Incompressible Viscous Flows

Q1. **Choose the correct answer**

(i) The maximum velocity of a one-dimensional incompressible fully developed viscous flow between two fixed parallel plates is 6m/s. The mean velocity of the flow is

(a) 2 m/s
(b) 3 m/s
(c) 4 m/s
(d) 5 m/s

[*Ans.(*c)]

(ii) Loss of head due to fluid friction for steady, fully developed, incompressible laminar flow through circular pipe is

(a) $h_f = \frac{32\mu QL}{\pi\rho g D^4}$

(b) $h_f = \frac{128\mu QL}{\rho g D^4}$

(c) $h_f = \frac{128\mu QL}{\pi\rho g D^4}$

(d) $h_f = \frac{64\mu QL}{\pi\rho g D^4}$

[*Ans.(*c)]

(iii) The velocity profile of a fully developed laminar flow in a straight circular pipe is given by $v_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left(1-\frac{r^2}{R^2}\right)$, where $\frac{dp}{dz}$ is the constant, R is the radius of the pipe and r is the radial distance from the centre of the pipe. The average velocity of fluid in the pipe is

(a) $-\frac{R^2}{\mu}\frac{dp}{dz}$

(b) $-\frac{R^2}{2\mu}\frac{dp}{dz}$

(c) $-\frac{R^2}{4\mu}\frac{dp}{dz}$

(d) $-\frac{R^2}{8\mu}\frac{dp}{dz}$

[*Ans.(*d)]

(iv) For fully developed, incompressible, laminar flow through circular pipes, shear stress varies

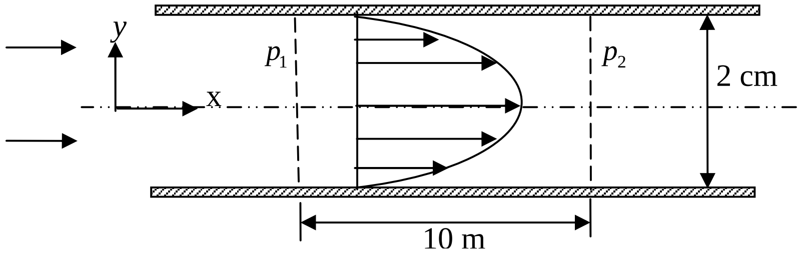
(a) linearly with the radial distance from the axis
(b) parabolically with the radial distance from the axis
(c) exponentially with the radial distance from the axis
(d) linearly with the radial distance from the pipe wall

[*Ans.(*a)]

Q2.
Water at $20°C$ flows between two large stationary parallel plates which are 2 cm apart. If the maximum velocity is 1m/s, determine (a) average velocity, (b) the velocity gradients at the plates and (c) the difference in pressure between two points 10 m apart. Viscosity of water at $20°C$ is 0.001 Pa-s. Consider the flow to be a fully developed one.

Solution

The flow geometry is shown in the figure below.



(a) The velocity distribution for fully developed flow between two stationary horizontal parallel plates can be expressed as

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - H^2\right)$$

where H is channel half height and y is measured from the centerline.
The maximum velocity is obtained as

$$u_{\max} = -\frac{1}{2\mu}\frac{dp}{dx}H^2$$

The mean velocity is obtained from the above velocity field as

$$\bar{u} = \frac{\int_{-H}^{H} udA}{A} = \frac{\int_{0}^{H} udy}{H} = -\frac{1}{3\mu}\frac{dp}{dx}H^2$$

From the above expressions for mean velocity and maximum velocity, one can write

$$\bar{u} = \frac{2}{3}u_{\max}$$

Substituting $u_{\max} = 1$ m/s in the above equation, we get

$$\bar{u} = \frac{2}{3}u_{\max} = \frac{2}{3}\times 1 = 0.667 \text{ m/s}$$

(b)
Substituting the values of $\bar{u}$ and H in the velocity profile, one can write

$$u = \frac{3}{2}\times 0.667\left(1 - \frac{y^2}{0.01^2}\right) = \left(1 - 10000y^2\right)$$

