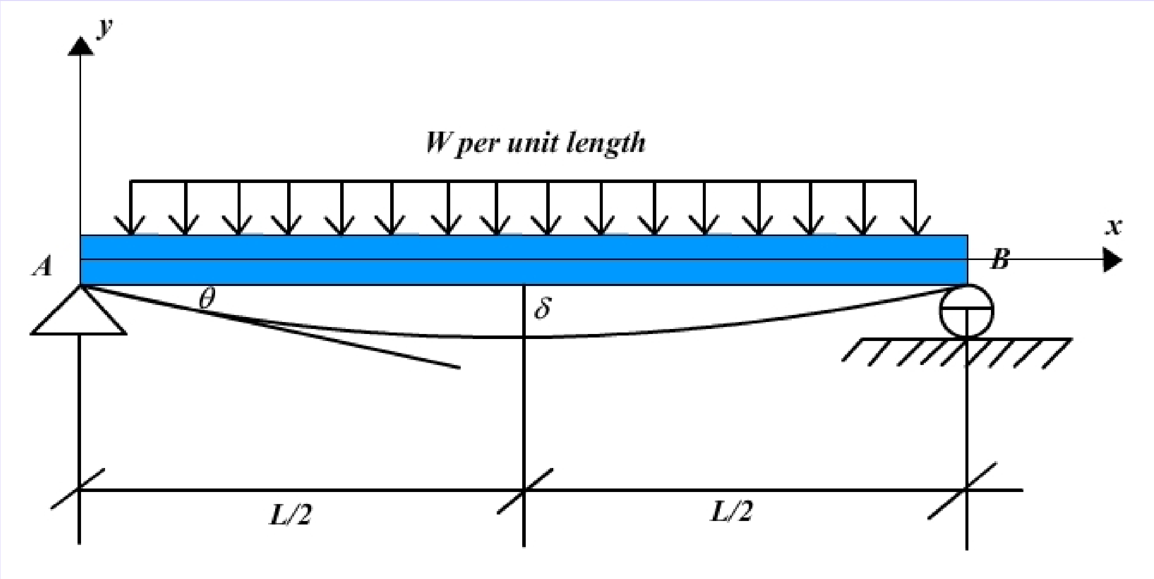
Problem 1: Calculating deflection by integration – uniform load

A simply supported prismatic beam AB carries a uniformly distributed load of intensity w over its span L as shown in figure. Develop the equation of the elastic line and find the maximum deflection δ at the middle of the span.

Figure:



Concepts involved:

Beam defection formula

Formulae used:

$d^2y/dx^2 = M/EI$

Solution:

Taking coordinate axes x and y as shown, we have for the bending moment at any point x.

$$M_x = wLx/2 - wx^2/2$$

And deflection equation becomes

$$EI\, d^2y/dx^2 = wLx/2 - wx^2/2.$$

Multiplying both sides by dx and integrating, we obtain

$$EI\ dy/dx = wLx^2/4 - wx^3/6 + C_1, \text{ ------------------ Equation 1}$$

Where C_1 is an integration constant. To evaluate this constant, we note from symmetry that when x = L/2, dy/dx = 0. From this condition, we find

$$C_1 - wL^3/24$$

And equation 1 becomes

$$EI\ dy/dx = -wLx^2/4 + wx^3/6 - wL^3/24 \text{ ------------------ Equation 2}$$

Again multiplying both sides by dx and integrating,

$$EIy = wLx^3/12 - wx^4/24 - wL^3x/24 + C_2$$

The integration constant C_2 is found from the condition that y = 0 when x = 0.

Thus C_2 = 0 and the required equation for the elastic line becomes

$$y = -wx / 24EI\ (L^3 - 2Lx^2 + x^3)$$

To find the maximum deflection at mid- span, we set x = L/2 in the equation and obtain

$$|\delta| = 5wL^4/384EI$$

The maximum slope θ_A at the left end of the beam can be found by setting x = 0 in the equation 2, which gives

