Lecture 1.4

Governing Equations in Stationary Frame of Reference

(Inviscid Flows)

Governing Equations of Fluid Motion for Inviscid flows

- The governing equations of the fluid motion are derived in the previous lecture. They are equations for conservation of mass, momentum and energy.

- To close the system, there is also a necessity of an additional equation called equation of state for compressible flows.

- In turbomachinery applications, it is somewhat common to neglect viscosity terms and thermal conductivity terms from the full equations, Eq. (1.3.24) and (1.3.42).

- The set of equations that result from these approximations are called unsteady, three-dimensional, compressible inviscid flow equations. They are also called Compressible Euler's equations.

- The compressible Euler's equations can be written in the non-conservative or conservative form as given in the following,

Continuity Equation

Non-conservation form

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathrm{V} = 0 \qquad (1.4.1)$$

Conservation form

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathrm{V}) = 0 \qquad (1.4.2)$$

Momentum Equation

Non-conservation form

x component: $$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \qquad (1.4.3)$$

y component: $$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \qquad (1.4.4)$$

z component: $$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \qquad (1.4.5)$$

Conservation form

x component: $$\frac{\partial(\rho u)}{\partial t}+\nabla\cdot(\rho u \mathrm{V})=-\frac{\partial p}{\partial x}+\rho f_x \quad (1.4.6)$$

y component: $$\frac{\partial(\rho v)}{\partial t}+\nabla\cdot(\rho v \mathrm{V})=-\frac{\partial p}{\partial y}+\rho f_y \quad (1.4.7)$$

z component: $$\frac{\partial(\rho w)}{\partial t}+\nabla\cdot(\rho w \mathrm{V})=-\frac{\partial p}{\partial z}+\rho f_z \quad (1.4.8)$$

Energy Equation

Non-conservation form

$$\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right)=\rho\dot{q}-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\rho \mathrm{f}\cdot\mathrm{V} \quad (1.4.9)$$

Conservation form

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\mathrm{V}\right]=\rho\dot{q}-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\rho \mathrm{f}\cdot\mathrm{V} \quad (1.4.10)$$

- In the vector form, the above equations can be written as:

$$\frac{\partial U}{\partial t}+\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}=S \qquad (1.4.11)$$

where U, E, F, G and S are given by

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho\left(e+\frac{V^2}{2}\right)\end{bmatrix}, E=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho u\left(e+\frac{V^2}{2}\right)+pu\end{bmatrix}, F=\begin{bmatrix}\rho u\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho v\left(e+\frac{V^2}{2}\right)+pv\end{bmatrix}, G=\begin{bmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho w\left(e+\frac{V^2}{2}\right)+pw\end{bmatrix} \qquad (1.4.12)$$

$$S = \{0, \rho f_x, \rho f_y, \rho f_z, \rho f_x u + \rho f_y v + \rho f_z w + \rho\dot{q}\ \}^{T} \qquad (1.4.13)$$

Combined First and Second Law of Thermodynamics

- Several of the auxiliary equations are based on the First and Second Laws of Thermodynamics, which provide the relation

$$Tds = de + pd\left(\frac{1}{\rho}\right) \qquad (1.4.14)$$

where s is the entropy. Using the definition of enthalpy,

$$h = e + \frac{p}{\rho}$$

it is possible to rewrite Eq. (1.4.14) as

$$Tds = dh - \frac{dp}{\rho} \qquad (1.4.15)$$

- This latter equation can also be written as

$$T\nabla s = \nabla h - \frac{\nabla p}{\rho}$$

since at any given instant, a fluid particle can change its state to that of a neighbouring particle.

- Upon combining this equation with Eqs. (5.170) and (1.4.14) and ignoring body forces, we obtain

$$\frac{\partial V}{\partial t} - V \times \zeta = T\nabla s - \nabla h - \nabla\left(\frac{V^2}{2}\right)$$

or

$$\frac{\partial V}{\partial t} - V \times \zeta = T\nabla s - \nabla H \qquad (1.4.16)$$

which is called Crocco's equation.

- This equation provides a relation between vorticity and entropy. For a steady flow it becomes

$$V \times \zeta = \nabla H - T\nabla s \qquad (1.4.17)$$

- Entropy remains constant along a streamline for a steady, nonviscous, nonconducting, adiabatic flow, i.e., isentropic flow. If we also assume that the flow is irrotational and isoenergetic, then Crocco's equation tells us that the entropy remains constant everywhere (i.e., homentropic flow).

