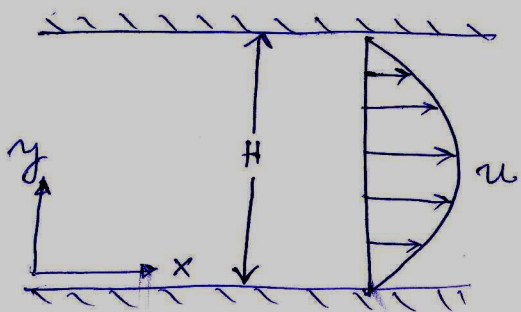


April-2015, ME 60310/ME41616

Q1 (a) The equation of motion, assuming u (x velocity) to be a function of y & t , can be written as:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \frac{dp}{dx} \Big|_0 + \rho A \sin \omega t. \quad \dots (1.1)$$



(Fig 1.1)

Because of linearity of equation (1.1), we can write:

$$u = u_0(y) + u_1(y, t) \dots (1.2)$$

where u_0 is the contribution from $-\frac{dp}{dx} \Big|_0$ & $u_1(y, t)$ is the

contribution from $A \sin \omega t$.

Hence write:

$$\left. \begin{aligned} \rho \frac{d^2 u_0}{dy^2} - \frac{1}{f} \frac{dp_0}{dx} \Big|_0 &= 0 & \text{subject to } u_0(y=0) &= 0 \\ & & u_0(y=H) &= 0 \end{aligned} \right\} \dots (1.3)$$

(Refer to fig 1.1)

The Solution is:
$$u_0(y) = -\frac{1}{2\mu} \frac{dp}{dx} \Big|_0 y(H-y) \quad \dots (1.4)$$

For u_1 , we write:

$$\frac{\partial u_1}{\partial t} = \rho \frac{\partial^2 u_1}{\partial y^2} + A \sin \omega t. \quad \dots (1.5)$$

Assuming u_1 to be a periodic function in time, one can

write:
$$u_1(t, y) = \text{Im}(\tilde{u}_1(y) e^{i\omega t}), \quad i = \sqrt{-1} \quad \dots (1.6)$$

Enforcing (1.6) into (1.5) we get:

$$\frac{d^2 \tilde{u}_1}{dy^2} - \frac{i\omega}{\nu} u_1 = -\frac{A}{\nu} \quad (1.7)$$

$$\text{Subject to, } \left. \begin{aligned} \tilde{u}_1(y=0) &= 0 \\ \tilde{u}_1(y=H) &= 0 \end{aligned} \right\}$$

The Solution to (1.7) is:

$$\tilde{u}_1(y) = a_1 \cosh\left[\sqrt{\frac{i\omega}{\nu}} y\right] + b_1 \sinh\left[\sqrt{\frac{i\omega}{\nu}} y\right] - \frac{iA}{\omega} \quad (1.8)$$

The constants in (1.8) can be obtained from the boundary conditions:

$$a_1 = \frac{iA}{\omega}; \quad b_1 = -\frac{iA}{\omega} \tanh\left[\frac{1}{2}\sqrt{\frac{i\omega}{\nu}}\right] \quad (1.9)$$

$$\text{hence, } \tilde{u}_1(y) = \frac{iA}{\omega} \left[-1 + \frac{\cosh\left(\sqrt{\frac{i\omega}{\nu}}\left(\frac{1}{2}-y\right)\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{i\omega}{\nu}}\right)} \right] \quad (1.10)$$

The total Solution for velocity can be obtained by combining (1.2), (1.4), (1.6) & (1.10):

$$u(y,t) = -\frac{1}{2\mu} \frac{dp}{dx}\bigg|_0 y(H-y) + \text{Im} \left[\frac{iA}{\omega} \left\{ -1 + \frac{\cosh\left(\sqrt{\frac{i\omega}{\nu}}\left(\frac{1}{2}-y\right)\right)}{\cosh\left(\sqrt{\frac{i\omega}{\nu}}\frac{1}{2}\right)} \right\} e^{i\omega t} \right]$$

