Lecture 1.5

Governing Equations in Rotating Frame of Reference

(Inviscid and Viscous Flows)

Rotating Frame of Reference

- It is mentioned in the Lecture 1.2 that the most convenient coordinate system for use in turbomachinery rotors is the relative or rotating coordinate system.

- In the previous lectures, however, the description of fluid motion is restricted only to the stationary parts such as nozzles, stators, guide vanes and the governing equations are written using a stationary frame of reference.

- The advantages of a relative system for flow through rotating passages are as follows:
  - The relative flow is steady in most cases
  - Boundary conditions are easier to apply;
  - Velocity profiles and boundary layers are similar to those observed in a stationary system.

- The general accelerating frame of reference is given in lecture 1.2. However, if the reference frame is only a rotating one, the coordinate system and the velocity diagram can be shown as in Fig. 1.5.1.

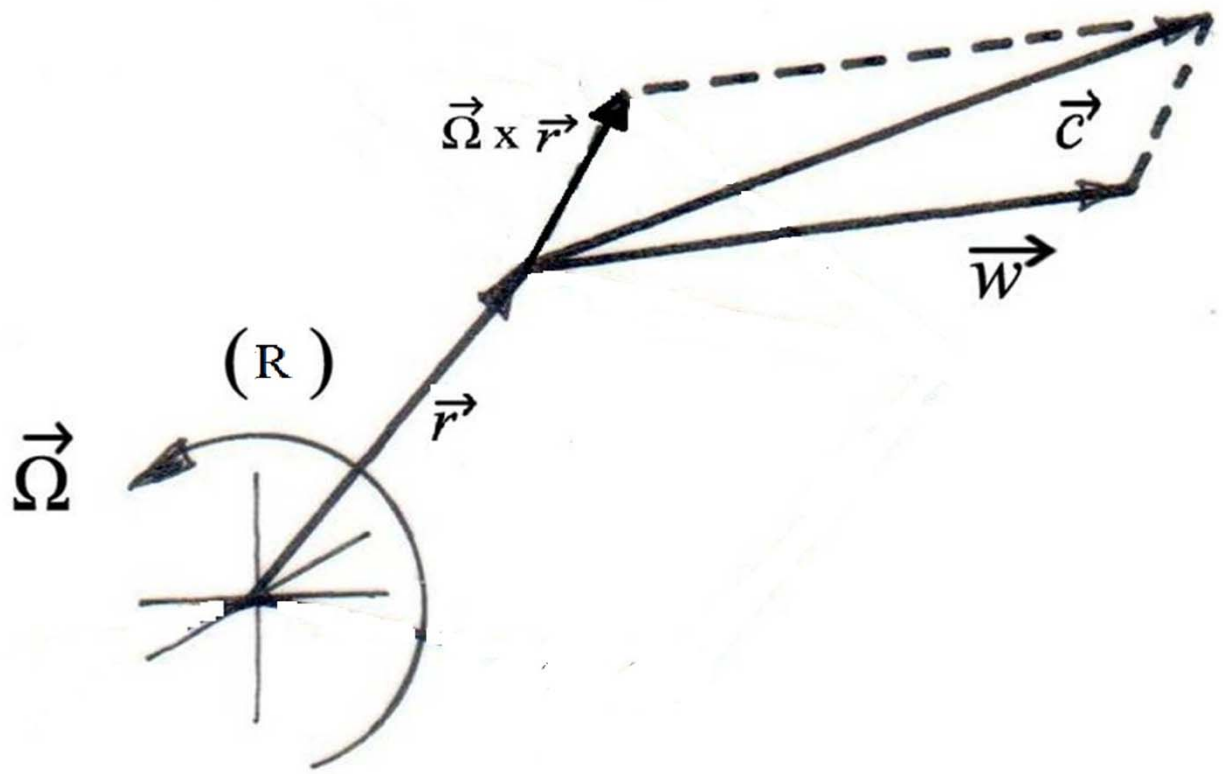


Fig. 1.5.1 Rotating coordinate system and velocity triangle

- If the frame of reference is attached to the rotor then it rotates with an angular velocity ω about the machine axis.

