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DIT

$n \in \mathbb{N}$ $\qquad$ $S_n \Rightarrow 2^n$ integers
$\{0, 1, 2, \ldots 2^n - 1\}$

Two discrete variables $x, y \in S_n$.

$K(x, y)$ (Complex, in general)

DIT of $f(x)$ (of discrete variables)

$$\boxed{\tilde{f}(y) = \sum_{x=0}^{N-1} \underbrace{K(y, x)}_{\text{Kernel}} f(x)}\text{,}$$

Matrix Eqn.

K is unitary

$$f(x) = \sum_{y=0}^{N-1} K^{+}(x,y)\,\tilde{f}(y)$$

$$|x\rangle = |x_{n-1}\, x_{n-2} \ldots x_0\rangle$$

$$x_i \in 0,1$$

$$U|x\rangle = \underbrace{\sum_{y=0}^{N-1} |y\rangle\langle y|}_{I}\, U|x\rangle$$

$$= \sum_{y=0}^{N-1} \underbrace{U(y,x)}\, |y\rangle$$

$$\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x)\, |x\rangle$$

$$U|x\rangle = \sum_{y=0}^{N-1} K(x,y)\, |y\rangle$$

$$U \cancel{\$} \sum_x f(x)\,|x\rangle$$

$$= \sum_x f(x)\, \underline{\underline{U|x\rangle}}$$

$$= \sum_x f(x) \sum_y K(\overset{x,y}{\cancel{y,x}}) \cdot |y\rangle$$

$$= \sum_y \left(\sum_x K(x,y)\, f(x) \right) |y\rangle$$

$$= \sum_y \tilde{f}(y)\, |y\rangle$$

QFT

$$K(x,y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i x y / N}{}}$$

$$= \frac{1}{\sqrt{N}} \omega_n^{xy}$$

$$e^{2\pi i/N} = \omega_n$$

xy is a usual decimal product NOT BITWISE product

$n = 1 \qquad N = 2$

$$K(x,y) = \frac{1}{\sqrt{2}} e^{2\pi i xy/2}$$

$$= \frac{1}{\sqrt{2}} (-1)^{xy}$$

$e^{\pi i} = -1$

$x = 0$
$y = 1$

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard matrix.

QFT in $\mathbb{C}^2$

$\alpha|0\rangle + \beta|1\rangle$

$n = 2 \qquad N = 4 \qquad x, y \in \{0, 1, 2, 3\}$

$$\omega_2 = e^{2\pi i/4} = e^{\pi i/2} = i$$

$$K(x,y) = (i)^{xy}$$

$$K = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

- $\cancel{7}_8$

$$\langle x | K K^{+} | y \rangle$$

$$= \sum_{z=0}^{N-1} \underbrace{\langle x | K | z \rangle} \underbrace{\langle z | K^{+} | y \rangle}$$

$$= \sum_{z=0}^{N-1} K(x, z)\, K^{+}(z, y)$$

$$= \frac{1}{N} \sum_{z=0}^{N-1} e^{2\pi i z \underline{\underline{(x-y)}}}$$

Finite Geometric Sum

$$\frac{e^{2\pi i (x-y)} - 1}{e^{2\pi i (x-y)/N} - 1} = 0$$

If $x=y$

$$\langle x | K K^{\dagger} | y \rangle = \delta_{x,y}$$