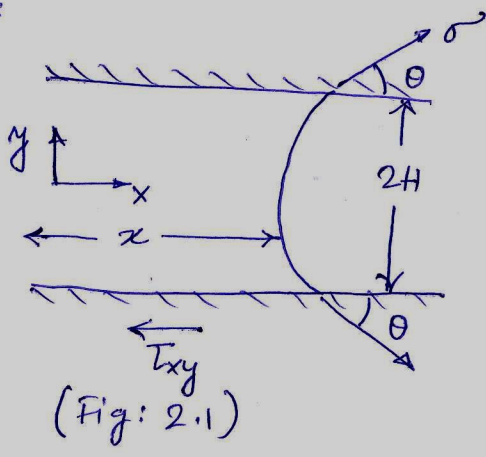
— (1.11)

(b) The major limitations of Stokes' law are as follows:

i) The velocity far away from the particle must be uniform. Otherwise one needs to apply Faxen's law to evaluate the drag: $F = 6\pi\mu a \left[U + \frac{a^2}{6} (\nabla^2 V_\infty)_0 \right]$, where $(\nabla^2 V_\infty)_0$ is evaluated at $r=0$, and $U = (V_\infty)_{r=0}$.

- ii) The fluid must be Newtonian/Stokesian.
- iii) The law is only valid for spherical particles.
- iv) The fluid surrounding the sphere must be unbounded and there must be a single particle in the fluid. If walls are present or other particles are present, one then has to apply the "method of Reflections" to get the drag & the velocity field.
- v) Despite the assumption $Re \rightarrow 0$, the inertia force may be large for large r/R , $R \rightarrow$ sphere radius.

Q2



Surface tension: σ
 Width of the plate: w
 Density of the fluid: ρ
 The Reduced order model for capillary filling of the liquid shown in figure (2.1) is as follows:

$$\frac{d}{dt} \left(2w\rho Hx \frac{dx}{dt} \right) = 2w\sigma \cos\theta - F_v \quad - (2.1)$$

$F_v \rightarrow$ viscous force and x is the location of the meniscus in the channel. For evaluating the viscous force we solve the momentum equation assuming the flow to be fully developed. The momentum equation reads:

$$0 = -\frac{dp}{dx} + a \frac{d}{dy} \left[\left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy} \right] \quad - (2.2)$$

Assuming $\frac{du}{dy} = 0$ at $y=0$ by ~~sym~~ symmetry, (2.2) can be integrated once to write:

$$\left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy} = \frac{1}{a} \frac{dp}{dx} y \quad - (2.3)$$

Now for $-H < y < 0$, $\frac{du}{dy} > 0$. Then from (2.3) we write:

$$\left(\frac{du}{dy}\right)^n = \frac{1}{a} \frac{dp}{dx} y$$

$$\text{or, } \frac{du}{dy} = \left[\frac{1}{a} \cdot \frac{dp}{dx} y \right]^{1/n} \quad - (2.4)$$

Equation (2.4) can be integrated subject to $u(-H) = 0$.
~~to get~~ The integration yields:

$$u(y) = \frac{n}{n+1} \left(-\frac{1}{a} \cdot \frac{dp}{dx} \right)^{1/n} \left[H^{\frac{n+1}{n}} - (-y)^{\frac{n+1}{n}} \right] \quad - (2.5)$$

The average velocity can be evaluated as:

$$u_{\text{avg}} = \frac{1}{H} \int_0^H u \, dy = \frac{2nH}{(n+1)(n+2)} \left(-\frac{H}{a} \cdot \frac{dp}{dx} \right)^{1/n} \quad - (2.6)$$

(2.5) & (2.6) can be combined to write:

$$u = \frac{n+2}{2H^{\frac{n+1}{n}}} u_{\text{avg}} \left[H^{\frac{n+1}{n}} - (-y)^{\frac{n+1}{n}} \right] \quad - (2.7)$$

The viscous stress at the wall is thus given by:

$$\tau_{xy} = a \left(\frac{du}{dy} \right)^n = a \left[\frac{(n+1)(n+2)}{2nH} u_{\text{avg}} \right]^n \quad - (2.8)$$

[as calculated from (2.7)]

$$\begin{aligned} \Rightarrow F_D &= 2x \tau_{xy} = 2ax \left[\frac{(n+1)(n+2)}{2nH} u_{\text{avg}} \right]^n \\ &= 2ax \left[\frac{(n+1)(n+2)}{2nH} \frac{dx}{dt} \right]^n \quad - (2.9) \end{aligned}$$

(2.9) is obtained by replacing $u_{\text{avg}} = \frac{dx}{dt}$.

