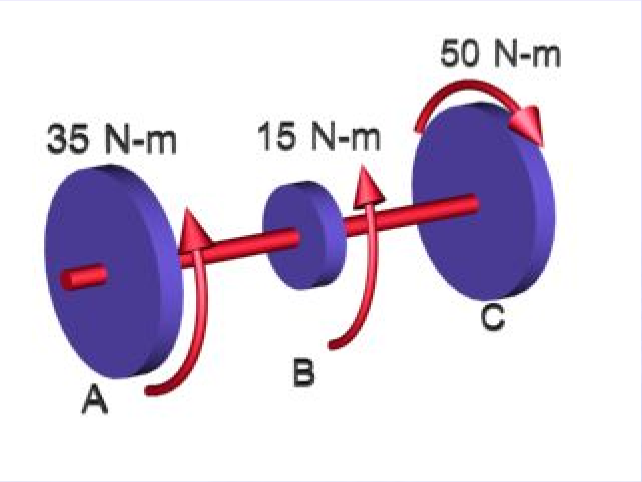
1. Internal torque in a Shaft

Find the internal torque at any section for the shaft shown in the figure. The forces are shown in the figure.

Figure:



Concepts Involved:

Force equilibrium

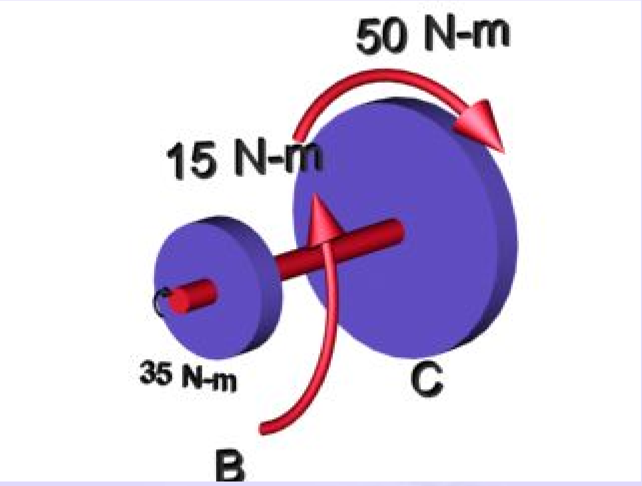
Solution :

Step 1

The 50N.m torque is balanced by the two torques of 35 and 15 N.m at A and B respectively. Therefore, the body as a whole is in equilibrium.

Step 2

We obtain the free body diagram of the part of the shaft, by passing a plane perpendicular to the shaft at any point between A and B. So we have $\Sigma M_x = 0$, this implies $T_{AB} = 35$N-m. The conclusion reached is that resisting torque developed between shaft A and B is 35N-m



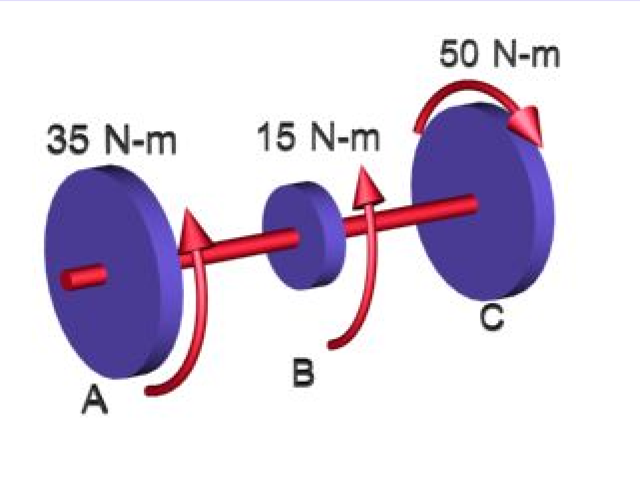
Step 3

On similar grounds the internal torque resisted between B and C is 50 N-m

Top

2. Torsion formula (circular elastic bars).

Find the maximum torsional stress in shaft AC (refer the figure). Assume the Diameter of AC is 15 mm.



Concepts involved:

1) Torsional stress

2) Torsion formula

Formulae used:

Polar moment of inertia

$$J = \int_A \rho^2 dA$$

Torsion formula

τ_{max} **= Tr/J**

Solution:

Step 1:

The maximum internal torque resisted by the shaft is known from the previous problem to be 50 N-m

---

Step 2:

Calculate J for the section using the first formula

$J = \pi d^4/32 = 4970 \text{ mm}^4$

Step 3:

Substitute in the torsion formula

$\tau_{max} = Tr/J = (50 \text{ X } 10^3 \text{ X}7.5)/ 4970 = 75.45\text{Mpa}$

Result variation:

