Module17:Coherence
Lecture 17: Coherence

We shall separately discuss "spatial coherence" and "temporal coherence".

17.1 Spatial Coherence

The Young's double slit experiment (Figure 17.1) essentially measures the spatial coherence. The wave $\tilde{E}(t)$ at the point P on the screen is the superposition of $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ the contributions from slits 1 and 2 respectively. Let us now shift our attention to the values of the electric field $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ at the positions of the two slits. We define the spatial coherence of the electric field at the two slit positions as

$$C_{12}(d) = \frac{\frac{1}{2}\langle \tilde{E}_1(t)\tilde{E}_2^*(t) + \tilde{E}_1^*(t)\tilde{E}_2(t)\rangle}{2\sqrt{I_1 I_2}} \tag{17.1}$$

The waves from the two slits pick up different phases along the path from the slits to the screen. The resulting intensity pattern on the screen can be written as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} C_{12}(d)\cos(\phi_2 - \phi_1) \tag{17.2}$$

where $\phi_2 - \phi_1$ is the phase difference in the path from the two slits to the screen. The term $\cos(\phi_2 - \phi_1)$ gives rise to a fringe pattern.

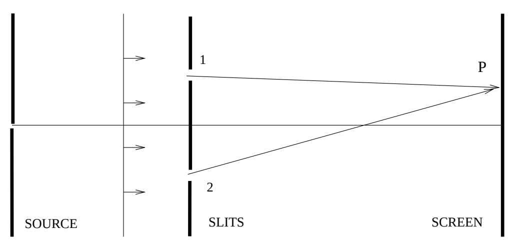

Figure 17.1: Young's double slit with a point source

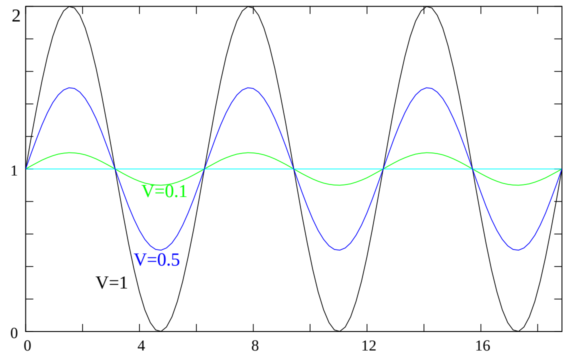

Figure 17.2: Fringe intensity for different visibilities

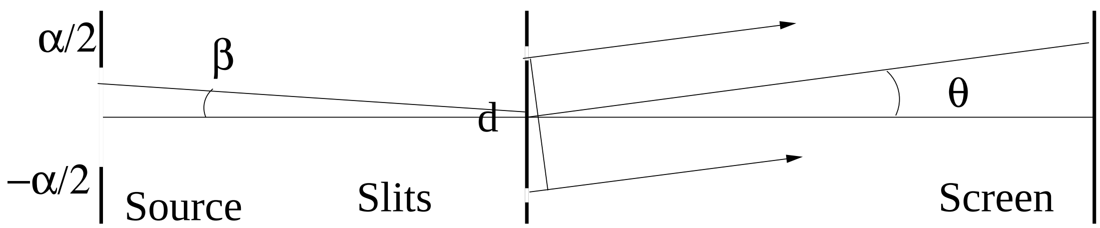

Figure 17.3: Double slit with a wide source

The fringe visibility defined as

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{17.3}$$

quantifies the contrast of the fringes produced on the screen. It has values in the range $1 \leq V \leq 0$. A value $V = 1$ implies very high contrast fringes, the fringes are washed away when $V = 0$. Figure 17.2 shows the fringe pattern for different values of V. It can be easily checked that the visibility is related to the spatial coherence as

$$V = \frac{2\sqrt{I_1 I_2} \mid C_{12}(d) \mid}{I_1 + I_2} \tag{17.4}$$

and the visibility directly gives the spatial coherence $V =\mid C_{12}(d) \mid$ when $I_1 = I_2$.

Let us first consider the situation when the two slits are illuminated by a distant point source as shown in Figure 17.1. Here the two slits lie on the same wavefront, and $\tilde{E}_1(t) = \tilde{E}_2(t)$. We then have

$$\frac{1}{2}\langle \tilde{E}_1(t)\tilde{E}_2^*(t)\rangle = \frac{1}{2}\langle \tilde{E}_1^*(t)\tilde{E}_2(t)\rangle = I_1 = I_2 \, . \tag{17.5}$$

whereby $C_{12}(d) = 1$ and the fringes have a visibility $V = 1$.

