

Module 22: Beats

Lecture 22: Beats

In this chapter we consider the superposition of two waves of different frequencies. At a fixed point along the propagation direction of the waves, the time evolution of the two wave are,

$$\tilde{A}_1(t) = A_0 e^{i\omega_1 t}, \quad (22.1)$$

and

$$\tilde{A}_2(t) = A_0 e^{i\omega_2 t}. \quad (22.2)$$

The superposition of these two waves of equal amplitude is

$$\tilde{A}(t) = \tilde{A}_1(t) + \tilde{A}_2(t), \quad (22.3)$$

which can be written as,

$$\begin{aligned} \tilde{A}(t) &= A_0 [e^{-i(\omega_2 - \omega_1)t/2} e^{i(\omega_1 + \omega_2)t/2} + e^{i(\omega_2 - \omega_1)t/2} e^{i(\omega_1 + \omega_2)t/2}], \\ &= 2A_0 \cos(\Delta\omega t/2) e^{i\bar{\omega}t}, \end{aligned} \quad (22.4)$$

where $\Delta\omega = \omega_2 - \omega_1$ and $\bar{\omega} = (\omega_1 + \omega_2)/2$ are respectively the difference and the average of the two frequencies. If the two frequencies ω_1 and ω_2 are very close, and the difference in frequencies $\Delta\omega$ is much smaller than $\bar{\omega}$, we can think of the resultant as a fast varying wave with frequency $\bar{\omega}$ whose amplitude varies slowly at a frequency $\Delta\omega/2$. The intensity of the resulting wave is modulated at a frequency $\Delta\omega$. This slow modulation of the intensity is referred to as beats. This modulation is heard when two strings of a musical instrument are nearly tuned and this is useful in tuning musical instruments.

In the situation where the two amplitudes are different we have,

$$\tilde{A}(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} = [a_1 e^{-i\Delta\omega t/2} + a_2 e^{i\Delta\omega t/2}] e^{i\bar{\omega}t}. \quad (22.5)$$

Again we see that we have a fast varying component whose amplitude is modulated slowly. The intensity of the resultant wave is,

$$I = AA^* = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\Delta\omega t). \quad (22.6)$$

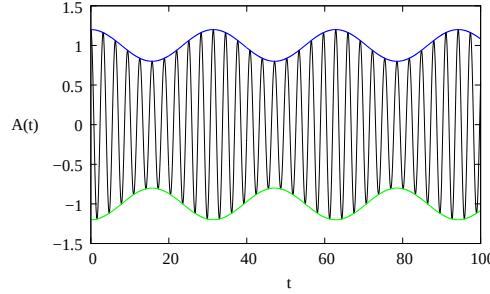


Figure 22.1: A modulated wave

We see that the intensity oscillates at the frequency difference $\Delta\omega$, and it never goes to zero if the two amplitudes are different.

Radio transmission based on “amplitude modulation” is the opposite of this. The transmitter has a generator which produces a sinusoidal electrical wave at a high frequency, say 800 kHz which is the transmission frequency. This is called the carrier wave. The signal which is to be transmitted, say sound, is a relatively slowly varying signal in the frequency range 20 Hz to 20 kHz. The sound is converted to an electrical signal and the amplitude of the carrier wave is modulated by the slowly varying signal. Mathematically,

$$\tilde{A}(t) = [1 + f(t)] e^{i\omega_c t}, \quad (22.7)$$

where ω_c is the angular frequency of the carrier wave, and $f(t)$ is the slowly varying signal. As an example we consider a situation where the signal has a single frequency component,

$$f(t) = a_m \cos(\omega_m t), \quad (22.8)$$

where a_m is the amplitude and ω_m the frequency of the modulating signal. The transmitted signal $A(t)$ is shown in Figure 22.1. The envelope contains the signal. This can be recovered at the receiver by discarding the carrier and retaining only the envelope. The transmitted signal can be expressed as,

$$A(t) = e^{i\omega_c t} + \frac{a_m}{2} e^{i(\omega_c + \omega_m)t} + \frac{a_m}{2} e^{i(\omega_c - \omega_m)t}. \quad (22.9)$$

We see that though the transmitter originally produces output only at a single frequency ω_c when there is no modulation, it starts transmitting two other frequencies $\omega_c + \omega_m$ and $\omega_c - \omega_m$ once the signal is modulated. These new frequencies are referred to as sidebands. If we plot the spectrum of the radiation from the transmitter, we see that the energy is distributed in three frequencies as shown in Figure 22.2.

For a more complicated sound signal, the sidebands will be spread over a range of frequencies instead of a few discrete frequencies. The audible frequency range extends upto 20 KHz but the transmitters and receivers usually do not work beyond 10 KHz. So a radio station transmitting at 800 KHz will

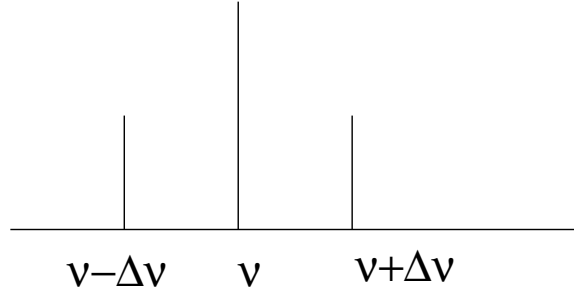


Figure 22.2: Sidebands

actually be transmitting modulated signal in the frequency range 790 KHz to 810 KHz.