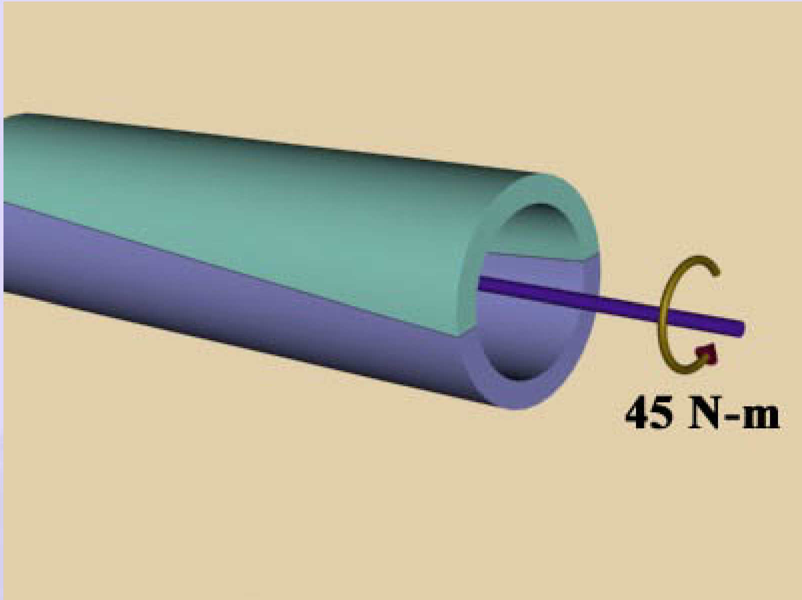
Consider the case when diameter changes. Guess how τ_{max} will change?

For answer: τ_{max} is inversely proportional to d^3

Top

3. Hollow cylinder

Consider a long tube of 25mm outside diameter, d_o, and of 20mm inside diameter d_i, twisted about its longitudinal axis with a torque T of 45 N-m Determine the shear stresses at the outside and inside of the tube



Concepts involved:

Torsional stress

Torsion formula

Formulae used:

Polar moment of inertia

$$J = \int_A \rho^2 dA = \frac{\pi\left(d_o^4 - d_i^4\right)}{32}$$

Torsion formula

τ_{max} **= Tr / J**

Solution:

Step 1:

Calculate J

$$J = \int_A \rho^2 dA = \frac{\pi\left(d_o^4 - d_i^4\right)}{32} = \frac{\pi\left(20^4 - 16^4\right)}{32}$$

$= 22641 mm^4$

Step 2:

Apply torsion formula for the two radii

$\tau_{max} = Tr_o / J = (50 \times 10^3 \times 12.5) / 22641 = 27.6$ MPa

$\tau_{max} = Tr_i / J = (50 \times 10^3 \times 10) / 22641 = 22.1$ MPa

---

Conclusion:

We see that hollow members are better transmitters of torsion than solid members.

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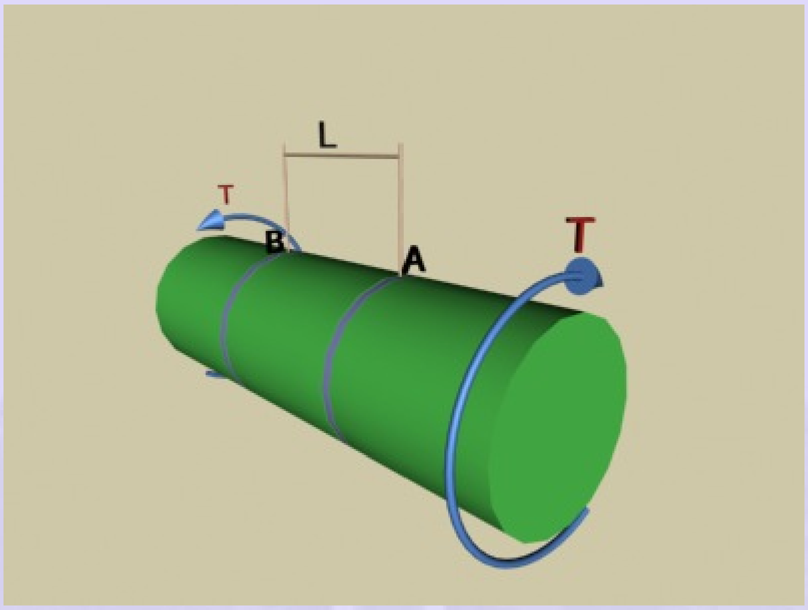
4. Angle of twist for circular members

Find the relative rotation of section B-B with respect to section A-A of the solid elastic shaft as shown in the figure when a constant torque T is being transmitted through it. The polar moment of inertia of the cross-sectional area J is constant.



