

8 Dispersion and Propagation of Wave Packets

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Keywords: Dispersion, group velocity, wave packet, amplitude equation

8.1 Dispersion

Consider a dispersion relation of the form $\omega = \omega(k)$. Superposing harmonic waves with Fourier amplitude $F(k)$ in such a medium gives

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(kx - \omega(k)t)} dk. \quad (8.1)$$

It is intuitively evident that, as time progresses, the wave-form is going to get modified, a phenomenon known as *dispersion*. This occurs because now the Fourier amplitude of the wave-form may be considered to be $F(k)e^{-i\omega(k)t}$, which is time-varying. However, in a very special case (or medium), if $\omega(k)/k = \beta$ (constant), then (8.1) may be simplified as

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ik(x - \beta t)} dk = f(x - \beta t). \quad (8.2)$$

Thus, in this special situation, the wave-form propagates without distortion at the speed $\beta = \omega/k$. Such a medium is known as a *non-dispersive* medium. When a wave-form travels through a dispersive medium it will distort since the phase speeds of the harmonic components constituting the wave-form are all different. Hence, we require a different definition of speed of the wave-form in a dispersive medium, which is discussed next.

8.2 Group Velocity

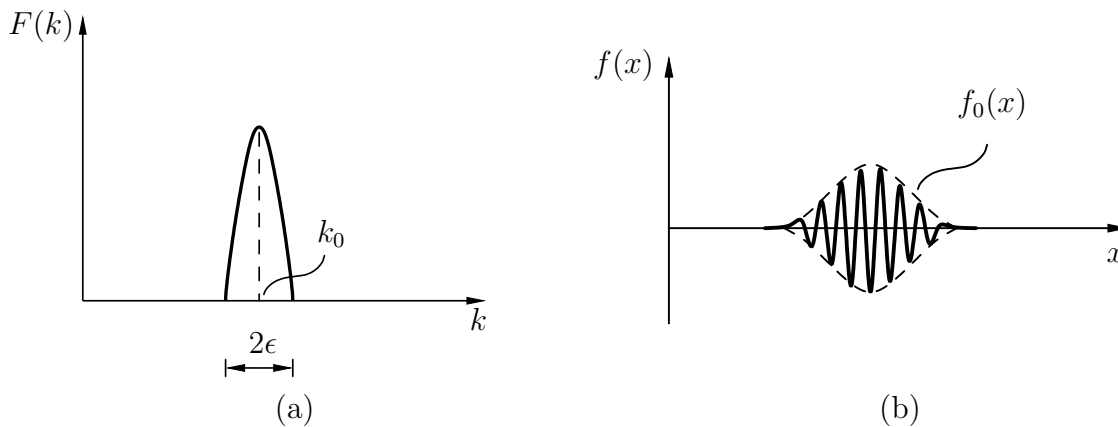


Fig. 8.1: (a) Fourier spectrum of a narrow-band wave-form, (b) Spatial representation of a narrow-band wave-form

Consider the Fourier representation of a traveling wave in a dispersive medium of the form

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(kx - \omega(k)t)} dk. \quad (8.3)$$

We will consider a narrow-band wave with a spectrum distribution similar to that shown in Fig. 8.1(a). The corresponding wave-form is an amplitude

modulated wave like the one shown in Fig. 8.1(b), and is termed as a wave-packet. Such a wave-packet has a functional representation of the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk = f_0(x) e^{ik_0 x}, \quad (8.4)$$

where $f_0(x)$ represents the modulating envelop, as shown in Fig. 8.1(b). It is of interest to determine the velocity of the profile of the wave-packet since the energy of the packet is localized in the high amplitude region.

Let us take the central wave number as k_0 , and write $k = k_0 + \xi$, where $\xi \in [-\infty, \infty]$. Then, one can easily show from (8.4) that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi) e^{i\xi x} d\xi = f_0(x). \quad (8.5)$$

Using this, one can now rewrite (8.3) as

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi) e^{i[k_0 x + \xi x - \omega(k_0 + \xi)t]} d\xi,$$

$$\text{or } f(x, t) \approx e^{i(k_0 x - \omega(k_0)t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi) e^{i\xi[x - \omega'(k_0)t]} d\xi,$$

$$\text{or } f(x, t) = f_0[x - \omega'(k_0)t] e^{i(k_0 x - \omega(k_0)t)} \quad (\text{using (8.5)}),$$

$$\text{or } f(x, t) = f_0[x - c_G t] e^{ik_0(x - c_P t)}, \quad (8.6)$$

where $c_P = \omega(k_0)/k_0$, and

$$c_G := \omega'(k_0) = \left. \frac{d\omega(k)}{dk} \right|_{k=k_0},$$

is defined as the *group velocity* of the wave. It is evident from (8.6) that the amplitude envelope $f_0[\cdot]$ of a wave-packet travels at the group velocity c_G , while the wave itself travels at the phase velocity given by $c_P = \omega(k_0)/k_0$. It should be noted that the solution (8.6) is valid only for a small time due to the approximation involved in deriving it. The concepts of c_P and c_G in a dispersive medium are graphically explained in Fig. 8.2, where the slopes of the dashed lines yield the respective values.