If our radio receiver were so sensitive that it picks up only a very small range of frequency around 800 KHz, we would not be able to hear the sound that is being transmitted as the higher frequency components would be missing.

Further, if there were two stations one at 800 KHz and another at 805 KHz, the transmissions from the two stations would overlap and we would get a garbage sound from our receivers. The stations should transmit at frequencies that are sufficiently apart so that they do not overlap. Typically the frequency range 500 KHz to 1500 KHz is available for AM transmission and it is possible to accommodate a large number of stations.

We next consider the full position and time dependence of the superposition of two waves of different frequencies. Assuming equal amplitudes for the two waves we have,

$$A(t) = A[e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)}]. \quad (22.10)$$

Proceeding in exactly the same way as when we considered only the time dependence, we now have,

$$A(t) = 2A \cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) e^{i(\bar{\omega}t - \bar{k}x)}, \quad (22.11)$$

where $\bar{\omega} = (\omega_2 + \omega_1)/2$ and $\bar{k} = (k_2 + k_1)/2$ are the mean angular frequency and wave number respectively, and $\Delta\omega = \omega_2 - \omega_1$ and $\Delta k = k_2 - k_1$ are the difference in the angular frequency and wave number respectively.

Let us consider a situation where the two frequencies are very close such that $\Delta\omega \ll \bar{\omega}$ and $\Delta k \ll \bar{k}$ the resultant (equation (22.11)) can then be interpreted as a travelling wave with angular frequency and wave number $\bar{\omega}$ and $\bar{k}$ respectively. This wave has a phase velocity,

$$v_p = \bar{\omega}/\bar{k}. \quad (22.12)$$

The amplitude of this wave undergoes a slow modulation. The modulation itself is a travelling wave that propagates at a speed $\frac{\Delta\omega}{\Delta k}$. The speed at which

the modulation propagates is called the group velocity, and we have

$$v_g = \frac{d\omega}{dk}. \quad (22.13)$$

As discussed earlier, it is possible to transmit signals using waves by modulating the amplitude. Usually (but not always) signals propagate at the group velocity.

There are situations where the phase velocity is greater than the speed of light in vacuum, but the group velocity usually comes out to be smaller. In all cases it is found that no signal propagates at a speed faster than the speed of light in vacuum. This is one of the fundamental assumptions in Einstein's Theory of Relativity.

Problems

1. Consider the superposition of two waves with different angular frequencies $\nu_1 = 200$ Hz and $\nu_2 = 202$ Hz. The two waves are in phase at time $t = 0$. [a.] After how much time are the two waves exactly out of phase and when are they exactly in phase again? [b.] What happens to the intensity of the superposed wave?
2. Consider the superposition of two fast oscillating signals

$$A(t) = 2 \cos(\omega_1 t) + 3 \sin(\omega_2 t)$$

with $\omega_1 = 1.0 \times 10^3 \text{ s}^{-1}$ and $\omega_2 = 1.01 \times 10^3 \text{ s}^{-1}$. The intensity of the resulting signal $A(t)$ is found to have beats where the intensity oscillates slowly.

- (i.) What is the time period of the beats?
- (ii.) What is the ratio of the minimum intensity to the maximum intensity?
3. The amplitude of a carrier wave of frequency $\nu = 1$ MHz is modulated with the signal $f(t) = \cos^3(\omega t)$ where $\omega = 2\pi \times 20 \text{ s}^{-1}$. What are the frequencies of the different side bands?
4. The two strings of a guitar which is being tuned are found to produce beats of time period $1/10$ s. Also, the minimum intensity is 20% of the maximum intensity. What is the frequency difference between the two strings? What is the ratio of the vibration amplitudes in the two strings?
5. The refractive index of x-rays inside materials, is

$$n = 1 - \frac{a}{\omega^2}, \quad (22.14)$$

where a is a constant whose value depends on the properties of the material. Calculate the phase velocity and the group velocity.

Solution

$$v_p = \frac{c}{n} = \frac{c}{1 - \frac{a}{\omega^2}} > c, \quad (22.15)$$

$$k = \frac{\omega}{c} - \frac{a}{c\omega}, \quad (22.16)$$

$$v_g = \frac{c}{1 + \frac{a}{\omega^2}} < c. \quad (22.17)$$

6. Consider the superposition of two waves,

$$\tilde{A}(x, t) = e^{i(\omega_1 t - k_1 x)} + e^{i(\omega_2 t - k_2 x)},$$

with wavelengths $\lambda_1 = 1$ m and $\lambda_2 = 1.2$ m. The wave has a dispersion relation,

$$\omega = c\sqrt{k^2 + 0.1k^4}.$$

Treating the wave as a slow modulation on a faster carrier wave,

- a. What are the angular frequency, wave number and phase velocity of the carrier wave?
 - b. What are the angular frequency and wave number of the modulation?
 - c. At what speed does the modulation propagate?
7. The dispersion relation for free relativistic electron waves is $\omega = (c^2 k^2 + \frac{m_e^2 c^4}{\hbar^2})^{1/2}$. Show that the product of phase and group velocity of the wave is a constant.
8. A wave packet in a certain medium is represented by the following

$$\tilde{A}(x, t) = 4 \cos(0.1x - 0.2t) \cos(x - 10t) \cos(0.05x - 0.1t).$$

Find group velocity and phase velocity for the packet. Plot the phase velocity in the medium as a function of wave number k , near $k=1$.