Concepts involved:

Angle of twist in circular members

Formulae used:

$$\theta = \int \frac{Tx}{JG} dx$$

Where,

ϕ= Angle of twist

T_x = torque at distance x

J_x = polar moment of area at distance x

G = Shear modulus

Solution:

Step 1:

Here neither torque nor J changes with x so,

T_x = T and J_x = J

$$\theta = \int_A^B \frac{T_A\,dx}{J_x G} = \int_0^L \frac{T\,dx}{JG} = \frac{T}{JG}\int_0^L dx = \frac{TL}{JG}$$

i.e , $\theta = \frac{TL}{JG}$

Note:

In applying the above equation, note particularly that the angle ϕ must be expressed in radians. Also observe the great similarity of this relation equation Δ =PL/AE, for axially loaded bars. Here $\phi \Leftrightarrow \Delta$. T$\Leftrightarrow$ P, J$\Leftrightarrow$ A, and G$\Leftrightarrow$ E. By the analogy, this equation can be recast to express the torsional spring constant, or torsional stiffness, k_t as

K_t = T/θ = JG /L [N-m/rad]

This constant torque required to cause a rotation of 1 radian, i.e.,ϕ = 1. It depends only on the material properties and the size of the member. As for axially loaded bars, one can visualize torsion members as springs.

The reciprocal of k_t defines the torsional flexibility f_t. Hence, for a circular solid or hollow shaft.

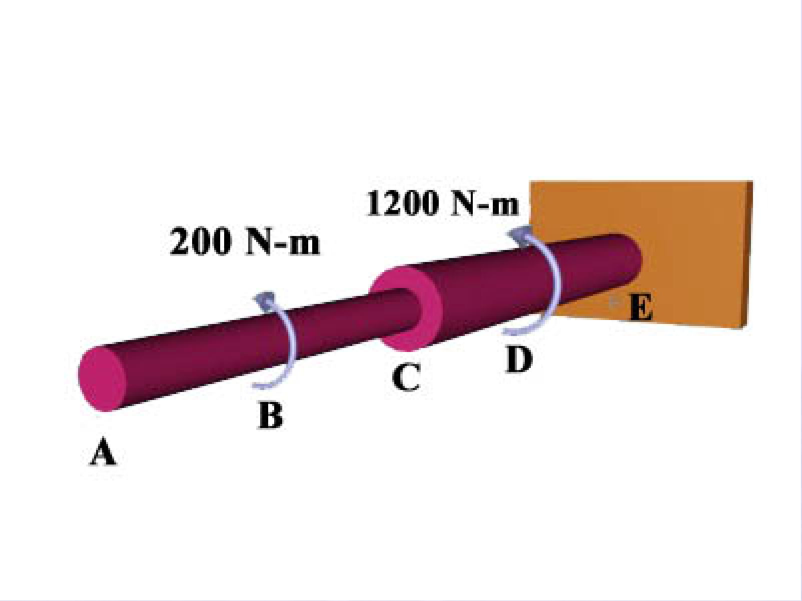
f_t = 1/k_t = L / JG [rad/N-m]

This constant defines the rotation resulting from application of a unit torque, i.e., T = 1. On multiplying by the torque T, one obtains the current equation

Top

5. Variable c/s and Torque

Consider the stepped shaft shown in figure rigidly attached to a wall at E. Determine the angle-of-twist of the end A when the torque at B and at D is applied. Assume the shear modulus G to be 80 GPa, a typical value for steels.



Concepts Involved:

Angle of twist

Formulae used:

$$\theta = \int \frac{Tx}{JG} dx$$

Where,

ϕ = Angle of twist

T_x = torque at distance x

J_x = polar moment of area at distance x

G = Shear modulus

Solution:

Step 1:

The torque at E is determined to assure equilibrium.

T_E = 1000 + 150 N-m

= 1150 N-m

Step 2:

