Lecture 6

MOMENTUM EQUATION

6.1 MOMENTUM EQUATION: INTEGRAL FORM

Newton's second law of motion states that the rate of change of linear momentum for a material region (system) is equal to the sum of external forces acting on the system. For a particle of mass dm, this law can written as

$$\mathrm{d}\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}\mathrm{d}m\right), \qquad \mathrm{d}m = \rho \mathrm{d}\Omega \tag{6.1}$$

Hence, for a finite material region, this law takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mathbf{v}\rho \mathrm{d}\Omega = \mathbf{F} \qquad \text{or} \quad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \mathbf{F} \tag{6.2}$$

where $\mathbf{P}$ is the linear momentum of the system. The intensive property corresponding to $\mathbf{P}$ is $\phi \equiv \mathbf{v}$. Hence, from Reynolds transport theorem for a fixed control volume

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \underbrace{\frac{\partial}{\partial t}\int_{\Omega} \rho \mathbf{v} \mathrm{d}\Omega}_{\substack{\text{Rate of change of}\\ \text{momentum in the CV}}} + \underbrace{\int_{S} \rho \mathbf{v}\mathbf{v}\cdot \mathrm{d}\mathbf{A}}_{\substack{\text{Rate of efflux of momentum}\\ \text{across the control surface}}} \tag{6.3}$$

Net force $\mathbf{F}$ on the control volume can be expressed as sum of the surface force, $\mathbf{F}_S$ (pressure, viscous stress), and body force, $\mathbf{F}_B$ (gravity, electromagnetic, centrifugal, Coriolis etc.), i.e.

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B \tag{6.4}$$

The surface force $\mathbf{F}_S$ essentially represents microscopic momentum flux across a surface and can be expressed as

$$\mathbf{F}_S = \int_{S} \boldsymbol{\tau}\cdot \mathrm{d}\mathbf{A} \tag{6.5}$$

where $\boldsymbol{\tau}$ is the stress tensor. Body force $\mathbf{F}_\mathrm{B}$ can be expressed as

$$\mathbf{F}_B = \int_{\Omega} \rho \mathbf{b}\, \mathrm{d}\Omega \tag{6.6}$$

where $\mathbf{b}$ is body force per unit mass. Combining (6.2)-(6.6), the integral form of momentum equation can be written as

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \mathbf{v}\, \mathrm{d}\Omega + \underbrace{\int_{S} \rho \mathbf{v}\mathbf{v}\cdot \mathrm{d}\mathbf{A}}_{\text{Convective flux}} = \underbrace{\int_{S} \boldsymbol{\tau}\cdot \mathrm{d}\mathbf{A}}_{\text{Diffusive flux}} + \int_{\Omega} \rho \mathbf{b}\, \mathrm{d}\Omega \tag{6.7}$$

Note that since momentum is a vector quantity, its convective and diffusive fluxes are the scalar products of second order tensors $\rho\mathbf{v}\mathbf{v}$ and $\boldsymbol{\tau}$ with the surface vector $\mathrm{d}\mathbf{A}$.

6.2 MOMENTUM EQUATION: DIFFERENTIAL FORM

For a fixed control volume, order of temporal differentiation and integration in Eq. (6.7) can be interchanged. Transform the convective and diffusive terms using Gauss divergence theorem, i.e.

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\mathbf{v}\mathrm{d}\Omega=\int_{\Omega}\frac{\partial(\rho\mathbf{v})}{\partial t}\mathrm{d}\Omega,\quad \int_{S}\rho\mathbf{v}\mathbf{v}\cdot\mathrm{d}\mathbf{A}=\int_{\Omega}\nabla\cdot(\rho\mathbf{v}\mathbf{v})\mathrm{d}\Omega\ \text{ and }\int_{S}\boldsymbol{\tau}\cdot\mathrm{d}\mathbf{A}=\int_{\Omega}\nabla\cdot\boldsymbol{\tau}\,\mathrm{d}\Omega \qquad (6.8)$$

Substitution of Eq. (6.8) into Eq. (6.7) yields

$$\int_{\Omega}\left[\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla.\left(\rho\mathbf{v}\mathbf{v}\right)-\nabla.\boldsymbol{\tau}-\rho\mathbf{b}\right]\mathrm{d}\Omega=0 \qquad (6.9)$$

Equation (6.9) holds for any control volume which is possible only if the integrand vanishes everywhere, i.e.