$$\left(\frac{dy}{dx}\right)_{x=0}$$

$$\Rightarrow \theta_A = wL^3/24EI.$$

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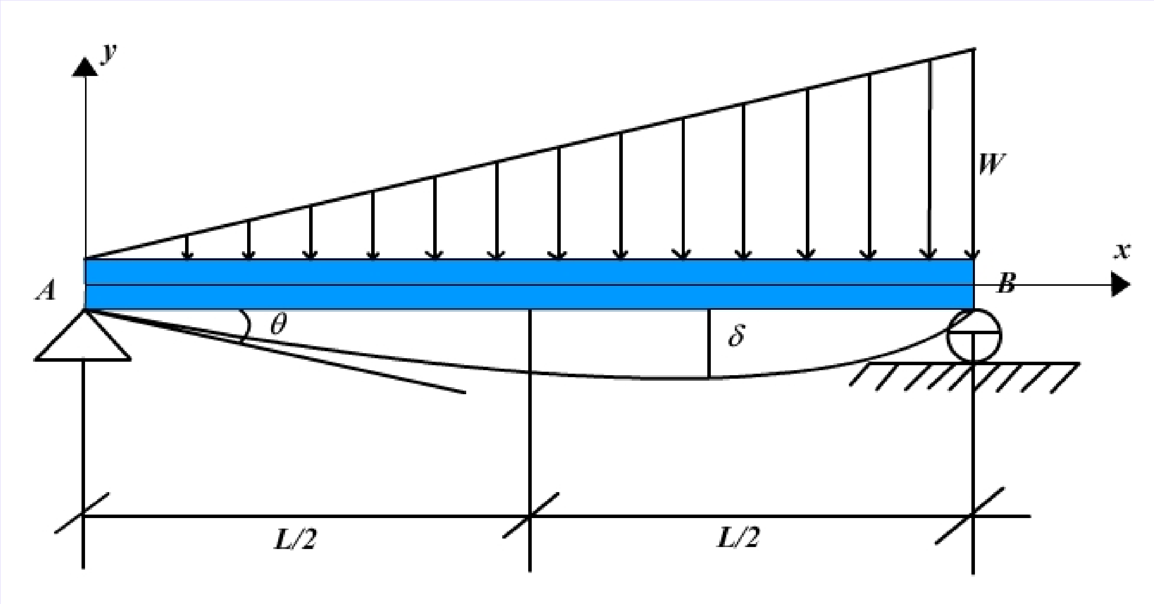
Problem 2 : Calculating deflection by integration - triangular load pattern

A simply supported beam AB carries a triangularly distributed load as shown in the fig. Find the equation of the deflection curve referred to the coordinate axes x and y as shown. Also determine the maximum deflection d .

Figure:



Solution:

In this case we begin directly with deflection equation. We have,

$$EI\, d^4y/dx^4 = - wx/L$$

Separating variables and integrating twice, we obtain

$$EIy\, d^2y/dx^2 = -wx^3/6L + C_1\, x + C_2 \text{ ------------------------1}$$

Again, separating variables and performing two more integrations, we obtain

$$EIy = -wx^5/120L + C_1\, x^3/6 + C_2\, x^2/2 + C_3\, x + C_4 \text{ ------------2}$$

To find the four constants of integration, we now note that the bending moment, represented by equation 1, and the deflection, represented by equation 2, both vanish when x = 0 and when x =L. From these four boundary conditions, we find

$$C_1 = wL/6,\ C_2 = 0,\ C_3 = -7wL^3/360,\ C_4 = 0$$

Substituting these values back into eq.2 and rearranging terms, we obtain

$$Y - (wx/360LEI)\ (7L^4 – 10L^2x^2 + 3x^4)$$

To find the maximum deflection d, we first set dy/dx = 0 and find x = 0.519l. Then using this value of x in the expression for y, the maximum deflection becomes