Find Torques at different c/s by drawing free body diagram

At any c/s between D and E is 1150 N.m

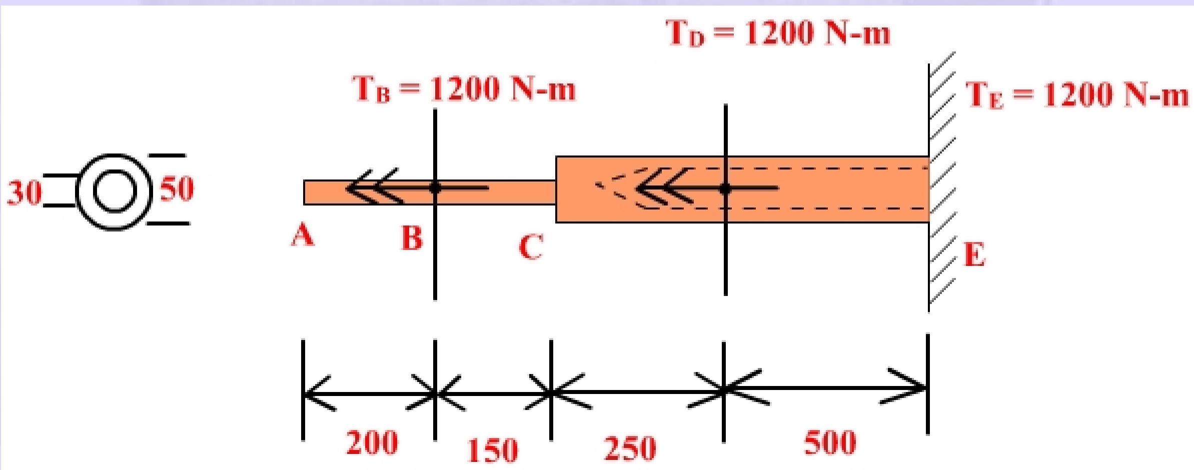
T_{DE} = 1150 N.m

At any c/s between BD,

T_{BD} = 150 N-m

At any c/s between AB

T_{AB} = 0 N-m



Step 3:

The polar moments of inertia for the two kinds of cross sections occurring in this problem are found using polar moment equation giving

$J_{AB} = J_{BC} = \dfrac{\pi d^4}{32} = \pi\ x25^4/32 = 38.3x10^3\ mm^4$

$J_{CD} = J_{DE} = \dfrac{\pi\left(d_o^4 - d_i^4\right)}{32}$

$= \pi(50^4 - 25^4)/32 = 575 \times 10^3\ mm^4$

Step 4:

Angle of twist is found at several c/s as shown

To find the angle-of-twist of the end A, angle-of-twist formula is applied for each segment and the results summered. The limits of integration for the segments occur at points where the values of T or J change abruptly.

Here total angle of twist is found out,

$$\phi = \int_A^E \frac{T_x dx}{J_x G} = \int_A^B \frac{T_{AB} dx}{J_{AB} G} + \int_B^C \frac{T_{BC} dx}{J_{BC} G} + \int_C^D \frac{T_{CD} dx}{J_{CD} G} + \int_D^E \frac{T_{DE} dx}{J_{DE} G}$$

In the last group of integrals, T's and J's are constant between the limits considered, so each integral reverts to a known solution. Hence,

$$\phi = \sum_i \frac{T_i L_i}{J_i G} = \frac{T_{AB} L_{AB}}{J_{AB} G} + \frac{T_{BC} L_{BC}}{J_{BC} G} + \frac{T_{CD} L_{CD}}{J_{CD} G} + \frac{T_{DE} L_{DE}}{J_{DE} G}$$

$$= 0 + \frac{150 \times 10^3 \times 200}{38.3 \times 10^3 \times 80 \times 10^3} + \frac{150 \times 10^3 \times 200}{575 \times 10^3 \times 80 \times 10^3} + \frac{1150 \times 10^3 \times 500}{575 \times 10^3 \times 80 \times 10^3}$$

$=0+9.8\times10^{-3}+1.0\times10^{-3}+12.5\times10^{-3}=23.3\times10^{-3}$ rad

As can be noted from above, the angle-of- twist for shaft segments starting from the left end are : 0 rad, 9.8×10^{-3} rad, 1.0×10^{-3}, rad 12.5×10^{-3} rad. Summing these quantities beginning from A, gives the angle-of-twist along the shaft. Since no shaft twist can occur at the built-in end, this function must be zero at E, as required by the boundary condition.

Therefore, according to the adopted sign convention the angle-of-twist at A is -23.3×10^{-3} rad occurring in the direction of applied torques.

No doubt local disturbances in stresses and strains occur at the applied concentrated torques and the change in the shaft size, as well as at the built-in end. However, these are local effects having limited influence on the overall behavior of the shaft.

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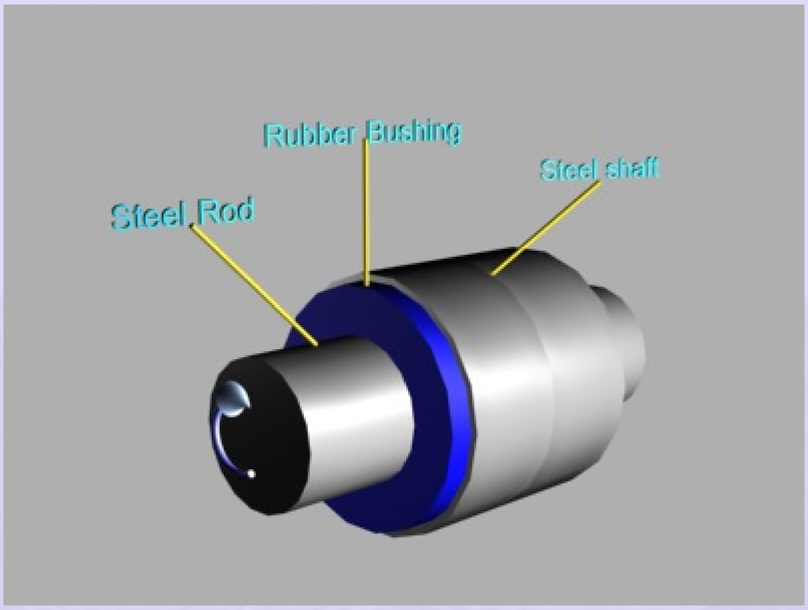
6. Composite bars

Determine the torsional stiffness k_t, for the rubber bushing shown in figure. Assume that the rubber is bonded to the steel shaft and the outer steel tube, which is attached to machine housing. The shear modulus for the rubber is G. Neglect deformations in the metal parts of the assembly.