We can now combine (2.1) & (2.9) to write:

$$\boxed{\frac{d}{dt} \left(\rho H x \frac{dx}{dt} \right) = \sigma \cos \theta - a x \left[\frac{(n+1)(n+2)}{2nH} \cdot \frac{dx}{dt} \right]^n} \quad (2.10)$$

Q3 We assume the drop to behave like a thin film, i.e., its radius is assumed to be much larger than its height. We will directly start from the thin film equation without deriving it. However we do mention the steps to be followed to derive the relevant equation in brief. These are as follows:

i) If h is the length scale in y direction & L is the same in x direction, assume $\frac{h}{L} = \varepsilon \ll 1$.

ii) Write the non-dimensional Navier-Stokes equation, non-dimensionalized with appropriate scales, for example $v_e \sim \varepsilon U_e$, $p_c \sim \frac{1}{\varepsilon^2} \frac{\mu U_e}{L}$ etc.

iii) Expand all the variables in a regular asymptotic series of ε and derive/deduce the leading order ($O(\varepsilon^0)$) equation.

iv) First solve for $p^{(0)}$, then solve the x -mom. equation for $u^{(0)}$, subject to appropriate b.c.'s.

v) Now integrate the continuity equation, along with Leibnitz theorem to get an equation for film height h , as a function of x & t .

In the present problem, the following assumptions are taken: i) $Bo \ll 1$, $Bo \rightarrow$ Bond number.

$$ii) u_c = \frac{\varepsilon^3 L^2 \rho g}{\mu}$$

The thin film equation is thus:

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right) \quad \text{--- (3.1)}$$

$$\text{Subject to, } \frac{\partial h}{\partial x} = 0 \text{ at } x = 0 \quad \text{--- (3.2.a)}$$

$$\text{at } x = L(t), h = 0 \quad \text{--- (3.2.b)}$$

L being the drop radius.

The lack of any natural length or time scale would naturally suggest a similarity solution.

$$\text{Let } L(t) = k t^\beta \text{ and } h(x,t) = t^\alpha f\left(\frac{ax}{t^\beta}\right) = t^\alpha f(\eta) \quad \text{--- (3.3)}$$

$$\text{where } \eta = \frac{ax}{t^\beta}. \text{ The } \text{~~var~~ mass conservation constraint would require: } \int_0^L h dx = 1 \quad \text{--- (3.4)}$$

$$\text{Enforcing (3.3) into (3.4) gives: } \alpha = -\beta. \quad \text{--- (3.5)}$$

$$\text{So } h = t^\alpha f(\eta) \text{ where } \eta = \frac{ax}{t^{-\alpha}} \quad \text{--- (3.6)}$$

$$\text{and } \int_0^{\eta_0} f(\eta) d\eta = a \quad \text{--- (3.7)}$$

$$\text{with } \eta_0 = \eta \text{ at } x = L(t).$$

Enforcing (3.6) into (3.1), and keeping in mind that for a similarity solution to exist, the equation

could require: $\alpha = -\frac{1}{5}$, $a = \sqrt{\frac{3}{5}}$

$$\text{and, } \frac{d}{d\eta} [\eta f] = \frac{d}{d\eta} \left[f^3 \frac{df}{d\eta} \right] \quad - (3.8)$$

$$\text{integrating once: } f^3 \frac{df}{d\eta} = -\eta f \quad \text{since } - (3.9)$$

[with aid of condition (3.2.a)]

integrating again, ~~and~~ and enforcing (3.2.b), we get:

$$f = \left[\frac{3}{2} (\eta_0^2 - \eta^2) \right]^{1/3} \quad - (3.10)$$

Now use the constraint, (3.7) to get:

$$\int_0^{\eta_0} \left[\frac{3}{2} (\eta_0^2 - \eta^2) \right]^{1/3} d\eta = a = \sqrt{\frac{3}{5}}$$

$$\text{This yields: } \eta_0 \approx \left[\frac{(3/5)^{1/2}}{(3/2)^{1/3} \int_0^{\pi/2} (\cos \theta)^{5/3} d\theta} \right]^{3/5} \approx 2.23.$$

$$\text{Thus } h = t^{-1/5} \left[\frac{3}{2} (\eta_0^2 - \eta^2) \right]^{1/3}$$

$$\text{and } \eta = \frac{\chi}{t^{1/5}} \sqrt{\frac{3}{5}} ;$$

$$\Rightarrow \frac{L(t)}{t^{1/5}} \sqrt{\frac{3}{5}} = \eta_0 \Rightarrow \boxed{L(t) = \sqrt{\frac{5}{3}} \eta_0 t^{1/5}}$$