The velocity gradients at the plates are

$$\left.\frac{\partial u}{\partial y}\right|_{y=\pm H} = -20000\,y\big|_{y=\pm H} = -20000\,y\big|_{y=\pm H}$$

$$= -20000 \times (\pm 0.01) = \mp\ 200\ \text{/s}$$

(c) The mean velocity is given by

$$\bar{u} = -\frac{1}{3\mu}\frac{dp}{dx}H^2$$

or

$$\frac{dp}{dx} = -\frac{3\mu\bar{u}}{H^2}$$

The pressure gradient can be expressed as

$$\frac{dp}{dx} = \frac{p_2 - p_1}{L}$$

where p_1 and p_2 are the pressures at point 1 and 2 respectively, and L is the distance between the two points. Then, the difference in pressure between two points L distance apart can be expressed as

$$p_1 - p_2 = \frac{3\mu\bar{u}L}{H^2}$$

Substituting the respective values in the above equation, we have

$$p_1 - p_2 = \frac{3 \times 0.001 \times 0.667 \times 10}{0.01^2} = 200\ \text{N/m}^2$$

Q3.

An oil of viscosity $0.1\ \text{Ns/m}^2$ and density $900\ \text{kg/m}^3$ is flowing between two stationary horizontal parallel plates 10 mm apart. The maximum velocity is 3 m/s. Find the mean velocity and the location at which this occurs. Also find the velocity at 3 mm from the wall of the plates. Consider the flow to be a fully developed one.

Solution

The velocity distribution for fully developed flow between two stationary horizontal parallel plates can be expressed as

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - H^2\right)$$

where H is channel half height and y is measured from the centerline.

The maximum velocity is obtained as

$$u_{\max} = -\frac{1}{2\mu}\frac{dp}{dx}H^2$$

The mean velocity is obtained from the above velocity field as

$$\bar{u} = \frac{\int_{-H}^{H} udA}{A} = \frac{\int_{0}^{H} udy}{H} = -\frac{1}{3\mu}\frac{dp}{dx}H^2$$

From the above expressions for mean velocity and maximum velocity, one can write

$$\bar{u} = \frac{2}{3}u_{\max}$$

Substituting $u_{\max} = 3\ \text{m/s}$ in the above equation, we get

$$\bar{u} = \frac{2}{3} \times 3 = 2\ \text{m/s}$$

Now, it is given that $u = \bar{u}$

$$\frac{1}{2\mu}\frac{dp}{dx}\left(y^2-H^2\right)=-\frac{1}{3\mu}\frac{dp}{dx}H^2$$

or $$y^2-H^2=-\frac{2}{3}H^2$$

or $$y=\frac{H}{\sqrt{3}}=0.577H$$

Substituting the value of H, we get

$$y=0.577\times0.005=0.002885\text{ m}=2.885\text{ mm}$$

The velocity distribution can be expressed in terms average velocity as

$$u=\frac{3}{2}\bar{u}\left(1-\frac{y^2}{H^2}\right)$$

Substituting the respective values in the above equation, we have

$$u=\frac{3}{2}\times2\left(1-\frac{y^2}{0.005^2}\right)=3\left(1-40000y^2\right)$$

The velocity at 3 mm from the plates or equivalently $5-4=1\text{ mm}=0.001\text{ m}$ from the centerline is

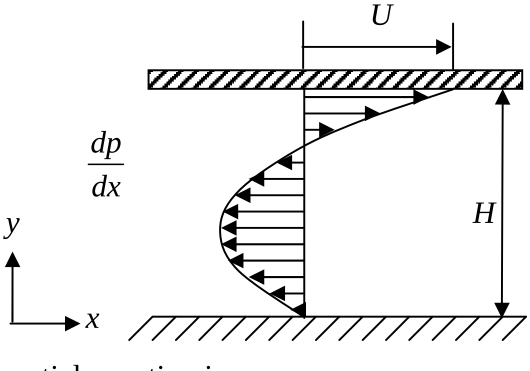
$$u=3\left(1-40000\times0.001^2\right)=2.88\text{ m/s}$$

Q4.

An oil with density ρ and viscosity μ flows between two horizontal parallel plates H apart. The upper plate is moving with a uniform speed U, while the lower one is kept stationary. A constant pressure gradient of $\frac{dp}{dx}$ drives the flow in such a way that the net flow rate across any section is zero. Find out the point where maximum velocity occurs. Also find the magnitude of maximum velocity.

Solution

The flow geometry and the velocity profile across a section are shown in the figure below.



The governing differential equation is

$$\mu\frac{d^2u}{dy^2}=\frac{dp}{dx}$$

Integrating the above equation twice with respect to y, we get

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants of integration.