Concepts involved:

Angle of twist in circular members

Formulae used:

$$\theta = \int \frac{Tx}{JG} dx$$

Where,

θ= Angle of twist

T_x = torque at distance x

J_x = polar moment of area at distance x

G = Shear modulus

Solution:

Step 1:

Calculate the torque acting at any surface.

Due to the axial symmetry of the problem, on every imaginary cylindrical surface of rubber of radius r, the applied torque T is resisted by constant shear stresses τ. The area of the imaginary surface is 2π rL. On this basis, the equilibrium equation for the applied torque T and the resisting torque developed by the shear stresses τ acting at a radius r is

T = (2π rL)τ r [area*stress*arm]

Step 2:

Get τ from the above relation and hence determine shear strain.

From this relation, $\tau = T/2\pi r^2 L$ Hence, by using Hooke's law, the shear strain γ can be determined for an infinitesimal tube of radius r and thickness dr, figure, from the following relation:

$$\gamma = \frac{\tau}{G} = \frac{T}{2\pi L G r^2}$$

Step 3:

Get the angle of twist by integrating throughout the surface.

This shear strain in an infinitesimal tube permits the shaft to rotate through an infinitesimal angle $d\theta$. Since in the limit r + dr is equal to r, the magnitude of this angle is $d\theta = \gamma$ (dr/r)

The total rotation θ of the shaft is an integral, over the rubber bushings, of these infinitesimal rotations, i.e.

$$\theta = \int d\theta = \frac{T}{2\pi L G}\int_{d/2}^{D/2} \frac{dr}{r^3} = \frac{T}{\pi L G}\left(\frac{1}{d^2} - \frac{1}{D^2}\right)$$

From which,

$$k_t = \frac{T}{\theta} = \frac{\pi L G}{\left(\frac{1}{d^2} - \frac{1}{D^2}\right)}$$

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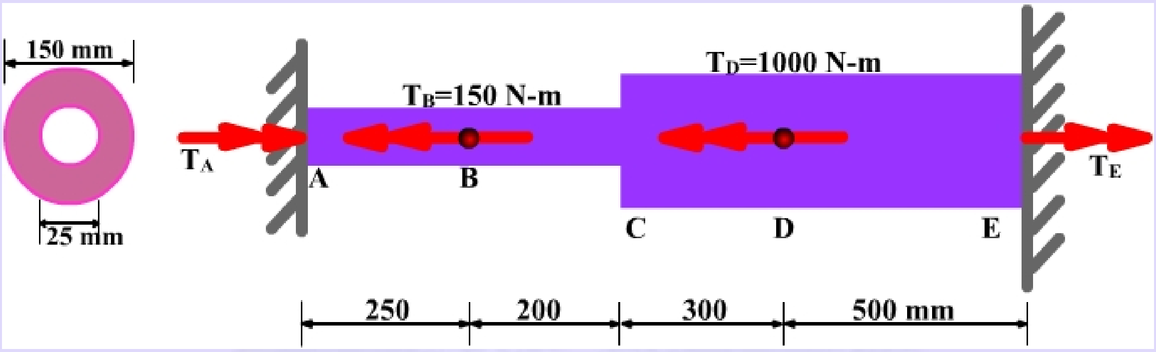
7. Statically indeterminate problem

Consider the stepped shaft shown in the figure. Determine the end reactions and plot the torque diagram for the shaft. Apply the force method.



Concepts involved:

Statically indeterminate structures

Force method of analysis

In this method the problems are reduced to statical determinancy by removing one of the redundant reactions and calculating the rotation θo at the released support. The required boundary conditions are then restored by twisting the member at the released end through an angle θ_1 such that $\theta_o + \theta_1 = 0$.