Q4

The charge distribution is given by the Poisson-Boltzmann model: (along with Debye-Hückel linearization)

$$\frac{d^2\psi}{dy^2} = -\frac{\rho_e}{\epsilon} = \lambda^2\psi, \quad \lambda^2 = \frac{2e^2c_0}{\epsilon kT} \rightarrow \text{inverse of Debye length} \quad \text{--- (A.1)}$$

The equation of motion is:

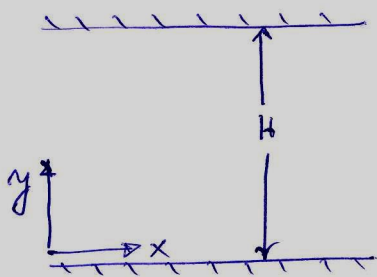
$$-\frac{dp}{dx} + \mu \frac{d^2u}{dy^2} + \rho_e E_0 = 0 \quad \text{--- (A.2)}$$

where, E_0 is the streaming potential, to be determined.

Non-dimensionalization:

Boundary Conditions: at $y=0, H, \psi = \zeta_0$ --- (A.3)

at $y=0, H; u = 0$ --- (A.4)



(Fig 1.1)

Non-dimensionalization:

$$\bar{\psi} = \frac{\psi}{\zeta_0}; \quad \bar{y} = \frac{y}{H}; \quad \bar{\lambda} = \lambda H;$$

$$u = u_p + u_E. \quad \text{--- (A.5)}$$

$$\bar{u}_p = \frac{u_p}{u_{p,c}}, \quad u_{p,c} = -\frac{1}{2\mu} \frac{dp}{dx}; \quad \bar{u}_E = \frac{u_E}{u_{E,c}}, \quad u_{E,c} = \frac{\epsilon \zeta_0 E_0}{\mu}.$$

With these non-dimensionalization scheme, the equations become:

$$\frac{d^2\bar{\psi}}{d\bar{y}^2} = \bar{\lambda}^2 \bar{\psi} \quad \text{with } \bar{\psi} = \zeta_0 \text{ at } \bar{y} = 0, 1 \quad \text{--- (A.6)}$$

$$\frac{d^2\bar{u}_p}{d\bar{y}^2} + 1 = 0 \quad \text{with } \bar{u}_p = 0 \text{ at } \bar{y} = 0, 1 \quad \text{--- (A.7)}$$

$$\frac{d^2\bar{u}_E}{d\bar{y}^2} - \frac{d^2\bar{\psi}}{d\bar{y}^2} = 0 \quad \text{with } \bar{u}_E = 0 \text{ at } \bar{y} = 0, 1 \quad \text{--- (A.8)}$$

(A.6) yields:

$$\bar{\psi} = \frac{\cosh\left[\bar{\lambda}\left(\frac{1}{2}-\bar{y}\right)\right]}{\cosh(\bar{\lambda}/2)} \quad - (A.9)$$

(A.7) yields: $\bar{u}_p = \bar{\psi}(1-\bar{y})$ - (A.10)

(A.8) yields: $\bar{u}_E = -(1-\bar{\psi})$ - (A.11)

The constraint: Total current = 0.

$$\bar{I} = \bar{I}_c + \bar{I}_p + \bar{I}_E = 0 \quad - (A.12)$$

$$\begin{aligned} \bar{I}_c = \text{conduction current} &= \int_0^H \frac{2z^2 e^2}{f} E_0 (c_+ + c_-) dy \\ &\approx \frac{2z^2 e^2 E_0 H c_0}{f} \quad - (A.13) \end{aligned}$$

f → friction factor.

