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Lec. 32

$$\vec{F} = \frac{\partial \vec{P}_{mech}}{\partial t} = \int_{vol.} \rho(\vec{E} + \vec{v} \times \vec{B})\, d^3x$$

$$= \int_{vol.} [\epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) + (\vec{J} \times \vec{B})]\, d^3x$$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\vec{D} = \epsilon_0 \vec{E}$$

$$\frac{\partial \vec{P}_{mech}}{\partial t} + \epsilon_0 \frac{d}{dt}\int (\vec{E}\times\vec{B})\, d^3x$$

$$= \epsilon_0 \int \vec{E}(\vec{\nabla}\cdot\vec{E})\, d^3x + \frac{1}{\mu_0}\int (\vec{\nabla}\times\vec{B})\times\vec{B}\, d^3x$$

$$+ \epsilon_0 \int (\vec{\nabla}\times\vec{E})\times\vec{E}\, d^3x$$

$$= \epsilon_0 \int \left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] d^3x$$

$$+ \frac{1}{\mu_0}\int \left[\vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right] d^3x$$

$$\epsilon_0(\vec{E}\times\vec{B}) = \epsilon_0\mu_0(\vec{E}\times\vec{H})$$
$$= \frac{1}{c^2}(\vec{E}\times\vec{H}) = \frac{1}{c^2}\vec{S}$$

$$\frac{\partial \vec{P}_{mech}}{\partial t} + \frac{1}{c^2}\int \vec{S}\, d^3x = \cdots\cdot$$

$\vec{S}$ = energy flux

$\frac{|\vec{S}|}{c}$ = energy density

$\frac{|\vec{S}|}{c^2}$ = momentum density

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_x$$

$$= E_x\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$- E_y(\vec{\nabla}\times\vec{E})_z + E_z(\vec{\nabla}\times\vec{E})_y$$

$$= E_x\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$- E_y\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right)$$

$$+ E_z\left(\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}E_x^2 + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z)$$

$$- \frac{1}{2}\frac{\partial}{\partial x}(E_y^2 + E_z^2)$$

$$\left(\frac{\partial}{\partial x} E_x^2 + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z) \right)$$

$$- \frac{1}{2} \frac{\partial}{\partial x} |E|^2$$

Tensor of Rank 2 $\quad T_{ij} \quad i = 1, 2, 3$

$\qquad\qquad\qquad\qquad\qquad j = 1, 2, 3$

$$\overleftrightarrow{T}$$

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta - \frac{1}{2}|E|^2\delta_{\alpha\beta}\right] + \frac{1}{\mu_0}\left[B_\alpha B_\beta - \frac{1}{2}|B|^2\delta_{\alpha\beta}\right]$$

Maxwell's Stress Tensor.

$B=0$

$$\overleftrightarrow{T} = \epsilon_0\begin{pmatrix} E_x^2-\frac{1}{2}|E|^2 & E_xE_y & E_xE_z \\ E_xE_y & E_y^2-\frac{1}{2}|E|^2 & E_yE_z \\ E_xE_z & E_yE_z & E_z^2-\frac{1}{2}|E|^2 \end{pmatrix}$$

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_\alpha = \sum_{\beta=1}^{3} \frac{\partial}{\partial x_\beta} T_{\alpha\beta}$$

Take $\vec{B}=0$

$$(\vec{\nabla}\cdot\overleftrightarrow{T})_x = \frac{\partial}{\partial x} T_{xx} + \frac{\partial}{\partial y} T_{xy} + \frac{\partial}{\partial z} T_{xz}$$

$$= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}E^2\right) + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z)$$

$$\frac{\partial \vec{P}_{mech}}{\partial t} + \int \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} d^3x = \int \vec{\nabla} \cdot \overleftrightarrow{T} \, d^3x$$

$$= \oint \overleftrightarrow{T} \cdot d\vec{S}$$

$$\overleftrightarrow{T} \cdot \hat{n}$$

$$\boxed{\frac{\partial \vec{p}_{mech}}{\partial t} + \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}}$$

Conservation of Linear Momentum

$\vec{E} = 0$

$$T_{zx} = \frac{1}{\mu_0} B_z B_x \qquad T_{zy} = \frac{1}{\mu_0} B_z B_y$$

$$T_{zz} = \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}B^2\right)$$

$$F_z = \oint (\overleftrightarrow{T} \cdot d\vec{S})_z$$

$$= \oint \left(T_{zx}\, dS_x + T_{zy}\, dS_y + T_{zz}\, dS_z\right)$$

$$= \frac{1}{\mu_0} \oint B_z (\vec{B} \cdot d\vec{S}) - \frac{1}{2\mu_0} \oint B^2\, dS_z .$$

$$F_z = -T_{zz}\, \pi R^2 = -\frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}B^2\right) \cdot \pi R^2$$

$$= -\frac{1}{2\mu_0} B_z^2 \cdot \pi R^2$$

$$= -\frac{2}{9} \pi \mu_0 \omega^2 \sigma^2 R^4 .$$