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla.\left(\rho\mathbf{v}\mathbf{v}\right)=\nabla.\boldsymbol{\tau}+\rho\mathbf{b} \qquad (6.10)$$

Equation (6.10) is referred as the *conservative* or *strong conservation* form of momentum equation. It is also known as *Cauchy equation of motion*.

The integral form of momentum equation (6.7) or its differential form represented by Eq. (6.10) is applicable to an inertial control volume. Similar forms can be derived for moving control volumes and non-inertial reference frames (Batchelor 1973, Panton 2005, Kundu and Cohen 2008).

Further, using the identity

$$\nabla\cdot\left(\rho\mathbf{v}\mathbf{v}\right)=\left[\nabla\cdot\left(\rho\mathbf{v}\right)\right]\mathbf{v}+\rho\mathbf{v}\cdot\nabla\mathbf{v} \qquad (6.11)$$

and chain rule of differentiation, we get

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla.\left(\rho\mathbf{v}\mathbf{v}\right)=\left[\frac{\partial\rho}{\partial t}+\nabla.\left(\rho\mathbf{v}\right)\right]\mathbf{v}+\rho\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]$$

Using continuity equation (5.4), the first term on the RHS of the preceding equation vanishes. Thus, Eq. (6.10) takes the following form:

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]=\nabla.\boldsymbol{\tau}+\rho\mathbf{b}\qquad\text{or}\qquad\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=\nabla.\boldsymbol{\tau}+\rho\mathbf{b} \qquad (6.12)$$

where the operator (D/D*t*) denotes the material or particle derivative. Equation (6.12) is referred to as the *non-conservative form* of the momentum equation.

Example 6.1
Derive the differential form of the momentum equation using an infinitesimal differential control volume in Cartesian coordinates.

Solution
Let us consider flow of a fluid through an infinitesimal differential control volume of dimensions d*x*, d*y* and d*z*. For the sake of clarity, Figure 6.2 depicts the flow through a three-

dimensional control volume, and the forces acting on the surfaces of the control volume in x-direction only. The resultant force in x-direction is

$$F_x = \rho b_x \mathrm{dxdydz} + \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\mathrm{dx}\right) - \tau_{xx}\right]\mathrm{dydz}$$

$$+\left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\mathrm{dy}\right) - \tau_{yx}\right]\mathrm{dxdz} + \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\mathrm{dz}\right) - \tau_{yx}\right]\mathrm{dxd}y$$

$$= \left[\rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\mathrm{dxdydz}$$

Similarly, components of the resultant force in y- and z-directions are

$$F_y = \left[\rho b_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right]\mathrm{dxdydz}$$

$$F_z = \left[\rho b_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]\mathrm{dxdydz}$$

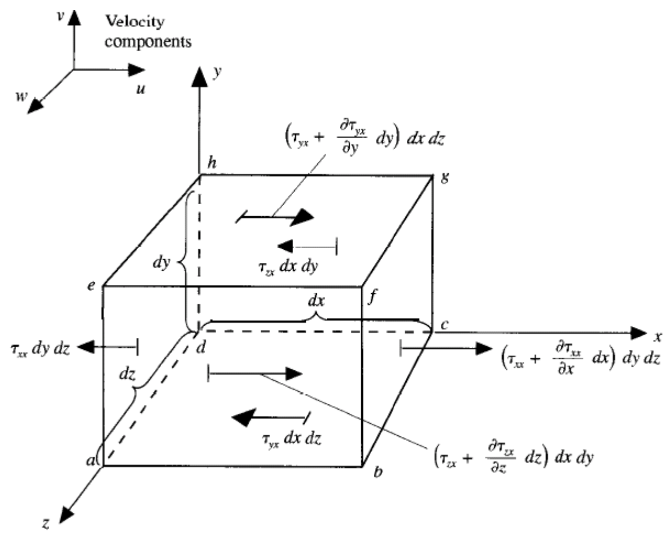

Figure 6.2 Forces acting on the faces of a differential control volume (only the x-components are shown in the figure for clarity)