- The thermodynamic relation given by Eq. (1.4.14) involves only changes in properties, since it does not contain path-dependent functions. For the isentropic flow of a perfect gas it can be written as

$$Tds = 0 = c_p\, dT - RT\, dp/p$$

or

$$dp/p = (\gamma/\ \gamma - 1)\ dT/T$$

- This equation can be integrated to yield

$$p/T^{\gamma/(\gamma - 1)} = \text{const}$$

which becomes

$$p/\rho^{\gamma} = \text{const} \qquad (1.4.18)$$

after substituting the perfect gas equation of state.

- The speed of sound is given by

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{1.4.19}$$

where the subscript s indicates a constant entropy process

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\left(\frac{\gamma p}{\partial \rho}\right)} = \sqrt{\gamma R T} \tag{1.4.20}$$

Boundary Conditions for Inviscid Flow

- As there is no friction, the flow velocity at the wall is a finite non zero value. The flow velocity vector adjacent to the wall must be tangent to the wall. If **n** is unit normal vector at a point on the surface, the wall boundary condition can be given as

$$\mathbf{V}.\mathbf{n} = 0 \quad \text{(at the surface)} \qquad (1.4.21)$$

- In other words the component of velocity perpendicular to the wall is zero, i.e. the flow at the surface is tangent to the wall. This is the only surface boundary condition for inviscid flow. The magnitude of the velocity, as well as values of the fluid temperature, pressure, and density at the wall, falls out as part of the solution.

- Depending on the problem at hand, whether it be viscous or inviscid, there are various types of boundary conditions elsewhere in the flow, away from the surface boundary.

- For example, for flow through a duct of fixed shape, there are boundary conditions which pertain to the inflow and outflow boundaries, such as at the inlet and exit of the duct.

- If the problem involves an aerodynamic body immersed in a known freestream, then the boundary conditions applied at a distance infinitely far upstream, above, below, and downstream of the body are simply that of the given freestream conditions.

Cascade Flows

- The coordinate system most commonly employed for the analysis of cascade flows is shown in Fig. 1.4.1. Here, x and y are aligned with the cascade axial and tangential directions; and X and Y are aligned with the direction of chord length and normal directions, respectively. The corresponding velocity components are also shown in Fig. 1.4.1.

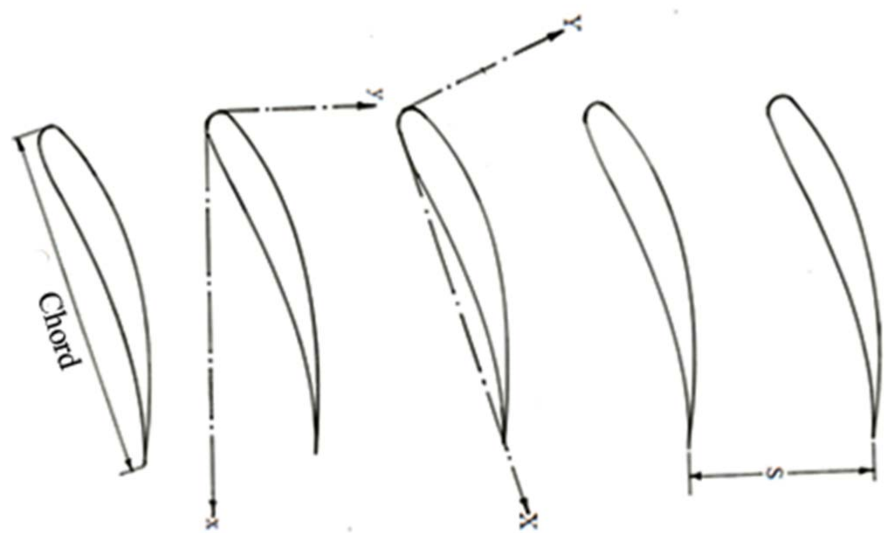

Fig. 1.4.1 Stationary linear cascade and the coordinate system

Irrotational Cascade Flow

- When the flow is incompressible and irrotational, the cascade flow is governed by the following Laplace's equations in stream function/potential function:

$$\psi_{xx} + \psi_{yy} = 0 \quad (1.4.22)$$

$$\phi_{xx} + \phi_{yy} = 0 \quad (1.4.23)$$

- The flow represented by these equations (Eqs. 1.4.22 and 1.4.23) and their solutions belong to a class termed "potential flow," which has historically been the most widely explored and most developed field in the subject area of fluid mechanics and turbomachinery.

- The boundary conditions to be satisfied along with these equations are that the flow velocity and pressure far upstream are equal to the values in the undisturbed free stream (hence, ψ_x, ψ_y or ϕ_x, ϕ_y are specified upstream) and that the blade surfaces are streamlines. They are expressed as follows:

(1) The flow is uniform far upstream and downstream
i.e., $V_x = V_{x1}$, $V_y = V_{y1}$ at far upstream,

(2) $(\partial \Psi/\partial s)|_{\text{surface}} = \nabla\psi \cdot \mathbf{s}$ or $\partial\phi/\partial n = \nabla\phi \cdot \mathbf{n} = 0$ or $\mathbf{V.n} = 0$
on the solid surface, and

(3) the Kutta – Joukowski condition at the trailing edge.