We next consider the effect of a finite source size. It is assumed that the source subtends an angle α as shown in Figure 17.3. This situation can be analyzed by first considering a source at an angle β as shown in the figure. This produces an intensity

$$I(\theta,\beta) = 2I_1 \left[1 + \cos\left(\frac{2\pi d}{\lambda}(\theta+\beta)\right)\right] \tag{17.6}$$

at a point at an angle θ on the screen where it is assumed that $\theta, \beta \ll 1$. Integrating β over the angular extent of the source

$$\begin{aligned} I(\theta) &= \frac{1}{\alpha}\int_{-\alpha/2}^{\alpha/2} I(\theta,\beta)\,d\beta \\ &= 2I_1\left[1 + \frac{\lambda}{\alpha 2\pi d}\left\{\sin\left[\frac{2\pi d}{\lambda}\left(\theta+\frac{\alpha}{2}\right)\right] - \sin\left[\frac{2\pi d}{\lambda}\left(\theta-\frac{\alpha}{2}\right)\right]\right\}\right] \\ &= 2I_1\left[1 + \frac{\lambda}{\pi d\alpha}\cos\left(\frac{2\pi d\theta}{\lambda}\right)\sin\left(\frac{\pi d\alpha}{\lambda}\right)\right] \end{aligned} \tag{17.7}$$

It is straightforward to calculate the spatial coherence by comparing eq. (17.7) with eq. (17.2). This has a value

$$C_{12}(d) = \sin\left(\frac{\pi d\alpha}{\lambda}\right) / \left(\frac{\pi d\alpha}{\lambda}\right) \tag{17.8}$$

and the visibility is $V =\mid C_{12}(d) \mid$. Thus we see that the visibility which quantifies the fringe contrast in the Young's double slit experiment gives a direct estimate of the spatial coherence. The visibility, or equivalently the spatial coherence goes down if the angular extent of the source is increased. It is interesting to note that the visibility becomes exactly zero when the argument of the Sine term the expression (17.8) becomes integral multiple of π. So when the width of the source is equal to $m\lambda/d,\ m = 1, 2\cdots$ the visibility is zero.

Why does the fringe contrast go down if the angular extent of the source is increased? This occurs because the two slits are no longer illuminated by a single wavefront, There now are many different wavefronts incident on the slits, one from each point on the source. As a consequence the electric fields at the two slits are no longer perfectly coherent $\mid C_{12}(d) \mid< 1$ and the fringe contrast is reduced.

Expression (17.8) shows how the Young's double slit experiment can be used to determine the angular extent of sources. For example consider a situation where the experiment is done with starlight. The variation of the visibility V or equivalently the spatial coherence $C_{12}(d)$ with varying slit separation d is governed by eq. (17.8). Measurements of the visibility as a function of d can be used to determine α the angular extent of the star.

17.2 Temporal Coherence

The Michelson interferometer measures the temporal coherence of the wave. Here a single wave front $\tilde{E}(t)$ is split into two $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ at the beam splitter. This is referred to as division of amplitude. The two waves are then superposed, one of the waves being given an extra time delay τ through the difference in the arm lengths. The intensity of the fringes is

$$\begin{aligned} I &= \frac{1}{2}\langle[\tilde{E}_1(t)+\tilde{E}_2(t+\tau)]\,[\tilde{E}_1(t)+\tilde{E}_2(t+\tau)]^*\rangle \qquad (17.9)\\ &= I_1+I_2+\frac{1}{2}\langle\tilde{E}_1(t)\,\tilde{E}_2^*(t+\tau)+\tilde{E}_1^*(t)\,\tilde{E}_2(t+\tau)\rangle \end{aligned}$$

where it is last term involving $\tilde{E}_1(t)\,\tilde{E}_2^*(t+\tau)$... which is responsible for interference. In our analysis of the Michelson interferometer in the previous chapter we had assumed that the incident wave is purely monochromatic *ie.* $\tilde{E}(t)=\tilde{E}e^{i\omega t}$ whereby

$$\frac{1}{2}\langle\tilde{E}_1(t)\,\tilde{E}_2^*(t+\tau)+\tilde{E}_1^*(t)\,\tilde{E}_2(t+\tau)\rangle = 2\sqrt{I_1I_2}\cos(\omega\tau) \qquad (17.10)$$

The above assumption is an idealization that we adopt because it simplifies the analysis. In reality we do not have waves of a single frequency, there is always a finite spread in frequencies. How does this affect eq. 17.10?

