. What is the dispersion relation for this wave equation?
- b. Calculate the phase velocity and the group velocity.
- c. Analyze the behaviour when $k \ll 1/L$ and $k \gg 1/L$

2. For the wave equation given below (where c_s is a constant)

$$\left[4 \frac{\partial^2}{\partial x^2} + 9 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right] \xi(\vec{r}, t) = 0$$

- a. What is the speed of a travelling wave solution propagating along the x axis?
- b. What is the speed of a travelling wave solution propagating along the y axis?
- c. For what value of b is the travelling wave given below a solution of the wave equation given above?

$$\xi(\vec{r}, t) = e^{-[bx+y-5c_s t]^2}$$

- d. What is the speed of the travelling wave given above ?