To evaluate the constants, we apply the boundary conditions. The boundary conditions are:

at $y = 0,\qquad u = 0$

at $y = H,\qquad u = U.$

Applying the boundary conditions, we get

$$C_1 = \frac{U}{H} - \frac{1}{2\mu}\frac{dp}{dx}H \text{ and } C_2 = 0$$

Therefore, the velocity distribution becomes

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yH\right) + \frac{U}{H}y$$

The volume flow rate is given by

$$Q = \int_0^H uwdy \qquad [w = \text{width of the plate}]$$

$$= w\int_0^H \left[\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - Hy\right) + U\frac{y}{H}\right]dy$$

$$= w\left(-\frac{1}{12\mu}\frac{dp}{dx}H^3 + \frac{UH}{2}\right)$$

For zero flow rate ($Q = 0$), we have

$$w\left(-\frac{1}{12\mu}\frac{dp}{dx} + \frac{UH}{2}\right) = 0$$

or $$\frac{dp}{dx} = \frac{6\mu U}{H^2}$$

For maximum velocity, $\frac{\partial u}{\partial y} = 0$

or $$\frac{1}{2\mu}\frac{dp}{dx}\left(2y - H\right) + \frac{U}{H} = 0$$

or $$\frac{1}{\mu}\frac{dp}{dx}y = \frac{1}{2\mu}\frac{dp}{dx}H - \frac{U}{H}$$

or $$y = \frac{1}{2}H - \frac{U\mu}{H\frac{dp}{dx}}$$

or $$y = \frac{1}{2}H - \frac{U\mu}{H \times \frac{6\mu U}{H^2}} = \frac{1}{2}H - \frac{1}{6}H = \frac{1}{3}H$$

The maximum velocity is

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yH\right) + \frac{U}{H}y$$

$$= \frac{1}{2\mu}\times\frac{6\mu U}{H^2}\left(\left(\frac{H}{3}\right)^2 - \frac{H}{3}H\right) + \frac{U}{H}\frac{H}{3}$$

$$= \frac{3U}{H^2}\left(\frac{H^2}{9} - \frac{H^2}{3}\right) + \frac{U}{3} = -\frac{2U}{3} + \frac{U}{3} = -\frac{U}{3}$$

Q5.

Find the radial location in a steady, fully developed, laminar flow in a circular horizontal pipe where the local velocity is equal to the average velocity

Solution

The velocity distribution for steady, fully developed, laminar flow in a circular pipe is given by

$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where R is the radius of the pipe, r is the radial distance measured from center of the pipe and dp/dz is the constant pressure gradient that drives the flow.

Average velocity is found to be

$$\bar{v}_z = \frac{Q}{A} = \frac{\int_o^R v_z 2\pi r dr}{\pi R^2}$$

$$= \frac{1}{\pi R^2}\int_o^R \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)2\pi r dr$$

$$= -\frac{R^2}{8\mu}\frac{dp}{dz}$$

Now, it is given that $v_z = \bar{v}_z$

$$\frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1 - \left(\frac{r}{R}\right)^2\right] = -\frac{R^2}{8\mu}\frac{dp}{dz}$$

or $$1 - \left(\frac{r}{R}\right)^2 = \frac{R^2}{2}$$

or $$r = \frac{R}{\sqrt{2}} = 0.707R$$

Q6.

An oil with density $900\ \text{kg/m}^3$ and viscosity $0.16\ \text{Ns/m}^2$ is flowing through a 20 cm diameter pipe. The maximum shear stress at the pipe wall is $2.5\ \text{N/m}^2$. Determine (a) the pressure gradient, (b) the average velocity of flow and (b) the maximum velocity of flow.