- For a stationary observer, the fluid particle on the rotating frame at an arbitrary time 't' has a position vector $\vec{r}$ and a relative velocity $\vec{W}$. In addition, it is subjected to an inherent rotation of the frame causing the fluid particle to rotate with the velocity $\vec{\omega} \times \vec{r}$.
- Therefore, the observer located outside the rotating frame observes the velocity

$$\vec{C} = \vec{W} + \vec{\omega} \times \vec{r} \qquad (1.5.1)$$

Note that Eq. (1.5.1) has been mentioned earlier in the context of a single blade, refer, Lecture 1.2.

- In the turbomachinery passages, the variation of the absolute velocity (C) is highly skewed, and the absolute velocity on the surface of the blade is not zero.

- Figure 1.5.2 shows (i) a typical blade passage, (ii) the absolute velocity in the passage, (iii) the absolute and relative streamlines and (iv) the velocity triangle at location B (in the blade passage).

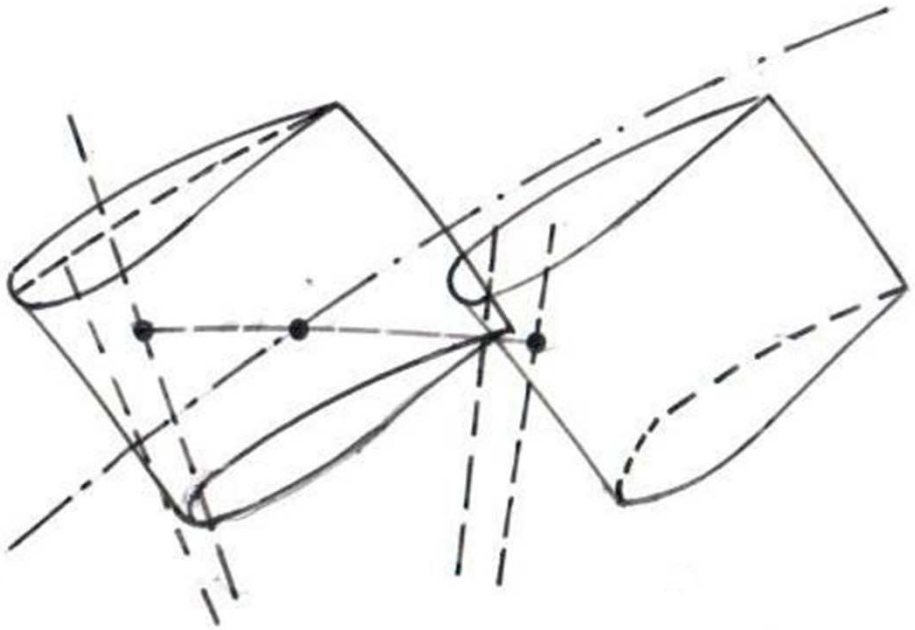


Figure 1.5.2 Absolute and relative velocities in a rotor blade cascade with viscous boundary conditions on the surface.

- Furthermore, it has been originally proved by Dean that the absolute velocity in a rotor has to be unsteady for the pressure rise or drop to occur across a rotor.

- Consider the absolute streamline s_A (total velocity, V) in Fig. 1.5.2. Along the streamline s_A, for an adiabatic inviscid flow we have

$$\left(\frac{DV}{Dt}\right) = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s_A} = -\frac{1}{\rho}\frac{\partial p}{\partial s_A} = -\frac{\partial h}{\partial s_A}$$

Hence

$$\left(\frac{\partial V}{\partial t}\right) = -\frac{\partial h_o}{\partial s_A} \qquad (1.5.2)$$

- Thus the absolute velocity has to be unsteady to produce a stagnation enthalpy change. Similarly, it can be proved that the static pressure in a stationary (fixed point) frame has to be unsteady.

- Let us consider the streamline s_A (fixed frame), for an inviscid flow:

$$\frac{Dh_o}{Dt} = \frac{\partial h_o}{\partial t} + V\frac{\partial h_o}{\partial s_A} = \frac{\partial h_o}{\partial t} - V\frac{\partial V}{\partial t} = \frac{\partial h}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

- Hence the stagnation enthalpy change across a blade can be achieved only when the pressure at a fixed point is unsteady. Thus for a rotor, it is essential to include time-dependent terms in the absolute reference frame.