$$\delta_{max} = |\ d_{\ max}\ | = y_{max} = 0.00652\ wL^4/EI$$

Setting x = L in eq.2, we obtain, the deflection,

$$\delta = |\ d\ | = PL^3/3EI$$

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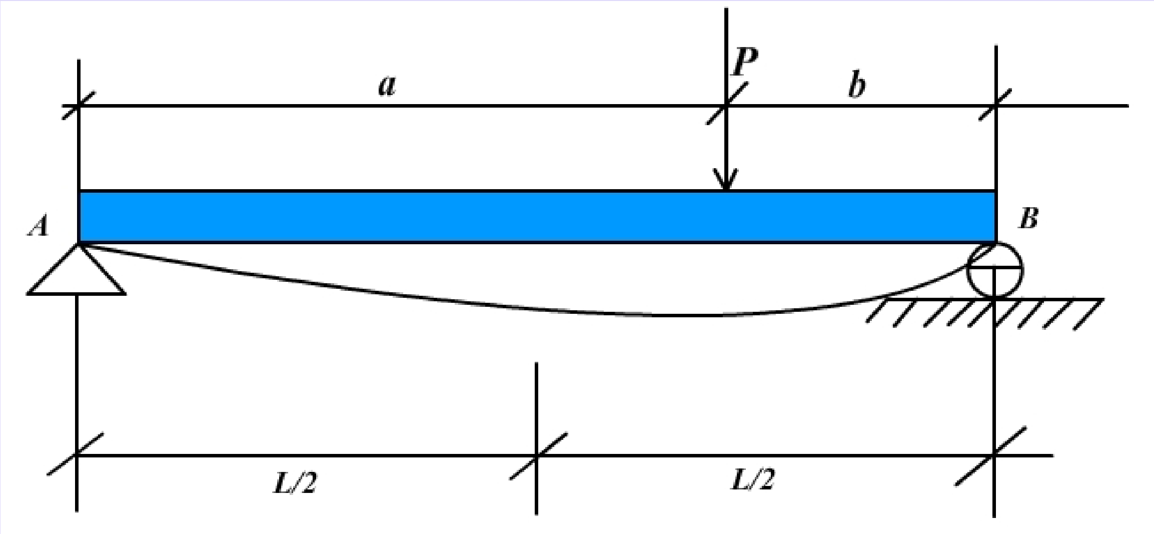
Problem 3 : Deflections - by differential equations, concentrated load

A simply supported prismatic beam AB carries a concentrated load P as shown in the figure. Locate the point of maximum deflection on the elastic line and find the value of this deflection.

Figure:



Solution:

we have for 0 < x < a,

$$M_x = \left(\frac{Pa}{L}\right)_x$$

While for a < x < L,

$$M_x = \frac{Pb}{L}(x - a).$$

Substituting these expressions for bending moment into *deflection equation*, we obtain for the two portions of the deflection curve, the following two differential equations

$$EI\, d^2y/dx^2 = (Pax/ L) \qquad \text{for } 0 \le x \le a$$

$$EI\, d^2y/dx^2 = (-Pa/L)\,(L-x) \qquad \text{for } a \le x \le L$$

Successive integration of these equations gives

$$EI\,dy/dx = Pax^2/2L + C_1 \text{ ------------------------------ (l)} \quad (\text{for } 0 \le x \le a)$$

$$EIy = Pax^3/6L + C_1x + C_2 \text{ ---------------------------- (m)} \quad (\text{for } 0 \le x \le a)$$

$$EI\,dy/dx = Pax - Pax^2/2L + D_1 \text{ ------------------------------ (n)} \quad (\text{for } a \le x \le L)$$

$$EIy = Pax^2/2 - Pax^3/6L + D_1x + D_2 \text{ ------------------------- (o)} \quad (\text{for } a \le x \le L)$$

Where C_1, C_2, D_1, D_2, are constants of integration. To find these four constants, we have the following conditions:

1. At $x = 0$, $y = 0$
2. At $x = L$, $y = 0$
3. $\left(\frac{dy}{dx}\right)_{x=a}$ is the same from equations (l) and (n)
4. $(y)_{x=a}$ is the same from equations (m) and (o)

Using these we get, $C_1 = Pb/6L\,(L^2 - b^2)$, $C_2 = 0$, $D_1 = Pa/6L\,(2L^2 + a^2)$, $D_2 = Pa^3/6EI$

Using the constants as determined eqs. (m) and (o) defining the two portions of the elastic line of the beam become

$$EIy = -\left(\frac{Pbx}{6L}\right)\left(L^2 - b^2 - x^2\right) \text{ --------------------------------------(p)}$$

$$EIy = \left(\frac{Pb}{6L}\right)\left[\left(L/b\right)(x-a)^3 + (L^2 - b^2)\,x - x^3\right] \text{ --------------------------(q)}$$

For a > b, the maximum deflection will occur in the left portion of the span, to which eq.(p) applies. Setting the derivative of this expression equal to zero gives

$$x = \sqrt{a\left(a+2b\right)/3} = \sqrt{\left(L-b\right)\left(L+b\right)/3}$$
$$= \sqrt{\left(L^2 - b^2/3\right)}$$