Solution

(a)

The velocity distribution for steady, fully developed, laminar flow in a circular pipe is given by

$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left[1-\left(\frac{r}{R}\right)^2\right]$$

Shear stress at any point of the pipe flow is given by

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) = \mu\frac{\partial v_z}{\partial z} = \mu\frac{1}{2\mu}\frac{dp}{dz}r$$

or $$\tau_{rz} = \frac{1}{2}\frac{dp}{dz}r$$

The wall shear stress is given by

$$\tau_w = \left|\tau_{rz}\big|_{wall}\right| = \frac{1}{2}\frac{dp}{dz}R$$

Substituting the respective values, we obtain

$$2.5 = \frac{1}{2}\frac{dp}{dz}\times 0.1$$

or $$\frac{dp}{dz} = 50 \text{ Pa/m}$$

(b)

The velocity distribution for steady, fully developed, laminar flow in a circular pipe can be expressed in terms of average velocity as

$$v_z = 2\bar{v}_z\left(1-\frac{r^2}{R^2}\right)$$

The wall shear stress can also be expressed in terms of average velocity as

$$\tau_w = \left|\tau_{rz}\big|_{wall}\right| = \left|\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\bigg|_{r=R}\right| = -\mu\bar{v}_z 2\left(-\frac{2r}{R^2}\bigg|_{r=R}\right)$$

or $$\tau_w = \frac{4\mu\bar{v}_z}{R}$$

Substituting the respective values, we obtain

$$2.5 = \frac{4\times 0.16\bar{v}_z}{0.1}$$

or $$\bar{v}_z = 0.39 \text{ m/s}$$

(c)

The maximum velocity of flow is found to be

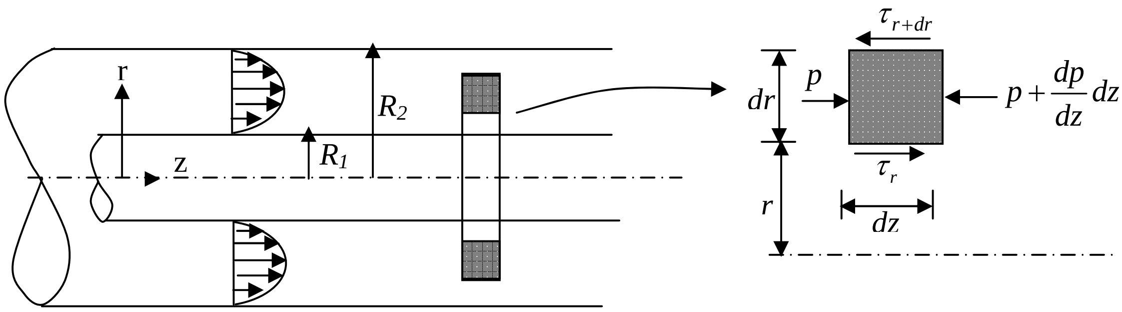
$$v_{z,\max} = 2\bar{v}_z = 2\times 0.39 = 0.78 \text{ m/s}$$

Q7.

Consider steady, incompressible fully developed flow through a horizontal concentric pipes of radii R_1 and $R_2\left(>R_1\right)$ respectively. Find an expression for the velocity distribution. Also show that the velocity distribution approaches that of pipe flow as $R_1 \to 0$.

Solution

A small element of thickness *dr* at a radius *r* from the centre is considered as shown in the figure below.



For steady, incompressible, fully developed flow in the annular space between two concentric pipes, the governing differential equation becomes

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dp}{dz}$$

or,
$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}$$

Integrating with respect to *r*, we get

$$r\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r^2 + C_1$$

where C_1 is a constant of integration.

Now,
$$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r + \frac{C_1}{r}$$

Integrating once more with respect to r, we have

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + C_1 \ln r + C_2$$

where C_2 is another constant of integration.

To evaluate the constants, we apply the boundary conditions. The boundary conditions are:

at $r = R_1, v_z = 0$

At $r = R_2, v_z = 0$

Applying the boundary conditions, we get

$$C_1 = \frac{1}{4\mu}\frac{dp}{dz}\frac{R_1^2 - R_2^2}{\ln(R_2/R_1)} \text{ and } C_2 = \frac{1}{4\mu}\frac{dp}{dz}\left[\frac{R_2^2 - R_1^2}{\ln(R_2/R_1)}\ln R_1 - R_1^2\right]$$

Substituting the values of C_1 and C_2 in the expression of velocity, we obtain

$$v_z = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)\left[\left(R_1^2 - r^2\right) + \frac{R_2^2 - R_1^2}{\ln(R_2/R_1)}\ln\frac{r}{R_1}\right]$$

This is the required expression for the velocity distribution for steady, incompressible, fully developed flow in the annular space between two concentric pipes.

As $R_1 \to 0$, the above expression for the velocity becomes

$$v_z = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)\left[R_2^2 - r^2\right]$$

This implies that the *velocity distribution approaches that of pipe flow.*