From Reynolds transport theorem,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \begin{pmatrix}\text{Rate of change of} \\ \text{momentum in the CV}\end{pmatrix} + \begin{pmatrix}\text{Rate of efflux of momentum} \\ \text{across the control surface}\end{pmatrix}$$

Rate of change of the momentum in CV = $\dfrac{\partial(\rho \mathrm{dxdydz}\mathbf{v})}{\partial t} = \dfrac{\partial(\rho \mathbf{v})}{\partial t}\mathrm{dxd}y\mathrm{dz}$

Rate of efflux of momentum in x-direction is given by

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{efflux in } x\text{-direction}\end{pmatrix} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\mathrm{d}x\right]\mathrm{d}y\mathrm{d}z\left(u + \frac{\partial u}{\partial x}\mathrm{d}x\right) - (\rho u\mathrm{d}y\mathrm{d}z)u$$
$$+\left[\rho v + \frac{\partial(\rho v)}{\partial y}\mathrm{d}y\right]\mathrm{d}x\mathrm{d}z\left(u + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{2}\right)\text{-}(\rho v\mathrm{d}x\mathrm{d}z)\left(u - \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{2}\right)$$
$$+\left[\rho w + \frac{\partial(\rho w)}{\partial z}\mathrm{d}z\right]\mathrm{d}x\mathrm{d}y\left(u + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{2}\right)\text{-}(\rho w\mathrm{d}x\mathrm{d}y)\left(u - \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{2}\right)$$
$$= \left[u\frac{\partial(\rho u)}{\partial x} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + \rho v\frac{\partial u}{\partial y} + u\frac{\partial(\rho w)}{\partial z} + \rho w\frac{\partial u}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Similarly, rate of momentum efflux in y- and z-directions are

$$\begin{pmatrix}\text{Rate of momentum}\\ \text{efflux in } y\text{-direction}\end{pmatrix} = \left[\frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho vw)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$\begin{pmatrix}\text{Rate of momentum}\\ \text{efflux in } z\text{-direction}\end{pmatrix} = \left[\frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

From Newton's second law, D**P**/Dt = **F**. The x-component of this equation is

$$\frac{\mathrm{D}P_x}{\mathrm{D}t} = \frac{\partial(\rho u)}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z + \left[\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= F_x = \left[\rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

which can be simplified to obtain the x-component of momentum equation. We can similarly obtain y- and z-components of the momentum equation, and these equations are given by

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = \rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$
$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = \rho b_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$
$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = \rho b_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

The preceding equations represent the *conservation form* of the momentum equation. Non-conservation form can be obtained using alternative form of the Newton's second law given by

$$m\mathbf{a} = (\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z)\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F}$$

Thus,

$$ma_x = (\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z)\frac{\mathrm{D}u}{\mathrm{D}t} = F_x = \left[\rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Simplification of the preceding equation and similar equations for y and z-components leads to the *non-conservation* form of momentum equation in Cartesian component form given by the following set of scalar equations:

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \rho b_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$
$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = \rho b_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$
$$\rho\frac{\mathrm{D}w}{\mathrm{D}t} = \rho b_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Exercise 6.1: Derive the differential form of momentum equation in polar coordinates by take an infinitesimal control volume in (a) cylindrical polar coordinates and (b) spherical polar coordinates.

REFERENCES

Batchelor, G. K. (1973). *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge.

Kundu, P. K. and Cohen, I. M. (2008). *Fluid Mechanics*, 4th Ed., Academic Press.

Panton, R. L. (2005). *Incompressible Flow*, 3rd Ed., Wiley.