Which defines the abscissa of the point having a horizontal tangent and hence the point of maximum deflection. Substituting this value of x into eq.(p), we find

$$y_{max} = \frac{Pb\left(L^2 - b^2\right)^{3/2}}{9\sqrt{3}EIL}$$

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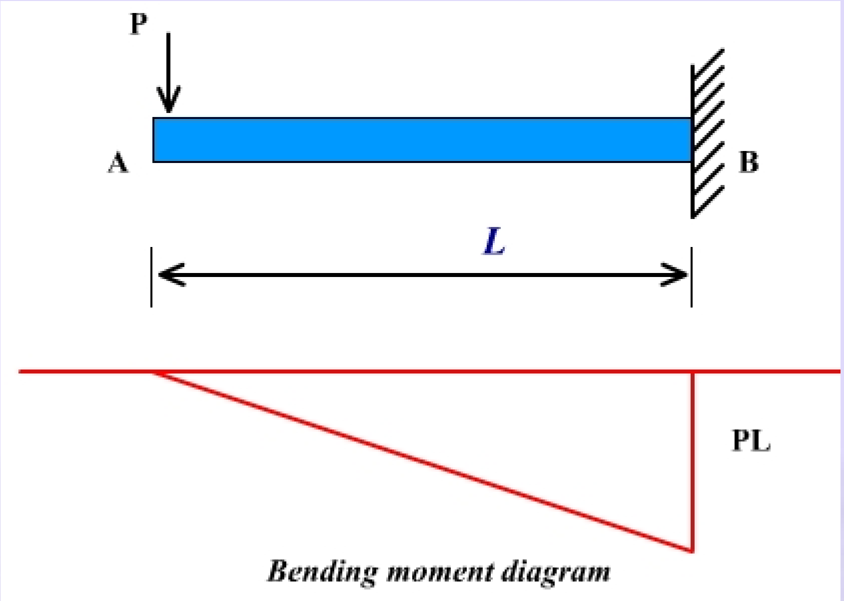
Problem 4: Deflections by Moment area method – Concentrated load

Determine the deflection δ and the slope θ at the free end A of the cantilever beam AB loaded as shown in the figure.

Figure:



Solution:

The bending moment diagram is shown in figure. Since the tangent to the elastic line at B coincides with the undeflected axis of the beam, the required deflection d will be the deflection of A from the tangent at B. Thus using Theorem 2 we have

$$d = \left(\frac{1}{2} * PL * L\right) * \left(\frac{2L}{3}\right) /EI$$

$$= PL^3/3EI \text{ --------------- } (1)$$

Likewise, the slope at A is the angle between tangents at A and at B and from Theorem 1, we have

$$q = \left\{\frac{1}{2} * PL * L\right\} / EI$$

$$= PL^2/2EI \text{ ----------}(2)$$

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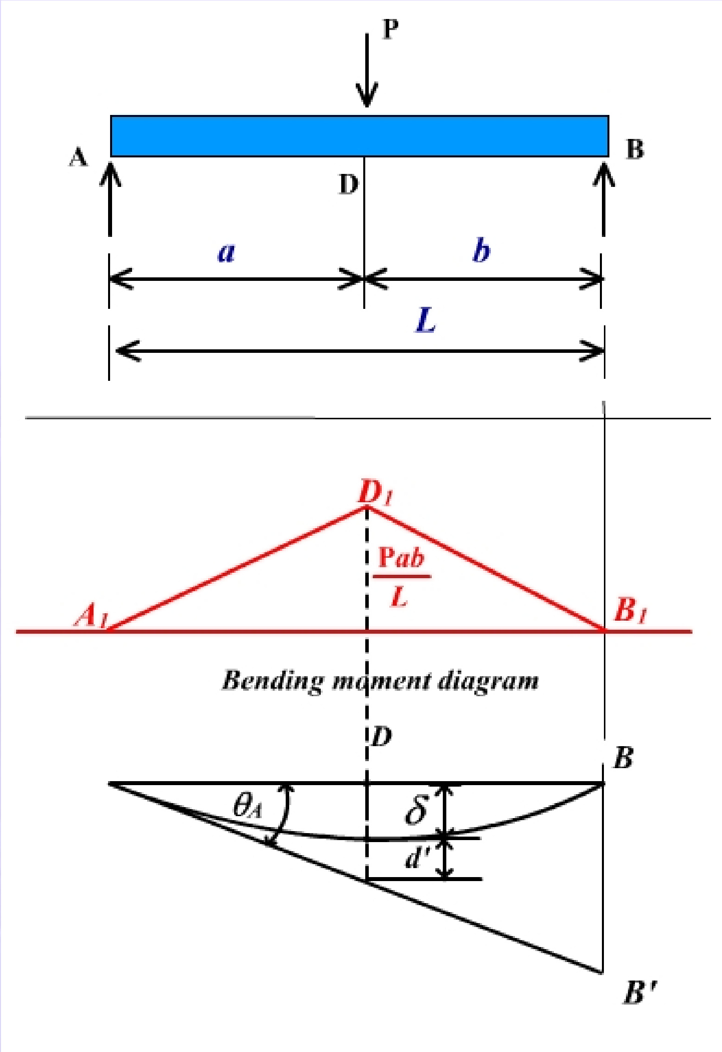
Problem 5: Deflections by Moment area method – Concentrated load