As an example let us consider two frequencies $\omega_1=\omega-\Delta\omega/2$ and $\omega_2=\omega+\Delta\omega/2$ with $\Delta\omega\ll\omega$

$$\tilde{E}(t)=\tilde{a}\left[e^{i\omega_1 t}+e^{i\omega_2 t}\right]\,. \qquad (17.11)$$

This can also be written as

$$\tilde{E}(t)=\tilde{A}(t)e^{i\omega t} \qquad (17.12)$$

which is a wave of angular frequency ω whose amplitude $\tilde{A}(t)=2\tilde{a}\cos(\Delta\omega t/2)$ varies slowly with time. We now consider a more realistic situation where we have many frequencies in the range $\omega-\Delta\omega/2$ to $\omega+\Delta\omega/2$. The resultant will again be of the same form as eq. (17.12) where there is a wave with angular frequency ω whose amplitude $\tilde{A}(t)$ varies slowly on the timescale

$$T\sim\frac{2\pi}{\Delta\omega}\,.$$

Note that the amplitude $A(t)$ and phase $\phi(t)$ of the complex amplitude $\tilde{A}(t)$ both vary slowly with timescale T. Figure 17.4 shows a situation where $\Delta\omega/\omega=0.2$, a pure sinusoidal wave of the same frequency is shown for comparison. What happens to eq. (17.10) in the presence of a finite spread in frequencies? It now gets modified to

$$\frac{1}{2}\langle\tilde{E}_1(t)\,\tilde{E}_2^*(t+\tau)+\tilde{E}_1^*(t)\,\tilde{E}_2(t+\tau)\rangle = 2\sqrt{I_1I_2}C_{12}(\tau)\cos(\omega\tau) \qquad (17.13)$$

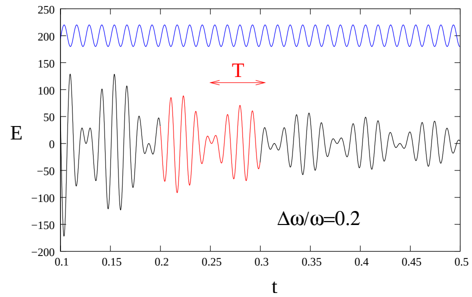

Figure 17.4: Variation of E with time for monochromatic and polychromatic light

where $C_{12}(\tau) \leq 1$. Here $C_{12}(\tau)$ is the temporal coherence of the two waves $\tilde{E}_1(t)$ and $\tilde{E}_2(t)$ for a time delay τ. Two waves are perfectly coherent if $C_{12}(\tau) = 1$, partially coherent if $0 < C_{12}(\tau) < 1$ and incoherent if $C_{12}(\tau) = 0$. Typically the coherence time τ_c of a wave is decided by the spread in frequencies

$$\tau_c = \frac{2\pi}{\Delta\omega}\,. \tag{17.14}$$

The waves are coherent for time delays τ less than τ_c *ie.* $C_{12}(\tau) \sim 1$for $\tau < \tau_c$, and the waves are incoherent for larger time delays *ie.* $C_{12}(\tau) \sim 0$ for $\tau > \tau_c$. Interference will be observed only if $\tau < \tau_c$. The coherence time τ_c can be converted to a length-scale $l_c = c\tau_c$ called the coherence length.

An estimate of the frequency spread $\Delta\nu = \Delta\omega/2\pi$ can be made by studying the intensity distribution of a source with respect to frequency. Full width at half maximum (FWHM) of the intensity profile gives a good estimate of the frequency spread.

The Michelson interferometer can be used to measure the temporal coherence $C_{12}(\tau)$. Assuming that $I_1 = I_2$, we have $V = C_{12}(\tau)$. Measuring the visibility of the fringes varying d the difference in the arm lengths of a Michelson interferometer gives an estimate of the temporal coherence for $\tau = d/c$. The fringes will have a good contrast $V \sim 1$ only for $d < l_c$. The fringes will be washed away for d values larger than l_c.