- It is necessary to include the time-dependent terms in the relative frame of reference only when the inflow (absolute or relative) velocity is circumferentially non-uniform or time-dependent. If the absolute flow is uniform at inlet to a rotor, the relative flow is usually steady.

Acceleration in the Rotating Frame

$$\frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t}\left(\vec{W} + \vec{\omega}\times\vec{r}\right) + \left(\vec{W} + \vec{\omega}\times\vec{r}\right) + \vec{\nabla}\left(\vec{W} + \vec{\omega}\times\vec{r}\right) \qquad (1.5.3)$$

$$= \frac{\partial\vec{\omega}}{\partial t} + \vec{r}\times\frac{\partial\vec{\omega}}{\partial t} + \vec{W}\,.\,\vec{\nabla}\,\vec{W} + \vec{W}\,.\,\vec{\nabla}\left(\vec{\omega}\times\vec{r}\right) + \left(\vec{\omega}\times\vec{r}\right).\vec{\nabla}\,\vec{W} + \left(\vec{\omega}\times\vec{r}\right).\vec{\nabla}\left(\vec{\omega}\times\vec{r}\right); \quad \left(\because\ \vec{\omega}\,\frac{\partial\vec{r}}{\partial t} = 0\right) \qquad (1.5.4)$$

Since we have

$$\vec{W}\,.\,\vec{\nabla}\left(\vec{\omega}\times\vec{r}\right) = W_1\,\frac{\partial}{\partial x}\left(\vec{\omega}\times\vec{r}\right) + W_1\,\frac{\partial}{\partial y}\left(\vec{\omega}\times\vec{r}\right) + W_1\,\frac{\partial}{\partial z}\left(\vec{\omega}\times\vec{r}\right) \qquad (1.5.5)$$

where

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} \qquad (1.5.6)$$

$$= i\left(\omega_2 z - \omega_3 y\right) - j\left(\omega_1 z - \omega_3 x\right) + k\left(\omega_1 y - \omega_2 x\right) \qquad (1.5.7)$$

$$\vec{W} \cdot \vec{\nabla}\left(\vec{\omega} \times \vec{r}\right) = i\left(W_3\omega_2 - W_2\omega_3\right) - j\left(W_3\omega_1 - W_1\omega_3\right) + k\left(W_2\omega_1 - W_1\omega_2\right)$$
$$= \vec{\omega} \times \vec{W} \qquad (1.5.8)$$

Similarly
$$\left(\vec{\omega} \times \vec{r}\right) \cdot \vec{\nabla}\vec{W} = \vec{\omega} \times \vec{W} \qquad (1.5.9)$$

$$\left(\vec{\omega} \times \vec{r}\right) \cdot \vec{\nabla}\left(\vec{\omega} \times \vec{r}\right) = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) \qquad (1.5.10)$$

- Therefore the acceleration in the rotating frame can be written as

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{W}}{\partial t} + \vec{r} \times \frac{\partial\vec{\omega}}{\partial t} + \vec{W} \cdot \vec{\nabla}\vec{W} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + 2\vec{\omega} \times \vec{W} \qquad (1.5.11)$$

- The terms on the right hand side of Eq. (1.5.11) $\partial\vec{W}/\partial t$ are the local acceleration within the relative frame of reference $r \times \frac{\partial \omega}{\partial t}$ is the angular acceleration, $\vec{w} \,.\, \vec{\nabla}\vec{w}$ is the convective term within the relative frame of reference.

- The other two terms $\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)$ and $2\vec{\omega} \times \overrightarrow{W}$ are the centrifugal and Coriolis forces respectively. These two are the additional terms due to the transformation from absolute into a relative frame of reference.

- The direction of Coriolis force is perpendicular to the vector $\vec{\omega}$ and $\overrightarrow{W}$ whereas $\vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{W}\right)$ is also perpendicular but points towards the axis of rotation.