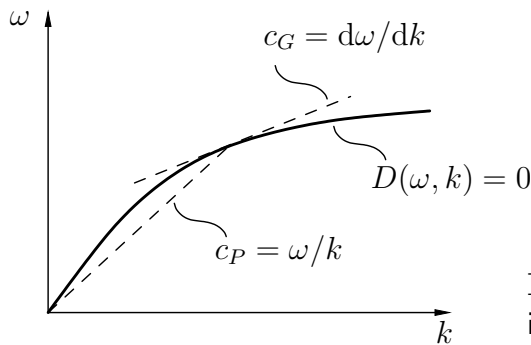


Fig. 8.2: The concept of phase and group velocities in a dispersive medium

8.3 Evolution of a Gaussian Wave Packet

Consider the propagation of a narrow-banded Gaussian wave packet represented in the Fourier domain as $F(k) = e^{-\alpha^2(k-k_0)^2}$. We are interested in evolution of the packet shape as it propagates in a dispersive medium. We

can improve upon the calculation of the Fourier integral presented in the previous section by expanding (8.3) as

$$\begin{aligned} f(x, t) &\approx e^{i(k_0 x - \omega(k_0)t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_0 + \xi) e^{i[x\xi - \omega'(k_0)t\xi - \omega''(k_0)t\xi^2/2]} d\xi, \\ &= e^{i(k_0 x - \omega(k_0)t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-[\beta^2 \xi^2 + 2\beta\gamma\xi]} d\xi, \end{aligned} \quad (8.7)$$

where $\beta = \alpha^2 - i\omega''(k_0)t/2$, and $2\beta\gamma = i(\omega'(k_0)t - x)$. Simplifying the above expression yields

$$\begin{aligned} f(x, t) &\approx e^{i(k_0 x - \omega(k_0)t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{[-(\beta\xi + \gamma)^2 + \gamma^2]} d\xi \\ &= \frac{1}{\sqrt{4\pi(\alpha^2 - i\omega''(k_0)t/2)}} e^{-\frac{(\omega'(k_0)t - x)^2}{4(\alpha^2 - i\omega''(k_0)t/2)}} e^{i(k_0 x - \omega(k_0)t)} \end{aligned} \quad (8.8)$$

This expression is composed of a harmonic wave of wave number k_0 and frequency $\omega(k_0)$, an envelop which is traveling at a speed $\omega'(k_0)$ (group velocity), and a temporal attenuating factor which falls as $\sim 1/\sqrt{t}$. It may be noted that the evolution of the amplitude $A(x, t)$ of the harmonic wave $e^{i(k_0 x - \omega(k_0)t)}$ in (8.7) is governed by the amplitude equation (Schrödinger equation)

$$A_{,t} - \omega'(k_0)A_{,x} - \frac{i\omega''(k_0)}{2}A_{,xx} = 0$$

which has a diffusive character. Thus, the spread of the packet will increase while its amplitude will fall as it propagates.

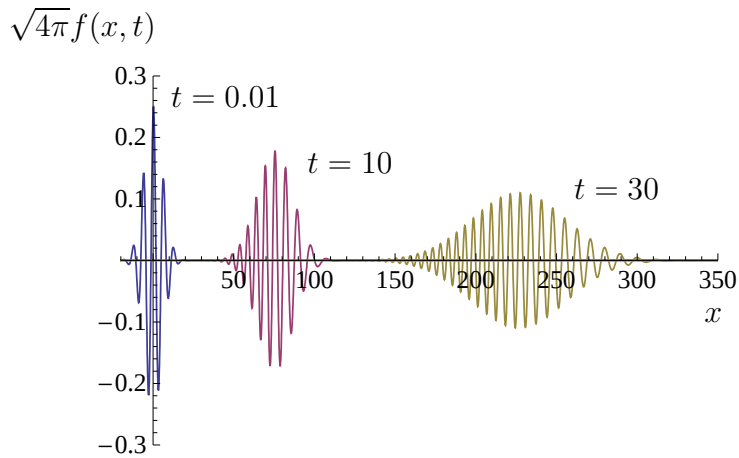


Fig. 8.3: Evolution of a Gaussian wave packet in a taut string on a foundation

For a string supported on a flexible foundation of stiffness density κ , the dispersion relation is given by $\omega^2 = c^2k^2 + \kappa/\rho A$. The propagation of a Gaussian wave packet with $\alpha = 4$ and $k_0 = 1$ is shown in Fig. 8.3. The values of the constants were taken as $c = \sqrt{200}$ m/s and $\kappa/\rho A = 500$ rad²/s².