Formulae used:

$\theta_o + \theta_1 = 0.$

where

θ_o = rotation at the released support

θ_1 = Rotation at the restored support

$$J=\pi\left(\frac{d_o^{4}-d_i^{4}}{32}\right)$$

where

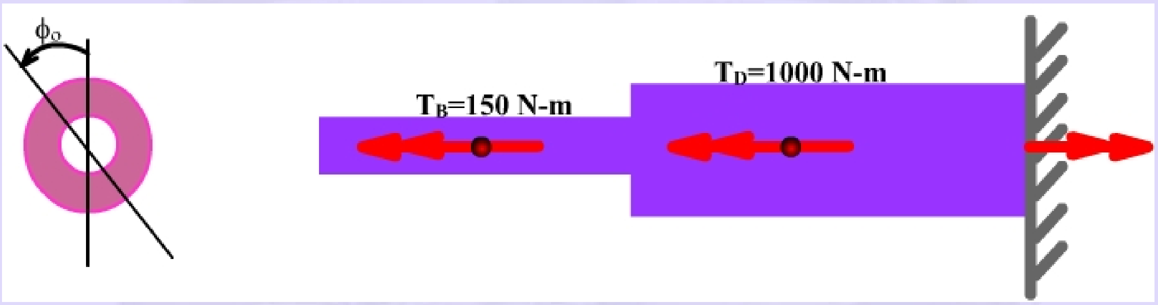
d_o = Outer diameter

d_i = Inner diameter

Solution:

Step 1:

Remove any of the redundant reaction and analyse the structure.



There are two unknown reactions, T_A and T_E. One of them can be considered as redundant, and, arbitrary, reaction T_A is removed. This results in the free-body diagram shown in the above figure. Then the end rotation is calculated as $\theta_o = 23.3\text{x}10^{-3}$ rad.

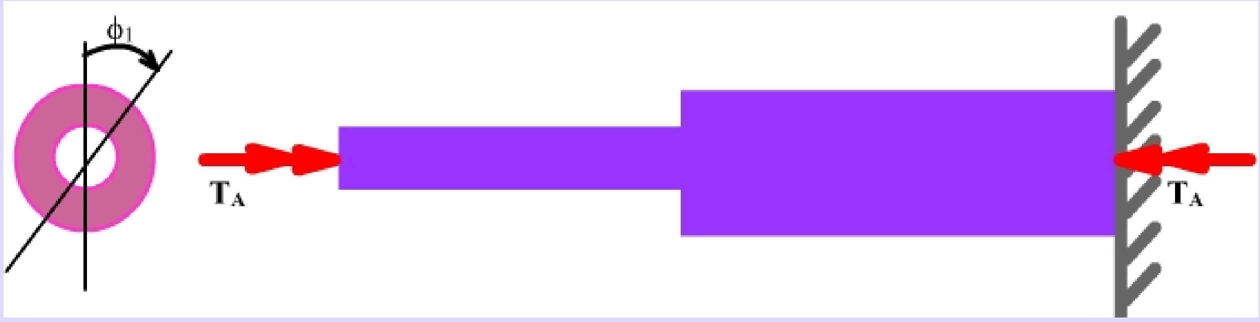
Step 2:

Get the J for all c/s

$J_{AC}=38.3\text{x}10^3\ \text{mm}^2$ and $J_{CE} = 575\text{x}10^3\ \text{mm}^2$.

Step 3:

Now apply the redundant reaction (T_A in this case).



By applying T_A to the unloaded bar, as shown in the above figure, end rotation θ_1 at end A is found.

$$\theta_1 = \sum_i \frac{T_i L_i}{J_i G_i}$$

$$= T_A \, x10^3 \left(\frac{450}{38.3 \, x \, 10^3 \, x \, 80 \, x \, 10^3} + \frac{800}{575 \, x \, 10^3 \, x \, 80 \, x \, 10^3} \right)$$

$$= (147 x 10^{-6} + 17 \, x \, 10^{-6}) T_A = 164 x 10^{-6} \, T_A \ \text{rad}$$

Where T_A has the units of N-m.

Step 4:

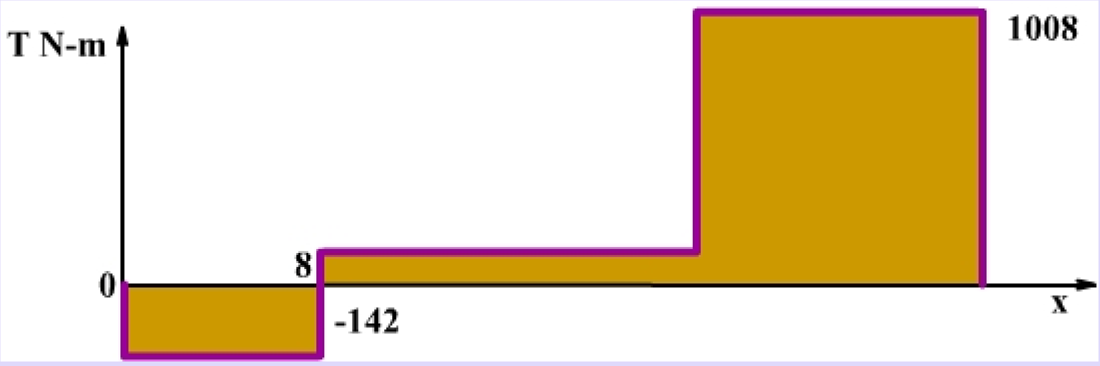
<u>Use compatibility condition to find out reactions.</u>

Using compatibility condition and defining rotation in the direction of T_A as positive, one has $-23.3x10^{-3}+164x10^{-6}T_A=0$

Hence T_A = 142N-m and T_B = 1150 -142 = 1008 N-m.