Continuity Equation in Rotating Frame of Reference

- Using the velocity vector $\vec{V} = \overrightarrow{W} + \vec{\omega} \times \vec{r}$ in the continuity Eq. (1.5.14), we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[\rho \left(\overrightarrow{W} + \vec{\omega} \times \vec{r} \right) \right] = 0$$

$$\frac{\partial \rho}{\partial t} + \left(\overrightarrow{W} + \vec{\omega} \times \vec{r} \right) \cdot \nabla \rho + \rho \nabla \cdot \omega + \rho \nabla \cdot \left(\vec{\omega} \times \vec{r} \right) = 0 \qquad (1.5.12)$$

$$\frac{\partial \rho}{\delta t} + \left(\vec{\omega} \times \vec{r} \right) \cdot \nabla \rho + \overrightarrow{W} \cdot \nabla \rho + \rho \nabla \cdot \omega + \rho \nabla \cdot \left(\vec{\omega} \times \vec{r} \right) = 0$$

- $\frac{\partial}{\partial t}$ and $(\vec{\omega} \times \vec{r} \cdot \nabla)$ are the local rate of change and spatial change respectively with respect to a stationary observer.

- These two terms together may be written as

$$\frac{\partial_R \rho}{\partial t} = \frac{\partial \rho}{\partial t} + \vec{\omega} \times \vec{r} \cdot \nabla \rho \qquad (1.5.13)$$

- This is nothing but the time rate of change within the rotating frame of reference. Therefore, the continuity equation in a rotating frame can be written as

$$\frac{\partial_R \rho}{\partial t} + \overrightarrow{W} \cdot \nabla \rho + \rho \nabla \cdot \vec{\omega} = 0 \qquad (1.5.14)$$

$$\frac{\partial_R \rho}{\partial t} + \nabla \cdot \left(\rho \vec{\omega} \right) = 0 \left(\nabla \cdot \left(\vec{\omega} \times \rho \right) \right) = 0 \qquad (1.5.15)$$

- This is similar to the continuity equation in an absolute frame with $\frac{\partial_R}{\partial t}$ in place of $\frac{\partial}{\partial t}$.

Equation of Motion in a Rotating Frame

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\vec{\omega}d\Omega+\oint_{S}\rho\vec{\omega}\left(\vec{\omega}\cdot\overrightarrow{ds}\right)=\int_{\Omega}\rho\,\vec{f}d\Omega+\int_{\Omega}-2\rho\left(\vec{\omega}\times\vec{W}\right)d\Omega \qquad (1.5.16)$$
$$+\int_{\Omega}\rho-\left(\vec{\omega}\times\left(\vec{\omega}\times\vec{W}\right)\right)d\Omega-\oint_{S}\rho\cdot\overrightarrow{ds}+\oint_{S}\overline{\overline{\tau}}\cdot\overrightarrow{ds}$$

- Integral equation (1.5.16) can be written in differential form as

$$\frac{\partial}{\partial t}\rho\vec{\omega}+\nabla\cdot\left(\rho\vec{W}\otimes\vec{W}\right)=\rho\vec{f}-\rho\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)-2\rho\left(\vec{\omega}\times\vec{W}\right)-\nabla p+\nabla\cdot\overline{\overline{\tau}} \qquad (1.5.17)$$

$$\Rightarrow\frac{\partial\vec{\omega}}{\partial t}+\vec{r}\times\frac{\partial\vec{\omega}}{\partial t}+\vec{\omega}\cdot\vec{\nabla}\,\vec{\omega}+\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)+2\vec{\omega}\times\vec{W}=\vec{f}-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\nabla\cdot\overline{\overline{\tau}} \qquad (1.5.18)$$

- The shear stress $\overline{\overline{\tau}}$ is a function of a relative velocity and the rotation of the system has no effect on the internal forces within the fluid, since they cannot be influenced by solid body motions of one system with respect to the other.

Euler's Equations of Motion

- Using the definition of $\overline{\overline{\tau}}$ RHS of (1.5.18) can be written as (for Newtonian fluids)

$$\vec{f} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \overline{\overline{\tau}} = -\frac{\nabla P}{\rho} + \vec{f} + \frac{\mu}{\rho}\left(\Delta\vec{v} + \frac{1}{3}\vec{v}\left(\vec{\nabla}.\vec{v}\right)\right) \quad (1.5.19)$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.

