

Lecture 5.4

Methods based on Coupling of Pressure and Velocity

Methods based on Coupling of Pressure and Velocity

- In these methods, the governing equations of the flow problem are solved in their primitive variable formulation, that is, in u , v , and p in two dimensions.
- These methods can be applied to both transient and stationary equations, therefore, to highlight the implementation in both the cases, some methods are described for transient equations and the other for the stationary equations.

Pressure Correction Methods

- Pressure correction methods, in general, start with a guessed pressure field to solve the momentum equations for an approximate velocity field.
- In the next step pressure is updated by enforcing divergence free velocity field which results in to a Poisson equation for pressure.
- The procedure is repeated until the velocity field obtained from the momentum equations satisfies the divergence free condition.

Pressure Correction Methods

- The Poisson equation for the pressure is obtained by taking the divergence on the momentum equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \nu (\Delta \vec{v}) \quad (5.4.1)$$

where Δ is the Laplacian operator. Enforcing the divergence free condition, the pressure Poisson equation is given by

$$\frac{1}{\rho} \Delta p = -\vec{\nabla} \cdot (\vec{v} \cdot \vec{\nabla}) \vec{v} \quad (5.4.2)$$

- For two-dimensions Eq. (5.4.2) is equivalent to

$$\frac{1}{\rho} \Delta p = \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) = \frac{\partial(u, v)}{\partial(x, y)} \quad (5.4.3)$$

Fractional Step Methods

- Fractional step methods, also known as projection methods, are in general applied to the transient equations.
- In this method, the pressure terms are neglected from the momentum equations and an intermediate velocity field $\vec{v}^*$ is obtained by solving them. Then the velocity field is updated by considering the pressure gradient terms.

- For transient equations, the scheme is given by

$$\begin{aligned}\vec{v}^* &= \vec{v}^n + \Delta t \left[-(\vec{v} \cdot \vec{\nabla}) \vec{v} + \nu (\Delta \vec{v}) \right]^n \\ \vec{v}^{n+1} &= \vec{v}^* - \Delta t \frac{1}{\rho} \vec{\nabla} p\end{aligned}\tag{5.4.4}$$

- Taking divergence on the second equation of Eq. (5.4.4) and enforcing the divergence free condition at (n+1) time level, gives

$$\Delta p = \frac{\rho}{\Delta t} \vec{\nabla} \cdot \vec{v}^*\tag{5.4.5}$$

Fractional Step Methods

- After solving Eq. (5.4.5), using any iteration method, for pressure (in general using Neumann boundary conditions), the second equation of Eq. (5.4.4) is used to compute the velocity field at the $(n+1)^{\text{th}}$ time level.
- In the pressure velocity (de)coupling methods, the pressure calculation which is realized using a Poisson equation is a crucial step which has to be implemented very efficiently.
- One may make use of multi-grid techniques in this step to enhance the convergence of the iterative procedures.

Staggered Grid

- It can be seen that checker board pressure and velocity fields (pressure and velocity are constant on alternating points) satisfy the continuity equation and make the momentum equations independent of the pressure terms.
- There is a danger of generating such unphysical solutions while solving the incompressible flow equations.
- However, such solutions can be avoided, if the computations are carried out on a staggered grid in which, the pressure (all scalar variables) is computed at the conventional grid points but velocity components are computed at the geometric centre of the two consecutive pressure points as shown in Fig. 5.4.1.

Staggered Grid

- In the cell centred finite volume approach, it is easier to state the same as, computing the pressure at the cell centres and the velocity components at the cell faces.

