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c. No. 15

$$\nabla^2 \phi = 0$$

$$\nabla^2 \Phi = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\right)\phi + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2}{\partial \phi^2}\Phi$$

$$\Phi(r, \theta, \phi) = R(r)\, P(\theta)\, F(\phi)$$

$$\underline{\frac{\sin^2\theta}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\Phi} + \frac{\sin\theta}{P}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right)$$

$$= -\frac{1}{F}\frac{\partial^2 F}{\partial \phi^2} \equiv m^2$$

$$\frac{\partial^2 F}{\partial \phi^2} + m^2 F = 0$$

$$F(\phi) \sim e^{\pm i m \phi}$$

$$e^{im\phi} = e^{im(\phi + 2\pi)}$$

$$m \Rightarrow \text{integers}.$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R = -\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\cancel{\Phi}P + \frac{m^2}{\sin^2\theta} = \ell(\ell+1)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

$$\cos\theta = \mu$$

$$-\sin\theta \, d\theta = d\mu.$$

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \left[\underline{\ell(\ell+1)} - \frac{m^2}{1-\mu^2}\right]P = 0$$

$$\boxed{P \sim P_{\ell m}(\theta)}$$

azimuthal Symmetry

$$\Rightarrow m = 0$$

$$\boxed{\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] + \ell(\ell+1)P = 0}$$

Legendre Polynomi

$$P_0(\cos\theta) = 1.$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] + 2P = 0 \checkmark$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1).$$

$$R \sim r^n$$

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = \ell(\ell+1)R$$

$$n(n-1)r^n + 2nr^n - \ell(\ell+1)r^n = 0$$

$$n(n-1) + 2n - (\ell+1)\ell = 0$$

$$\left(n+\frac{1}{2}\right)^2 = \left(\ell+\frac{1}{2}\right)^2$$

$$n+\frac{1}{2} = \pm\left(\ell+\frac{1}{2}\right) \Rightarrow \underline{n = \ell, -(\ell}$$

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \underline{\underline{\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)}} P_l(\cos\theta)$$

$$\Phi(R,\theta) = \Phi_0 \cos^2\theta .$$

$$\Longrightarrow \Phi(r>R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \Longleftarrow$$

$$\Phi(R,\theta) = \frac{\Phi_0}{3}\left[\underbrace{3\cos^2\theta - 1}_{2P_2(\cos\theta)} + \underset{P_0(\cos\theta)}{1}\right]$$

$$= \frac{\Phi_0}{3}\left[2P_2 + P_0\right]. \Longleftarrow$$

$$\Phi(r=R,\theta) \Rightarrow \quad \underline{\underline{B_0 = \frac{\Phi_0}{3}R \; ; \; B_2 = \frac{2\Phi_0}{3}}}$$

$$\Phi(r > R, \theta) = \frac{\Phi_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right].$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R^-} - \left.\frac{\partial \phi}{\partial r}\right|_{r=R^+} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \sigma_0 \sin 2\theta . \sin\phi$$
$$= 2\sigma_0 \sin\theta . \cos\theta \left[\frac{e^{i\phi} - e^{-i\phi}}{2i} \right]$$
$$= i\sigma_0 \sqrt{\frac{8\pi}{15}} \left(Y_{21} + Y_{2-1} \right).$$

$$\Phi(r,\theta,\phi) = \sum_{\ell m} \left(\underline{\underline{A_{\ell m} r^\ell}} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi)$$

Inside $\sum_{\ell m} A_{\ell m} r^\ell . Y_{\ell m}(\theta,\phi)$

Outside $\sum_{\ell m} \frac{B_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta,\phi)$

$$\frac{\partial}{\partial r} \sum_{\ell m} A_{\ell m}\, r^{\ell}\, Y_{\ell m}(\theta, \phi)$$

$$\Rightarrow \sum_{\ell m} A_{\ell m}\, \ell\, R^{\ell-1} \cdot Y_{\ell m}(\theta, \phi)$$