- For incompressible flows ($\nabla \cdot \vec{v} = 0$), Equation (1.5.18) takes the form

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{r}\times\frac{\partial\vec{\omega}}{\partial t} + \vec{\omega}\cdot\vec{\nabla}\vec{\omega} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + 2\vec{\omega}\times\vec{W} = -\frac{\nabla P}{\rho} + \vec{f} + \frac{\mu}{\rho}\left(\Delta\vec{v}\right) \quad (1.5.20)$$

- Finally, for inviscid flows the Euler's equations can be written as

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{r}\times\frac{\partial\vec{\omega}}{\partial t} + \vec{\omega}\cdot\vec{\nabla}\vec{\omega} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + 2\vec{\omega}\times\vec{W} = -\frac{\nabla P}{\rho} + \vec{f} \quad (1.5.21)$$

Equation of Energy in a Rotating Frame

- The work done by the centrifugal force has to be added to the energy equation in absolute frame since the Coriolis term does not contribute to the energy balance.

- In conservative form it can be written as

$$\frac{\partial}{\partial t}\rho\left(e+\frac{\vec{\omega}^2}{2}-\frac{\vec{v}^2}{2}\right)+\vec{\nabla}\left[\rho\vec{W}\left(h+\frac{\vec{\omega}^2}{2}-\frac{\vec{v}^2}{2}\right)-k\,\vec{\nabla}T-\bar{\bar{\tau}}\cdot\vec{\omega}\right]=\rho\vec{f}\cdot\vec{\omega} \quad (1.5.22)$$

- The governing equations for turbomachinery flows are often written in the rotating cylindrical coordinate system, especially when the geometric model is amenable to the rotational symmetry. These equations are given in the next section.

Navier - Stokes Equations in Rotating Cylindrical Coordinate System

$$\frac{\partial U}{\partial t}+\frac{1}{r}\frac{\partial\left(rE\right)}{\partial r}+\frac{1}{r}\frac{\partial F}{\partial\theta}+\frac{\partial G}{\partial z}=\frac{1}{r}S \qquad (1.5.23)$$

$$U=\begin{bmatrix}\rho\\ \rho W_r\\ \rho W_\theta\\ \rho W_z\\ \rho e_0\end{bmatrix},\quad E=\begin{bmatrix}\rho W_r\\ \rho W_r^2+p-\tau_{rr}\\ \rho W_r W_\theta-\tau_{r\theta}\\ \rho W_r W_z-\tau_{rz}\\ \rho W_r h_0-W_r\tau_{rr}-W_\theta\tau_{r\theta}-W_z\tau_{zr}+Q_r\end{bmatrix}$$

$$F=\begin{bmatrix}\rho W_\theta\\ \rho W_\theta W_r-\tau_{\theta r}\\ \rho W_\theta^2+p-\tau_{\theta\theta}\\ \rho W_\theta W_z-\tau_{\theta z}\\ \rho W_\theta h_0-W_r\tau_{\theta r}-W_\theta\tau_{\theta\theta}-W_z\tau_{\theta z}+Q_\theta\end{bmatrix},\quad G=\begin{bmatrix}\rho W_z\\ \rho W_z W_r-\tau_{zr}\\ \rho W_z W_\theta-\tau_{z\theta}\\ \rho W_z^2+p-\tau_{zz}\\ \rho W_z h_0-W_r\tau_{zr}-W_\theta\tau_{z\theta}-W_z\tau_{zz}+Q_z\end{bmatrix}$$

$$S = \begin{bmatrix} 0 \\ p + \rho W_\theta^2 + \rho \Omega^2 r^2 + 2\rho r \Omega W_\theta \\ -2\rho r \Omega W_r \\ 0 \\ 0 \end{bmatrix}$$

$$\tau_{\theta\theta} = \frac{2}{3}\mu\left[2\left(\frac{1}{r}\frac{\partial W_\theta}{\partial \theta} + \frac{W_\theta}{r}\right) - \frac{\partial W_z}{\partial z} - \frac{\partial W_r}{\partial r}\right] - \overline{\rho\left(w_\theta'\right)^2},$$

$$Q_\theta = -\gamma\left(\frac{\mu}{PR}\right)\frac{1}{r}\frac{\partial e}{\partial \theta} + \overline{\rho w_\theta' e'}$$