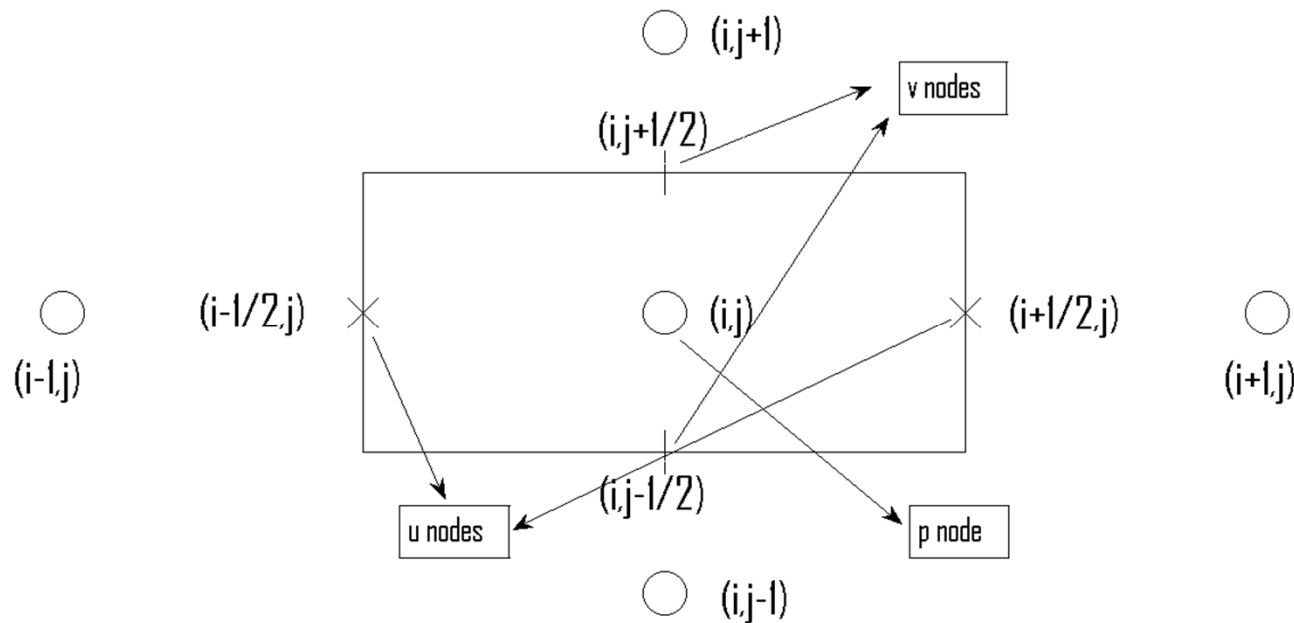


Fig. 5.4.1 p, u and v nodes in staggered approach

Staggered Grid

- If momentum equations are approximated at their velocity component nodes and continuity equation is evaluated at the p-node, then the problems encountered like unphysical checker board kind of solutions can be avoided while solving the incompressible flow equations.
- Evaluation of x-momentum equation at the u-node (i-1/2, j): Consider the x-momentum equation in 2D given by

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (5.4.6)$$

- Each term of Eq. (5.4.6) at $(i-1/2, j)$ is evaluated as follows (assuming constant step lengths $\Delta x, \Delta y$):

$$\left(u \frac{\partial u}{\partial x} \right)_{\left(i-\frac{1}{2}, j \right)} = u_{\left(i-\frac{1}{2}, j \right)} \frac{u_{\left(i+\frac{1}{2}, j \right)} - u_{\left(i-\frac{3}{2}, j \right)}}{2\Delta x} \quad (5.4.7)$$

$$\left(v \frac{\partial u}{\partial y} \right)_{\left(i-\frac{1}{2}, j \right)} = \frac{1}{4} \left(v_{\left(i, j+\frac{1}{2} \right)} + v_{\left(i, j-\frac{1}{2} \right)} + v_{\left(i-1, j+\frac{1}{2} \right)} + v_{\left(i-1, j-\frac{1}{2} \right)} \right) \frac{u_{\left(i-\frac{1}{2}, j+1 \right)} - u_{\left(i-\frac{1}{2}, j-1 \right)}}{2\Delta y} \quad (5.4.8)$$

$$\left(\frac{\partial p}{\partial x} \right)_{\left(i-\frac{1}{2}, j \right)} = \frac{p_{(i,j)} - p_{(i-1,j)}}{\Delta x} \quad (5.4.9)$$