Problems

1. Consider a situation where Young's double slit experiment is performed using light of wavelength 550 nm and $d = 1$ m. Calculate the visibility assuming a source of angular width $1'$ and 1°. Plot $I(\theta)$ for both these cases.

2. A small aperture of diameter 0.1 mm at a distance of 1m is used to illuminate two slits with light of wavelength $\lambda = 550$ nm. The slit separation is $d = 1$ mm. What is the fringe spacing and the expected visibility of the fringe pattern? (5.5×10^{-4} rad, $V = 0.95$)

3. A source of unknown angular extent α emitting light at $\lambda = 550\,\text{nm}$ is used in a Young's double slit experiment where the slit spacing d can be varied. The visibility is measured for different values of d. It is found that the fringes vanish ($V = 0$) for $d = 10\,\text{cm}$. [a.] What is the angular extent of the source? (5.5×10^{-6})

4. Estimate the coherence time τ_c and coherence length l_c for the following sources

Source	λ nm	$\Delta\lambda$ nm
White light	550	300
Mercury arc	546.1	1.0
Argon ion gas laser	488	0.06
Red Cadmium	643.847	0.0007
Solid state laser	785	10^{-5}
He-Ne laser	632.8	10^{-6}

5. Assume that Kr^{86} discharge lamp has roughly the following intensity distribution at various wavelengths, λ (in nm),

$$I(\lambda) = \frac{36 I_0}{36 + (\lambda - 605.616)^2 \times 10^8} \quad .$$

Estimate the coherence length of Kr^{86} source.(Ans. 0.3m)

6. An ideal Young's double slit (i.e. identical slits with negligible slit width) is illuminated with a source having two wavelengths, $\lambda_1 = 418.6\ nm$ and $\lambda_2 = 421.4\ nm$. The intensity at λ_1 is double of that at λ_2.

   a) Compare the visibility of fringes near order $m = 0$ and near order $m = 50$ on the screen [visibility $= (I_{max} - I_{min})/(I_{max} + I_{min})$].(Ans. 1:0.5)

   b) At what order(s) on the screen visibility of the fringes is poorest and what is this minimum value of the visibility. (Ans. 75, 225 etc. and 1/3)

7. An ideal Young's double slit (separation d between the slits) is illuminated with two identical strong monochromatic point sources of wavelength λ. The sources are placed symmetrically and far away from the double slit. The angular separation of the sources from the mid point of the double slit is θ_s. Estimate θ_s so that the visibility of the fringes on the screen is zero. Can one have visibility almost 1 for a non zero θ_s.

   Hint: See the following figure 17.5,

   (Further reading: Michelson's stellar interferometer for estimating angular separation of double stars and diameters of distant stars)

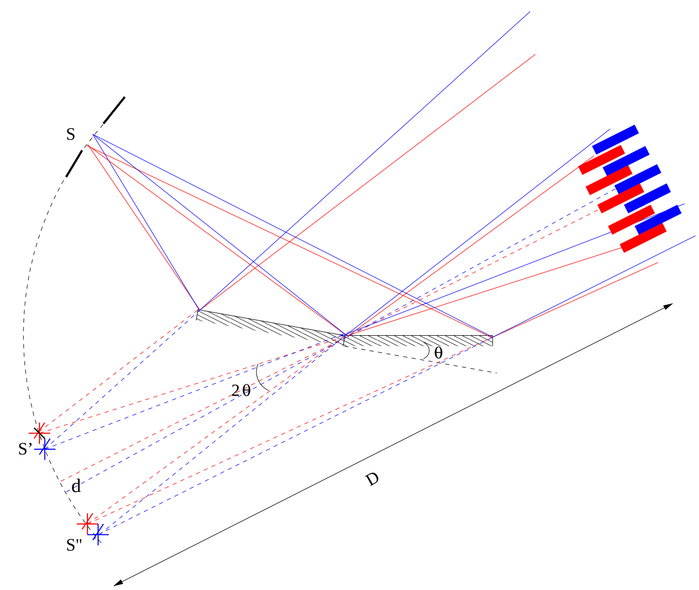

Figure 17.5: Two source vanishing visibility condition