Conservation Equations in Fluid Flow

Q1. Choose the correct answer

(i) The continuity equation is the result of application of the following law to the flow field

(a) Conservation of momentum
(b) Conservation of energy
(c) Conservation of mass
(d) Conservation of force

[*Ans.*(c)]

(ii) A flow field satisfying $\nabla.\vec{V} = 0$ as the continuity equation represents always a

(a) steady flow
(b) unsteady and non-uniform flow
(c) steady and uniform flow
(d) incompressible flow

[*Ans.*(*d*)]

(iii) The 2-D flow with velocity $\vec{V} = \left(x + 2y + 2\right)\hat{i} + \left(4 - y\right)\hat{j}$ is

(a) compressible and irrotational
(b) compressible and rotational
(c) incompressible and irrotational
(d) incompressible and rotational

[*Ans.*(d)]

Q2.

Check whether the following sets of velocity components represent a possible incompressible flow

(i) $u = 3xy$, $v = x^3 - xy^3$

(ii) $u = 2x^2 - xy + z^2$, $v = x^2 - 4xy + y^2$, $w = 2xy - yz + y^2$

Solution

(i)Here, $u = 3xy$ and $v = x^3 - xy^3$

Hence, $\frac{\partial u}{\partial x} = 3y$ and $\frac{\partial v}{\partial y} = -3xy^2$

For a two-dimensional, incompressible flow the continuity equation can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation, we find

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 3y - \left(-3xy^2\right) \neq 0$$

Hence the continuity equation for an incompressible flow is not satisfied. Therefore, it is not a possible incompressible flow.

(ii)

Here, $u = 2x^2 - xy + z^2$, $v = x^2 - 4xy + y^2$ and $w = 2xy - yz + y^2$

Hence, $\frac{\partial u}{\partial x} = 4x - y$, $\frac{\partial v}{\partial y} = -4x + 2y$, and $\frac{\partial w}{\partial z} = -y$

For a three-dimensional, incompressible flow the continuity equation can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation, we find

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4x - y - 4x + 2y - y = 0$$

Hence the continuity equation for an incompressible flow is satisfied. Therefore, it is a possible incompressible flow.

Q3.

For a three-dimensional velocity field, x and y component of velocity are given by

$$u = (x + y + z), v = -(xy + yz + zx)$$

Determine a possible z component of velocity, *w*, for an incompressible flow. How many possible z components are there?

Solution

For an incompressible flow the continuity equation can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Given $u = x + y + z$, and $v = -xy - yz - zx$

Hence, $\frac{\partial u}{\partial x} = 1$, and

$$\frac{\partial v}{\partial y} = -x - z$$

Substituting $\frac{\partial u}{\partial x}$, and $\frac{\partial v}{\partial y}$ in the continuity equation, we get

$$1 - x - z + \frac{\partial w}{\partial z} = 0$$

or $\frac{\partial w}{\partial z} = x + z - 1$

Integrating with respect to *z*, we have

$$w = xz + \frac{z^2}{2} - z + f(x, y)$$

There are infinite number of possible z components, since $f(x, y)$ is arbitrary. the simplest one would be found by setting $f(x, y) = 0$

Q4.

The velocity components in a two-dimensional flow field in cylindrical polar coordinate are given by $v_r = r\sin 2\theta$, $v_\theta = -2r\sin^2\theta$

Check whether the velocity field describes the motion of an incompressible flow or not?

Solution

For a two-dimensional, incompressible flow the continuity equation in cylindrical polar coordinate can be written in differential form as

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

Here, $v_r = r\sin 2\theta$ and $v_\theta = -2r\sin^2\theta$

Hence, $\frac{\partial v_r}{\partial r} = \frac{\partial}{\partial r}\left(r\sin 2\theta\right) = \sin 2\theta$, and

$$\frac{\partial v_\theta}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-2r\sin^2\theta\right) = -r\frac{\partial}{\partial \theta}\left(2\sin^2\theta\right) = -r\frac{\partial}{\partial \theta}\left(1-\cos 2\theta\right)$$

$$= -2r\sin 2\theta$$

Substituting $\frac{\partial v_r}{\partial r}$, and $\frac{\partial v_\theta}{\partial \theta}$ in the continuity equation, we get

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \sin 2\theta + \frac{r\sin 2\theta}{r} + \frac{1}{r}\left(-2r\sin 2\theta\right) = 0$$

Hence the continuity equation for an incompressible flow is satisfied. Therefore, it is a possible incompressible flow.

Q5.

The velocity field in a fluid flow is given by

$$\vec{V} = \left(\frac{y^3}{3} + 2x - x^2y\right)\hat{i} + \left(xy^2 - 2y - \frac{x^3}{3}\right)\hat{j}$$

(a) Check whether the flow is (i) incompressible or compressible, (ii) irrotational or rotational.

(b) If the flow is incompressible, obtain an expression for stream function.

Solution

(a) (i) For a two-dimensional, incompressible flow the continuity equation can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Given $u = \frac{y^3}{3} + 2x - x^2y$, and $v = xy^2 - 2y - \frac{x^3}{3}$

Hence, $\frac{\partial u}{\partial x} = 2 - 2xy$, and

$$\frac{\partial v}{\partial y} = 2xy - 2$$

Substituting $\frac{\partial u}{\partial x}$, and $\frac{\partial v}{\partial y}$ in the continuity equation, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0$$

Hence the continuity equation for an incompressible flow is satisfied. Therefore, the flow is incompressible.

(ii) For a two-dimensional flow, rotation is given by

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For the given velocity field

$$\frac{\partial u}{\partial y} = y^2 - x^2, \text{ and}$$

$$\frac{\partial v}{\partial x} = y^2 - x^2$$

Thus,

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[\left(y^2 - x^2\right) - \left(y^2 - x^2\right)\right] = 0$$

Since $\omega_{xy} = 0$, the flow is irrotational.

(b) From the definition of stream function ψ, we get

$$u = \frac{\partial \psi}{\partial y}$$

or

$$\psi = \int u dy = \int \left(\frac{y^3}{3} + 2x - x^2 y\right) dy$$

or

$$\psi = \frac{y^4}{12} + 2xy - \frac{x^2 y^2}{2} + f\left(x\right) + C_1$$

$$v = \frac{\partial \psi}{\partial x}$$

$$\psi = -\int v dx = -\int \left(xy^2 - 2y - \frac{x^3}{3}\right) dx$$

or

$$\psi = -\frac{x^2 y^2}{2} + 2xy + \frac{x^4}{12} + g\left(y\right) + C_2$$

Comparing the above two equations, we have

$$f(x) = \frac{x^4}{12}, \; g(y) = \frac{y^4}{12}$$

Hence, the stream function for the flow is

$$\psi = \frac{y^4}{12} + 2xy - \frac{x^2 y^2}{2} + \frac{x^4}{12} + C$$

where C is a constant.

Q6.

The stream function in a two-dimensional, incompressible flow field is given as

$$\psi = 3xy.$$

(a) Determine the velocity components.

(b) Check whether the flow is irrotational or rotational.

Solution

(a)From the definition of stream function ψ, we get

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Thus, the velocity components become

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(3xy) = 3x$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(3xy) = -3y$$

The velocity is then $\vec{V} = u\hat{i} + v\hat{j} = 3x\hat{i} - 3y\hat{j}$

(b)For the above velocity components, we have

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0$$

The rotation is given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Since the rotation is zero, the flow is irrotational.