- The Kutta – Joukowski condition ensures that the flow leaves the sharp trailing edge smoothly and that the velocity there is finite. In an inviscid flow, this is possible only when there is a stagnation point at the trailing edge and the pressure difference across the trailing edge is zero.

- It may be noted that the flow around rounded trailing edges cannot be uniquely solved until the role of viscosity is introduced.

- This has led to such useful and practical solutions as flow around an aircraft wing, cascades of blades, and flow around other stream-lined bodies.

- The methods available for solving the potential flow through a cascade or designing a cascade for a prescribed pressure distribution can be briefly classified as follows:

- **Conformal Transformation Method**: In this method, the flow around a cascade of blades is transformed into the flow around a cylinder. Because the flow in the latter plane is known exactly, the reverse transformation provides the flow in the cascade plane. The procedure is reversed for a design problem.

- **Method of Singularities**: This is an approximate method where the blade is replaced by a set of singularities such as sources, sinks, and vortices. In the method of surface singularities or panel method, the blade surface is replaced by panels of source or vortex sheets of infinitesimal length.

- **Numerical Methods**: In these methods, equations are solved numerically using a finite difference or finite volume method. These are the main focus of this course and will be discussed in Module 4.

- **Graphical Method**: In the method developed by Wislicenus the deviation between the camber line and the mean streamline is derived empirically using cascade experimental data. The mean streamline and thickness effects are determined from an assumed pressure distribution.

Subsonic Inviscid Cascade Flows

- In many types of turbomachinery, the compressibility effect (high Mach number flow) is substantial. These effects result in a compact turbomachine due to increased pressure ratio and mass flow per unit area.

- From the definition of the pressure rise (or drop) coefficient across a blade row, the pressure ratio can be expressed as

$$p_2/p_1 = 1 + C_p\,(\gamma M_{1R}^2/2) \qquad (1.4.24)$$

- Thus, for a given C_p, the pressure ratio increases with the Mach number. However, excessive Mach number results in shock waves, thus decreasing the efficiency of the turbomachinery owing to entropy production.

- Depending on the inlet Mach number, the subsonic flow through a cascade can be classified as:
  - Subcritical cascade ($M_1 < M_{cr}$)
  - Supercritical cascade ($M_{cr} < M_1 < M^*$) or ($M_{cr} < M_1 < 1.0$) and
  - Choked cascade ($M_1 = M^*$)

- For shock-free operation and minimum profile loss, the cascade must be operated below M_{cr} . The values of critical M_{1cr} and choking M^* Mach numbers are given by the gas dynamic equations:

$$M_{1cr} = \sqrt{\frac{\gamma+1}{\gamma-1}\left\{1-\left[\frac{\left(\frac{2}{\gamma+1}\right)^{\gamma/\gamma-1}-C_{p\min}}{1-C_{p\min}}\right]^{(\gamma-1)/\gamma}\right\}}$$

$$\frac{A^*}{S\cos\alpha_1} = M^*\left\{\frac{\gamma+1}{2}\left(1-\frac{\gamma-1}{\gamma+1}M^{*2}\right)\right\}^{1/(\gamma-1)} \qquad (1.4.25)$$

- The Mach number range to achieve lower profile losses can be extended beyond M_{cr} by carefully designing the blade shape, for example, by the use of supercritical blades or controlled diffusion blades.

- Numerical methods/schemes suggested in Modules 4 and 5 can be used for solving the governing two-dimensional inviscid compressible flow equations in turbomachinery cascades.

- The transonic/supersonic flows in compressor/turbine cascades are classified on the basis of inlet/exit Mach number values.

Transonic and Supersonic Cascade Flows

- If the inlet (approaching) flow to the cascade is more than unity ($M_1 > 1.0$) it is a supersonic compressor cascade.
- If the exit flow to the cascade is greater than unity ($M_2 > 1.0$) it is a supersonic turbine cascade.
- If $M_{cr} < M_1 < 1.0$, the flow is in a transonic compressor cascade.
- If $M_{cr} < M_2 < 1.0$, the flow is in a transonic turbine cascade.
- The development of the flow as the entry Mach number is increased from subsonic to supersonic flow (illustrated in Fig. 1.4.2 for a compressor cascade).

