

37 Non topological Solitons:The KdV equation

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37.1 The KdV equation

We shall present the equation as obtained by Korteweg and de Vries with justified hydrodynamic assumptions in their 1895 paper. Consider the fig. 37.1, where the average depth of the water, l is assumed to be large compared to the fluctuations η . The flow is also considered inviscid. The $l + \eta$ is the height of the surface of water above the bottom at a distance of x from some

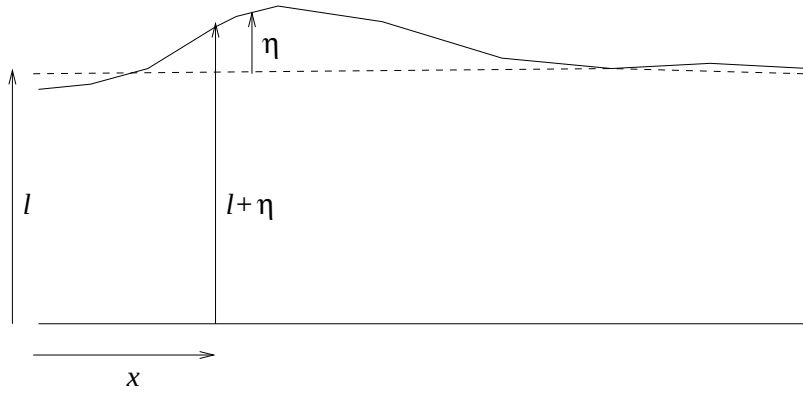


Fig. 37.1: A travelling water wave

origin. The KdV equation is then given by,

$$\frac{\partial \eta}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{l}} \partial_x \left(\frac{2}{3} \alpha \eta + \frac{1}{2} \eta^2 + \frac{1}{3} \sigma \partial_{xx} \eta \right) \quad (37.1)$$

where, $\sigma = \frac{l^3}{3} - \frac{Tl}{g\rho}$ with T as surface tension and α is a small arbitrary constant connected with the velocity of the uniform motion given to the liquid. The equation (37.1) can be written as,

$$\eta_t = \alpha \sqrt{\frac{g}{l}} \eta_x + \frac{3}{2} \sqrt{\frac{g}{l}} \eta \eta_x + \frac{\sigma}{2} \sqrt{\frac{g}{l}} \eta_{xxx} \quad (37.2)$$

Let us scale $\eta = aw$, $t = b\tau$ and $x = c\xi$,

$$\begin{aligned} \frac{a}{b} w_\tau &= \sqrt{\frac{g}{l}} \left(\alpha \frac{a}{c} w_\xi + \frac{3}{2} \frac{a^2}{c} w w_\xi + \frac{\sigma}{2} \frac{a}{c^3} w_{\xi\xi\xi} \right) \\ w_\tau &= \sqrt{\frac{g}{l}} \left(\alpha \frac{b}{c} w_\xi + \frac{3}{2} \frac{ab}{c} w w_\xi + \frac{\sigma}{2} \frac{b}{c^3} w_{\xi\xi\xi} \right) \end{aligned} \quad (37.3)$$

Now setting

$$\alpha \sqrt{\frac{g}{l}} \frac{b}{c} = 1 \quad \text{and} \quad \frac{3}{2} \sqrt{\frac{g}{l}} \frac{ab}{c} = \frac{3}{2\alpha} a = 6 \quad \text{gives } a = 4\alpha. \quad (37.4)$$

$$w_\tau = w_\xi + 6ww_\xi + \frac{\sigma}{2\alpha} \frac{1}{c^2} w_{\xi\xi\xi}. \quad (37.5)$$