\bar{I}_p = Streaming current due to pressure driven flow:

$$\begin{aligned} &= \int_0^H f_e u_p dy = A \int_0^1 \bar{\lambda}^2 \bar{\psi} \bar{u}_p d\bar{y} \\ &= A \bar{I}_1 \quad - (A.14) \end{aligned}$$

where $A = -\frac{\epsilon S_0 u_{p,e}}{H}$ and $\bar{I}_1 = \int_0^1 \bar{\lambda}^2 \bar{\psi} \bar{u}_p d\bar{y}$

\bar{I}_E = Streaming current due to electric field:

$$\begin{aligned} &= \int_0^H f_e \bar{u}_E dy = B \int_0^1 \bar{\lambda}^2 \bar{\psi} (\bar{\psi} - 1) d\bar{y} \\ &= B \bar{I}_2 E_0 \quad - (A.15) \end{aligned}$$

where $B = -\frac{\epsilon^2 S_0^2}{\mu H}$; $\bar{I}_2 = \int_0^1 \bar{\lambda}^2 \bar{\psi} (\bar{\psi} - 1) d\bar{y}$

Combining (4.12) - (4.15) we can write:

$$\frac{2e^2 E_0 c_0}{f} + A I_1 + B I_2 E_0 = 0$$

$$\alpha, \quad E_0 = - \frac{A I_1}{B I_2 + \frac{2e^2 c_0}{f}} \quad - (4.16)$$

where $A = - \frac{\epsilon \epsilon_0 u_{pc}}{H} = - \frac{\epsilon \epsilon_0 H}{2\mu} \cdot \frac{dp}{dx}$

$$B = - \frac{\epsilon^2 \epsilon_0^2}{\mu H} \quad ; \quad I_1 = \int_0^1 \bar{x}^2 \bar{\eta} \bar{y} (1-\bar{y}) d\bar{y}$$

$$I_2 = \int_0^1 \bar{x}^2 \bar{\eta} (\bar{\eta} - 1) d\bar{y} \quad \text{and} \quad \bar{\eta} = \frac{\cosh[\bar{x}(\frac{1}{2} - \bar{y})]}{\cosh(\bar{x}/2)}$$

Q5 Given $\frac{u}{u_{avg}} = \frac{3}{2} \left(1 - \frac{y^2}{H^2}\right)$ - (5.1)

The species transport equation is given by:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) \quad - (5.2)$$

In conjunction with Taylor-dispersion problem, we define the following:

$$\bar{x} = x - u_{avg} t \quad \text{and} \quad \bar{u} = u - u_{avg} = \frac{1}{2} u_{avg} \left(1 - \frac{3y^2}{H^2}\right)$$

Thus one ~~gets~~ has,

$$\frac{\partial c}{\partial t} + \bar{u} \frac{\partial c}{\partial \bar{x}} = \quad - (5.3)$$

Further assume $\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$

Equation (5.2) thus reads:

$$\bar{u} \frac{\partial c}{\partial \bar{x}} = D \frac{\partial^2 c}{\partial y^2}, \quad D \text{ is diffusivity} \quad - (5.4)$$

assuming $\frac{\partial c}{\partial \bar{x}}$ to be independent of y , one can integrate (5.4) twice to write: [with $\frac{\partial c}{\partial y} = 0$ at $y = 0$]

$$c = C_1 + \frac{u_{avg} H^2}{4D} \frac{\partial c}{\partial \bar{x}} \left(\frac{y^2}{H^2} - \frac{y^4}{2H^4} \right) \quad - (5.5)$$

Writing $\int_0^H c dy = \bar{c}$ [average], we can determine C_1 from (5.5), which has the form:

$$C_1 = \bar{c} - \frac{7 u_{avg} H^2}{120 D} \frac{\partial c}{\partial \bar{x}} \quad - (5.6)$$

Hence,
$$c = \bar{c} + \frac{u_{avg} H^2}{4D} \left[-\frac{7}{30} + \frac{y^2}{H^2} - \frac{y^4}{2H^4} \right] \frac{\partial c}{\partial \bar{x}} \quad - (5.7)$$

Thus flux of the species in x direction then reads:

$$\bar{J} = \frac{1}{H} \int_0^H \bar{c} \bar{u} dy = - \frac{2 u_{avg}^2 H^2}{105 D} \frac{\partial c}{\partial \bar{x}} \quad - (5.8)$$

In conjunction with Taylor dispersion & Fick's law of diffusion we can thus write from (5.8):

$$\boxed{D_{eff} = \frac{2 u_{avg}^2 H^2}{105 D} = \frac{2 Pe^2 D}{105}} \quad - (5.9)$$