Step 5:

<u>Now torques at all c/s can be found out using equilibrium.</u>



The torque diagram for the shaft is shown in the above figure. The direction of the internal torque vector T on the left part of an isolated shaft segment coincides with that of the positive x axis, it is taken as positive. Note that most of the applied torque is resisted at the end E. Since the shaft from A to C is more flexible than from C to E, only a small torque develops at A.

In this problem is indeterminate only to the first degree, it has three kinematics degrees of the freedom. Two of these are associated with the applied torques and one with the change in the shaft size. Therefore, an application of the displacement method would be more cumbersome, requiring three simultaneous equations.

Tips while solving similar problems:

Identification of the redundant force is one of the key points in this type of problem. The choice is usually simple.

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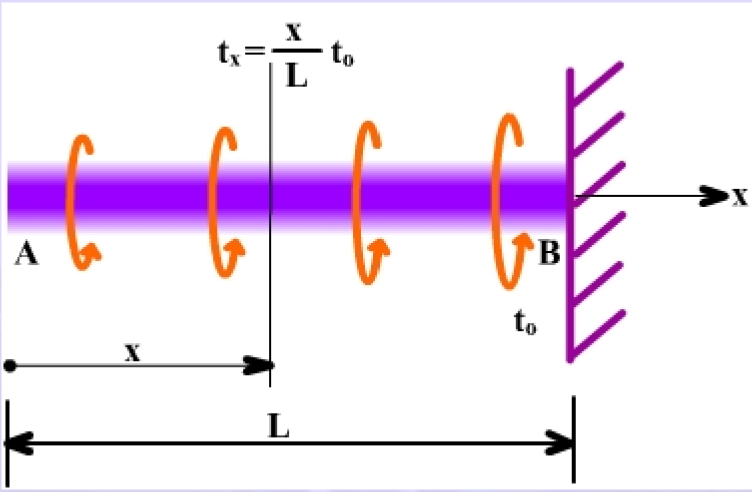
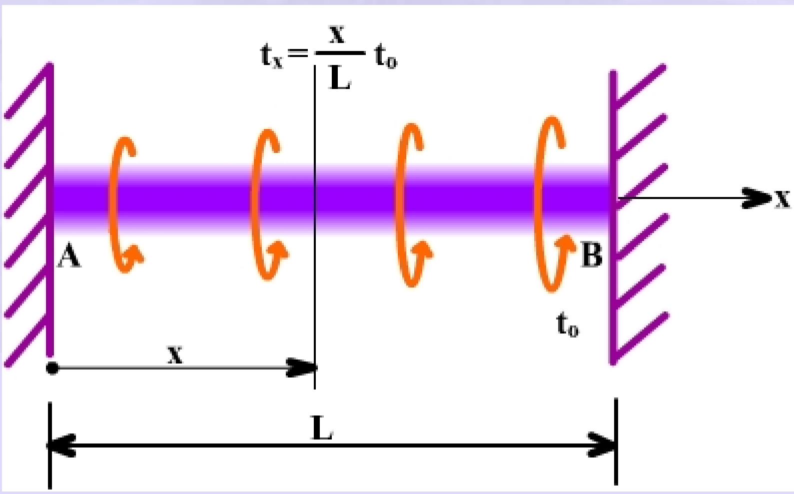
8. Alternative differential approach to Torsion

Consider an elastic circular bar having a constant JG subjected to a uniformly varying torque t_x, as shown in the figure. Determine the rotation of the bar along its length and the reactions at the ends A and B for two cases:

(a) assume that end A is free and the end B is built-in



(b) assume that both ends of the bar are fixed.



Concepts involved:

Differential approach

Formulae used:

$$JG\frac{d^2\phi}{dx^2} = -t_x = -\frac{x}{L}t_o$$

t_x = Distributed torque in a small infinitesimal element

T = Torque developed at the ends of the element due to the distributed torque t_x

Solution:

Part 1:

By integrating the above equation twice and determining the constants of integration C_1 and C_2 from the boundary conditions, the required conditions are determined

$$JG\frac{d^2\phi}{dx^2} = -t_x = -\frac{x}{L}t_o$$

$$JG\frac{d\phi}{dx} = T = -\frac{t_o x^2}{2L} + C_1$$

Apply boundary condition to get C_1 and C_2

T_A=T(0) = 0 hence, C_1=0

T_B = T(L) = - t_o (L / 2)

$$JG\phi = T = -\frac{t_o x^3}{6L} + C_2$$

$$\phi_B = \phi(L) = 0 \quad \text{hence} \quad C_2 = \frac{t_o L}{6}$$

$$JG\phi = \frac{t_o L^2}{6} - \frac{t_o x^3}{6L}$$

The negative sign for TB means that the torque vector acts in the direction opposite to that of the positive x-axis.