We now set $c = \sqrt{\frac{\sigma}{2\alpha}}$ and obtain,

$$w_\tau = w_\xi + 6ww_\xi + w_{\xi\xi\xi}. \quad (37.6)$$

Substituting,

$$u = w + \frac{1}{6} \quad (37.7)$$

in (37.6), we have

$$u_\tau = u_\xi + 6(u - \frac{1}{6})u_\xi + u_{\xi\xi\xi}. \quad (37.8)$$

Now taking, $\xi \rightarrow -\xi$, we get the KdV equation in the standard form as introduced in the last chapter, i.e.,

$$u_\tau + 6uu_\xi + u_{\xi\xi\xi} = 0 \quad (37.9)$$

Soliton solution to (37.9) is given by

$$u = \frac{c}{2} \text{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - c\tau - \xi_0)\right], \quad (37.10)$$

that is

$$\eta = 2c\alpha \text{sech}^2\left[\sqrt{\frac{c\alpha}{2\sigma}}(x - c\alpha\sqrt{\frac{g}{l}}t - x_0)\right]. \quad (37.11)$$

Problem 1: In (37.2) consider $T = 0.073$, $g = 9.8$ & $\rho = 1000$ in SI units for water. By appropriate scaling, $\eta = au$, $t = b\tau$ and $x = c\xi$ in (37.2), one obtains,

$$u_\tau + u_\xi + 6uu_\xi + u_{\xi\xi\xi} = 0,$$

which admits a solitary wave solution, $u = \frac{A}{2} \text{sech}^2[\frac{\sqrt{A}}{2}\{\xi - (1+A)\tau + \delta\}]$. In Scott Russell's description the solitary wave was moving with speed approximately 4 m/s and its height was $\frac{1}{3}$ m to $\frac{1}{2}$ m. It was approximately 10 m long. Find a , b and c and hence estimate approximately the depth, l , of the water in the canal.

37.2 Conservation Laws

Existence of new conservations laws is actually responsible for the non-dispersive solitonic solution.

Let us start with ordinary continuity equation in one dimension,

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial \mathbf{j}(\mathbf{x}, \mathbf{t})}{\partial x} = 0 \quad (37.12)$$

The continuity equation leads to a conservation law if the current $\mathbf{j}$ respects some appropriate boundary conditions. If $\mathbf{j}(x \rightarrow \infty, t) = \mathbf{j}(x \rightarrow -\infty, t) = 0$, we integrate the continuity equation from t_1 to t_2 in the time domain and $-\infty$ to ∞ in the space domain,

$$\int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dx \frac{\partial \rho(x, t)}{\partial t} + \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dx \frac{\partial \mathbf{j}(\mathbf{x}, \mathbf{t})}{\partial x} = 0 \quad (37.13)$$

Let us define, the charge,

$$Q(t) = \int_{-\infty}^{\infty} dx \rho(x, t), \quad (37.14)$$

then

$$\int_{t_1}^{t_2} dt \frac{\partial Q(t)}{\partial t} + \int_{t_1}^{t_2} dt \mathbf{j}(x, t) \Big|_{-\infty}^{\infty} = 0, \quad (37.15)$$

and which gives the charge conservation, using the boundary condition on $\mathbf{j}$,

$$Q(t_2) - Q(t_1) = 0 \quad \Rightarrow \quad Q(t_1) = Q(t_2). \quad (37.16)$$

The energy is one such charge, KdV we shall now find several such conserved charges. We would write continuity equation and from where we could read off the the conserved charges. One such charge is $Q_1 = \int_{-\infty}^{\infty} u(x, t) dx$, which is basically mass conservation. The KdV equation can be recast in a continuity equation,

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= 0, \\ u_t + (3u^2 + u_{xx})_x &= 0 \end{aligned} \quad (37.17)$$

and from the above we find $\rho = u$ and $\mathbf{j} = 3u^2 + u_{xx}$ and since $\mathbf{j}$ is a function of u and its space derivative u_{xx} which follow the boundary condition (36.15).