A simply supported beam AB carries a concentrated load P at point D as shown in figure. Find the deflection d of point D from the cord line and the tangent at A.

Figure:



Solution:

The bending moment diagram is shown in following figure. The area of this diagram is Pab/2L and the distance of its centroid C from B is 1/3(L +b) as shown. Taking the statical moment of this area with respect to point B, we obtain the deflection B'B of B away from the tangent at A. thus

$$B'B = \frac{Pab}{2EI}\left(\frac{L+b}{3}\right)$$

Then noting from the figure that θ_A = B'B /l, we have

$$\theta_A = \frac{Pab}{6EIL}(L+b) \quad \text{-------------------------------------- (1)}$$

We see also from the figure that the required deflection of point D from the cord line AB is

$$\delta = (a * \theta_A) - d' \quad \text{--------------------------------------(2)}$$

Where d' is the deflection of D away from the tangent at A. This deflection d' can be found by using theorem 2 for the portion A_1D_1 of the bending moment diagram. The area of this is $Pa^2b/2L$ and its centroid is at the distance a/3 to the left of D. Thus

$$d' = (Pa^2b/2LEI) * a/3 \quad \text{-------------------------------------(3)}$$

Substitutions of expressions (3) and (1) into eq.(2) gives

$$\delta = \frac{Pa^2b}{6EIL}(L+b) - \frac{Pa^3b}{6LEI} = \frac{Pa^2b^2}{3LEI} \quad \text{-------------------------------------- (4)}$$

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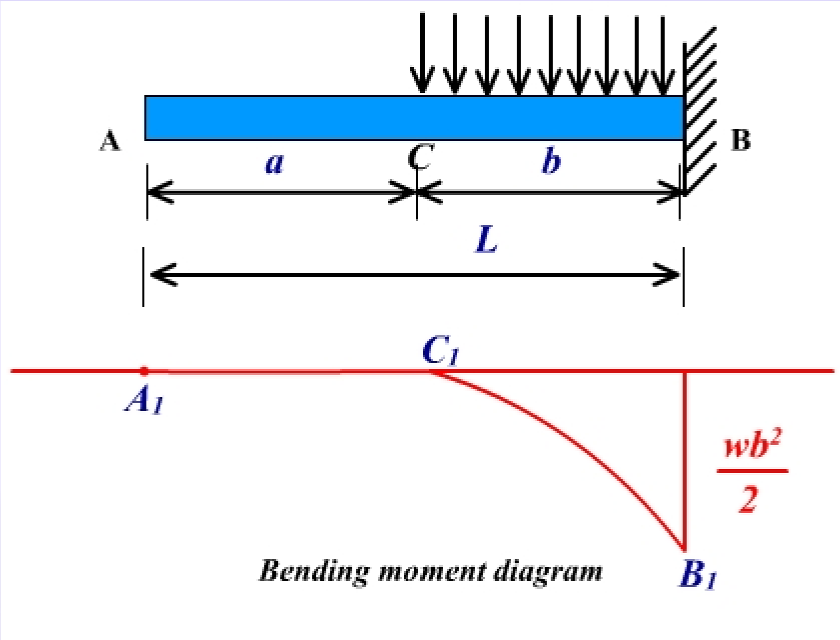
Problem 6: Moment Area Method - udl

A prismatic cantilever beam AB carries a uniformly distributed load over the portion b of its length as shown in the figure. Find the deflection δ of the free end A

Figure:



Solution:

The bending moment diagram is shown in the figure. Its area $wb^3/6$ from the position of its centroid C are found by reference to the figure. Now from the second theorem the deflection will be obtained by dividing by EI the static moment of this area with respect to point A_1. Thus

$$\delta = \frac{wb^3}{6EI} \times \left\{ a + \frac{3b}{4} \right\}$$

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