$$\tau_{zz} = \frac{2}{3}\mu\left[2\frac{\partial W_z}{\partial z} - \frac{\partial W_r}{\partial r} - \frac{W_r}{r} - \frac{1}{r}\frac{\partial W_\theta}{\partial \theta}\right] - \overline{\rho\left(w_z'\right)^2},$$

$$Q_z = -\gamma\left(\frac{\mu}{PR}\right)\frac{\partial e}{\partial z} + \overline{\rho w_z' e'}$$

$$\tau_{zz} = \frac{2}{3}\mu\left[2\frac{\partial W_r}{\partial r} - \frac{W_r}{r} - \frac{1}{r}\frac{\partial W_\theta}{\partial \theta} - \frac{\partial W_z}{\partial z}\right] - \overline{\rho\left(w_r'\right)^2},$$

$$Q_r = -\gamma\left(\frac{\mu}{PR}\right)\frac{\partial e}{\partial r} + \overline{\rho w_r' e'}$$

$$\tau_{\theta\theta} = \frac{2}{3}\mu\left[2\left(\frac{1}{r}\frac{\partial W_\theta}{\partial\theta} + \frac{W_\theta}{r}\right) - \frac{\partial W_z}{\partial z} - \frac{\partial W_r}{\partial r}\right] - \overline{\rho\left(w_\theta'\right)^2},$$

$$Q_\theta = -\gamma\left(\frac{\mu}{PR}\right)\frac{1}{r}\frac{\partial e}{\partial\theta} + \overline{\rho w_\theta' e'}$$

$$\tau_{zz} = \frac{2}{3}\mu\left[2\frac{\partial W_z}{\partial z} - \frac{\partial W_r}{\partial r} - \frac{W_r}{r} - \frac{1}{r}\frac{\partial W_\theta}{\partial\theta}\right] - \overline{\rho\left(w_z'\right)^2},$$

$$Q_z = -\gamma\left(\frac{\mu}{PR}\right)\frac{\partial e}{\partial z} + \overline{\rho w_z' e'}$$

$$\tau_{zz} = \frac{2}{3}\mu\left[2\frac{\partial W_r}{\partial r} - \frac{W_r}{r} - \frac{1}{r}\frac{\partial W_\theta}{\partial\theta} - \frac{\partial W_z}{\partial z}\right] - \overline{\rho\left(w_r'\right)^2},$$

$$Q_r = -\gamma\left(\frac{\mu}{PR}\right)\frac{\partial e}{\partial r} + \overline{\rho w_r' e'}$$

and

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{\partial W_\theta}{\partial z} + \frac{1}{2}\frac{\partial W_z}{\partial \theta}\right] - \overline{\rho w_z' w_\theta'}$$

$$\tau_{rz} = \tau_{zr} = \mu\left[\frac{\partial W_z}{\partial r} + \frac{\partial W_r}{\partial z}\right] - \overline{\rho w_r' w_z'}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[\frac{1}{r}\frac{\partial W_r}{\partial \theta} + \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{z}\right] - \overline{\rho w_r' w_\theta'}$$

Treatment of Stator-Rotor Interface

- The governing equations for the problem of rotor-stator interaction involving stationary and moving frames of the stator and rotor domains respectively, can be expressed in two ways:
  - Using relative quantities (velocity components and total internal energy) or
  - Using absolute quantities.
- If the equations are written based on the relative quantities, the flux terms have different values for the stationary and moving frames. In this approach, the flux quantities at the interface region of the stator were transformed to moving frame of reference before it was exchanged to rotor frame and vice versa.

- If the governing equations are written in terms of the absolute quantities, the flux terms arising due to motions of the frames can be considered separately. In that case, the values of the inviscid and viscous fluxes remain the same in both moving and stationary frames.

- Therefore the inviscid and viscous flux can be exchanged between the stationary and moving domains without any transformation.

- It is therefore clear that the use of governing equations in terms of the components of the absolute velocity is straight forward with respect to the evaluation of flux exchanged through the interface between stationary and moving domain. This is especially convenient for finite volume formulations given in Module 4.

Summary of Lecture 1.5

The governing equations in rotating frame of reference are written. It is shown that the centrifugal and Coriolis terms are a natural outcome from the formulation with reference to rotating frame. The equations written in rotating cylindrical coordinate system are also presented.

END OF LECTURE 1.5