We can construct another conserved charge $Q_2 = \int_{-\infty}^{\infty} u^2(x, t) dx$ (*momentum conservation*) in the following fashion. We have now $\rho = u^2$ and we find the corresponding current $\mathbf{j}$, by assuming $\mathbf{j} = au^3 + buu_{xx} + cu_x^2$. We need to determine the constants a, b and c . Putting ρ and $\mathbf{j}$ in the continuity equation

$$2uu_t + 3au^2u_x + buu_{xxx} + bu_xu_{xx} + 2cu_xu_{xx} = 0, \quad (37.18)$$

equating $b + 2c = 0$ and $a = 4$, we have,

$$u_t + 6uu_x + \frac{b}{2}u_{xxx} = 0. \quad (37.19)$$

Now for $b = 2$ the above equation is satisfied since it becomes KdV equation, so we have the required current $\mathbf{j} = 4u^3 + 2uu_{xx} - u_x^2$. Further one can construct a third conserved charge, $Q_3 = \int_{-\infty}^{\infty} (u^3 - \frac{u_x^2}{2}) dx$. In fact one has infinitely many independent conserved quantities. This is why after the collisions solitons come out with their original shapes, since infinite number of quantities are to be conserved simultaneously.

Problem 2: Show that the current density for the conserved charge Q_3 is

$$\mathbf{j} = \frac{9}{2}u^4 - u_x u_{xxx} + \frac{1}{2}u_{xx}^2 + 3u^2 u_{xx} - 6u u_x^2. \quad (37.20)$$

37.3 Boussinesq equation

The Boussinesq equation (1871) preceded the KdV equation by almost a quarter century. It is given by

$$u_{tt} = gl \partial_{xx} [u + \frac{3}{2} \frac{u^2}{l} + l^2 3u_{xx}], \quad (37.21)$$

where l is the depth of the water and g is the acceleration due to gravity.

The Boussinesq equation (37.21) admits a solitary wave solution of the form

$$u = h \operatorname{sech}^2 \left[\sqrt{\frac{3h}{4l^3}} (x \pm \sqrt{g(l+h)}t) \right]. \quad (37.22)$$

Again we see that the velocity is proportional to the amplitude h of the wave.

In the following we establish its relation with the KdV equation. Let us scale the variables in the following fashion,

$$u = \delta^a \eta, \quad \xi = \delta^b (x + c_s t) \quad \tau = \delta^c t \quad (37.23)$$

where, $c_s = \sqrt{gl}$, speed of sound. The derivatives transform to

$$\partial_x = \delta^b \partial_\xi, \quad \partial_t = \delta^b c_s \partial_\xi + \delta^c \partial_\tau, \quad (37.24)$$

and the Boussinesq equation takes the form

$$\delta^{2c} \partial_{\tau\tau} \eta + 2c_s \delta^{b+c} \partial_{\xi\tau} \eta = \frac{3}{2} \frac{c_s^2}{l} \delta^{a+2b} \partial_{\xi\xi} \eta^2 + c_s^2 \delta^{4b} \frac{l^2}{3} \partial_{\xi\xi\xi} \eta. \quad (37.25)$$

equating the scaling exponents of last three terms of the above equation

$$b + c = a + 2b = 4b \Rightarrow a = 2b \text{ and } c = 3b. \quad (37.26)$$

Substituting a and c in (40.20) we have,

$$\begin{aligned} \delta^{6b} \partial_{\tau\tau} \eta + 2c_s \delta^{4b} \partial_{\xi\tau} \eta &= \frac{3}{2} \frac{c_s^2}{l} \delta^{4b} \partial_{\xi\xi} \eta^2 + c_s^2 \delta^{4b} \frac{l^2}{3} \partial_{\xi\xi\xi} \eta \\ \delta^{2b} \partial_{\tau\tau} \eta + 2c_s \partial_{\xi\tau} \eta &= \frac{3}{2} \frac{c_s^2}{l} \partial_{\xi\xi} \eta^2 + c_s^2 \frac{l^2}{3} \partial_{\xi\xi\xi} \eta. \end{aligned} \quad (37.27)$$

Taking $b > 0$ and $\delta \rightarrow 0$, the first term vanishes (one could take $b = 1/2$ and that would make the first term an order higher in δ and could be neglected).

We now integrate the dressed equation w.r.t. ξ to have (37.1),

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2} \sqrt{\frac{g}{l}} \partial_\xi \left[\frac{1}{2} \eta^2 + \frac{l^3}{3} \partial_{\xi\xi} \eta \right]. \quad (37.28)$$

The above equation is now in the form of KdV equation