Part 2:

Except for the change in the boundary conditions, the solution procedure is the same as in part (a).

$$JG\frac{d^2\phi}{dx^2} = -t_x = -\frac{x}{L}t_o$$

$$JG\frac{d\phi}{dx} = T = -\frac{t_o x^2}{2L} + C_1$$

$$JG\phi = T = -\frac{t_o x^3}{6L} + C_1 x + C_2$$

$\phi_A = \phi\ (0) = 0$ hence $C_2 = 0$

$\phi_B = \phi\ (L) = 0$ hence $C_1 = t_o (L / 6)$

$$JG\phi = \frac{t_o L^2}{6} - \frac{t_o x^3}{6L}$$

$T_A = T(0) = t_o (L / 6)$

$T_B = T(L) = -t_o (L / 2) + t_o (L / 6) = -t_o (L / 3)$

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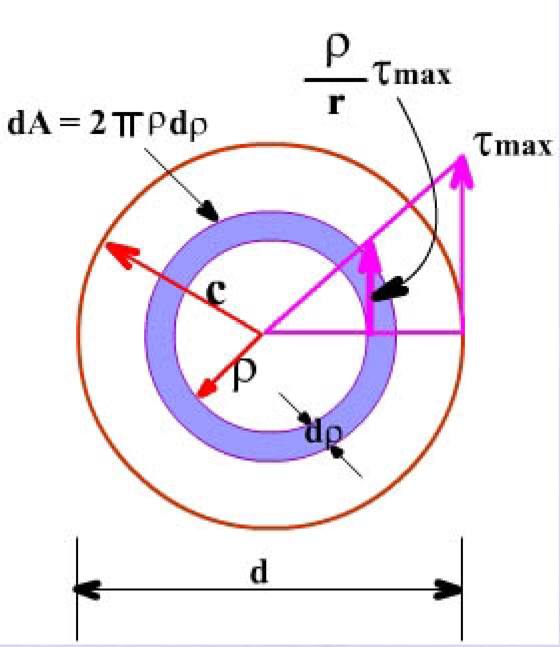
9. Energy and load impacts

Find the energy absorbed by an elastic circular shaft subjected to a constant torque in terms of maximum shear stress and the volume of materials (Refer to the figure). Find the rotation of an elastic circular shaft with respect to the built in end when a torque T is applied at the free end.



Concepts involved:

Energy methods

Formulae used:

Shear energy formula

$$U_s = \int_V \frac{\tau^2}{2G} dV$$

Where,

U_s = Shear energy stored in the body

τ = Shear stress

Solution:

Part 1:

The shear stress in an elastic circular shaft subjected to torque varies linearly from the longitudinal axis. Hence the shear stress acting on an element at a distance ρ from the center of the cross-section is

$$\frac{(\tau_{max}\rho)}{r}$$

Then

$$U_s = \int_V \frac{\tau^2}{2G} dV$$

$$U_s = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{\tau_{max}^2 \rho^2}{2Gr^2} 2\pi\rho \, d\rho \, L$$

$$= \frac{\tau_{max}^2}{2G} \frac{2\pi L}{r^2} \int_0^c \rho^3 d\rho = \frac{\tau_{max}^2}{2G} \frac{2\pi L}{r^2} \frac{r^2}{4}$$

$$= \frac{\tau_{max}^2}{2G} \left(\frac{1}{2} \text{vol} \right)$$

If there were uniform shears throughout the member, a more efficient arrangement for absorbing energy would be obtained. Rubber bushings with their small G values provide an excellent device for absorbing shock torque from a shaft.

Part 2:

If torque T is gradually applied to the shaft, the external work We = (1/2) Tθ , Where θ is the angular rotation of the free end in radians. The expressions for the internal strain energy Us, which was found in part (a), may be written in a more convenient form by noting that τ_{max} = Tr / J, the volume of the rod πr^2L, and J = $\pi r^4/2$. Thus

$$U_s = \frac{{\tau_{max}}^2}{2G}\left(\frac{1}{2}\text{vol}\right)$$

$$= \frac{\tau^2 r^2}{2J^2 G}\frac{1}{2}\pi r 2L$$

$$= \frac{\tau^2 L}{2JG}$$

Then from $W_e = U_s$

$T\theta/2 = T^2L/2JG$ and $\theta = TL/JG$

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