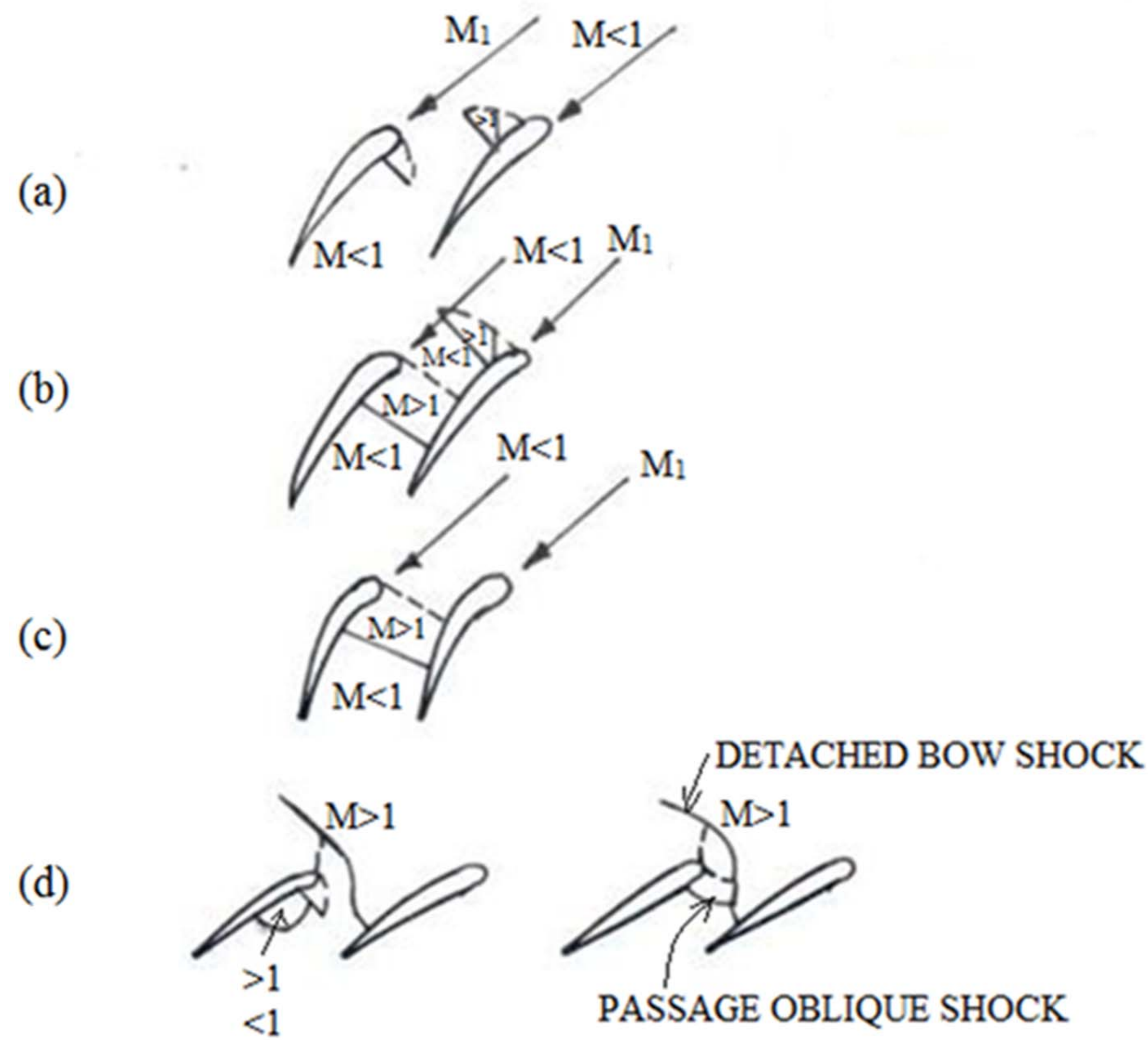

Fig.1.4.2 Transition from subsonic to supersonic flow in a compressor cascade

- Referring to Fig. 1.4.2, a supersonic bubble may exist near the leading edge or inside the passage or both, depending on the inlet Mach number and the rate of acceleration of the flow near the suction surface.

- As M_1 approaches M_{cr}, change in suction peak pressure occurs in a transonic cascade.

- In Fig. 1.4.2a, the sonic region is initiated very near the leading edge, and the shock wave is located further downstream. As the Mach number is increased further (without changing the flow inlet angle), transition takes place from (a) to (d) and then to (e). The condition at (b) represents a choked condition.

- An increase in M_1 from (a) to (b) or (c) is possible only with change in inlet angle (α_1).

Summary of Lecture 1.4

More often the flow in turbomachinery is treated as inviscid. The governing equations and boundary conditions for inviscid compressible flows in the subsonic, transonic and supersonic regions are presented in this lecture. A description of treating the turbomachinery cascade flows is given.

END OF LECTURE 1.4