$$\left(\frac{\partial^2 u}{\partial x^2} \right)_{\left(i-\frac{1}{2}, j \right)} = \frac{u_{\left(i-\frac{3}{2}, j \right)} - 2u_{\left(i-\frac{1}{2}, j \right)} + u_{\left(i+\frac{1}{2}, j \right)}}{\Delta x^2} \quad (5.4.10)$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{\left(i-\frac{1}{2},j\right)} = \frac{u_{\left(i-\frac{1}{2},j+1\right)} - 2u_{\left(i-\frac{1}{2},j\right)} + u_{\left(i-\frac{1}{2},j-1\right)}}{\Delta y^2} \quad (5.4.11)$$

- Substitution of Eqs. (5.4.7) to (5.4.11) in Eq. (5.4.6) completes the discretisation of x-momentum equation at the u-node $(i - 1/2, j)$. The final discretised equation is given by

$$\begin{aligned} u_{\left(i-\frac{1}{2},j\right)} \frac{u_{\left(i+\frac{1}{2},j\right)} - u_{\left(i-\frac{3}{2},j\right)}}{2\Delta x} + \frac{1}{4} \left(v_{\left(i,j+\frac{1}{2}\right)} + v_{\left(i,j-\frac{1}{2}\right)} + v_{\left(i-1,j+\frac{1}{2}\right)} + v_{\left(i-1,j-\frac{1}{2}\right)} \right) \frac{u_{\left(i-\frac{1}{2},j+1\right)} - u_{\left(i-\frac{1}{2},j-1\right)}}{2\Delta y} = \\ - \frac{p_{(i,j)} - p_{(i-1,j)}}{\Delta x} + \frac{\mu}{\rho} \left(\frac{u_{\left(i-\frac{3}{2},j\right)} - 2u_{\left(i-\frac{1}{2},j\right)} + u_{\left(i+\frac{1}{2},j\right)}}{\Delta x^2} + \frac{u_{\left(i-\frac{1}{2},j+1\right)} - 2u_{\left(i-\frac{1}{2},j\right)} + u_{\left(i-\frac{1}{2},j-1\right)}}{\Delta y^2} \right) \end{aligned} \quad (5.4.12)$$

- In a short notation Eq. (5.4.12) can be written as

$$u_{i-\frac{1}{2},j} = \sum_{i,j} a_{i,j} u_{i,j} - c_{i-\frac{1}{2},j} (p_{i,j} - p_{i-1,j}) + b_{i-\frac{1}{2},j} \quad (5.4.13)$$

- In Eq. (5.4.13), the summation of u on the RHS contains the terms at $(i-3/2, j)$, $(i+1/2, j)$, $(i-1/2, j-1)$ and $(i-1/2, j+1)$ with their coefficients $a_{i,j}$. b_{ij} contains the source terms, if any, and boundary conditions. c_{ij} are the coefficients of the pressure terms after simplification.
- A similar approach to discretize the y-momentum equation at v-node $(i, j-1/2)$ is very straight forward and hence is not repeated here.
- In the discretization of the convective terms of Eqs. (5.4.7) and (5.4.8), central differences have been used. (In the approximation of the first partial derivatives).
- For convection dominant problems, the discretization of type Eq. (5.4.12) may diverge, therefore, for those problems, the central differences in Eqs. (5.4.7) and (5.4.8) may be replaced with the upwind approximations.

Exercise Problems

1. Derive the coefficients a_{ij} , b_{ij} and c_{ij} in Eq. (5.4.13)
2. Change the central difference approximations in Eq. (5.4.7) and Eq. (5.4.8) to upwind approximations and then derive the coefficients a_{ij} , b_{ij} and c_{ij} in Eq. (5.4.13).
3. Discretize the y-momentum equation at v-node $(i, j-1/2)$
4. Consider the conservative form of Eq. (5.4.6) in the cell formed around the u-node $(i-1/2, j)$ and discuss the cell centred finite volume discretisation.

Summary of Lecture 5.4

- Incompressible flow computations over staggered grids are introduced in this lecture.

END OF LECTURE 5.4