1 Continuum Models - I

Contents

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1.1 Transverse Vibrations of Strings

A string is a one-dimensional elastic continuum that does not transmit or resist bending moment. Consider a string, stretched along the x-axis to a

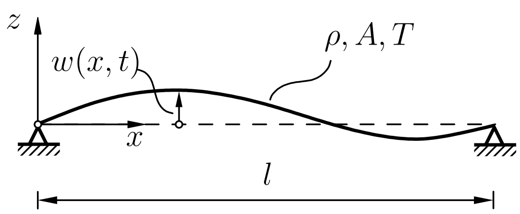

Fig. 1.1: Schematic representation of a taut string

length l by a tension T, as shown in Fig. 1.1. A small element of the string is shown in Fig. 1.2. Neglecting longitudinal motion and assuming $w_{,x} \ll 1$,

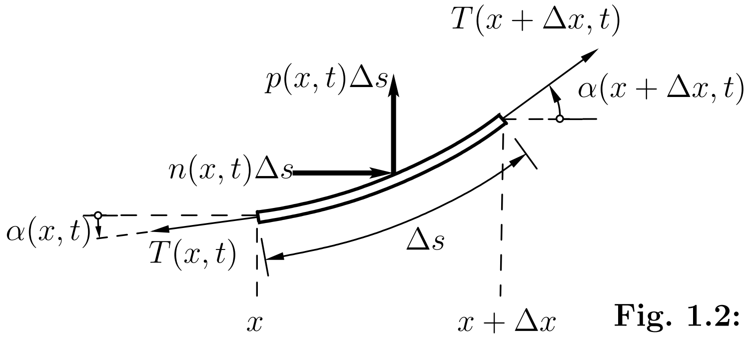

Fig. 1.2: Free body diagram of a string element

the governing equations for the element can be written as

$$\text{Longitudinal dynamics:} \qquad [T(x,t)]_{,x} = -n(x,t) \tag{1.1}$$

$$\text{Transverse dynamics:} \qquad \rho A(x) w_{,tt} - [T(x) w_{,x}]_{,x} = p(x,t). \tag{1.2}$$

When $n(x,t) = p(x,t) = 0$, and ρA is constant, we can rewrite (1.2) as

$$w_{,tt} - c^2 w_{,xx} = 0, \tag{1.3}$$

where $c = \sqrt{T/\rho A}$ is a constant having the dimension of speed. This represents the unforced transverse dynamics of a uniformly tensioned string.

A stretched infinite string on a compliant foundation modeled as a distributed stiffness is shown in Fig. 1.3. The equation of motion reads

$$\rho A w_{,tt} - T w_{,xx} + \kappa w = 0, \tag{1.4}$$

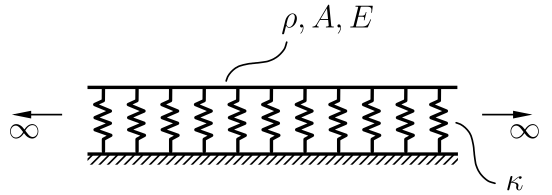

Fig. 1.3: An infinite stretched string on a compliant foundation

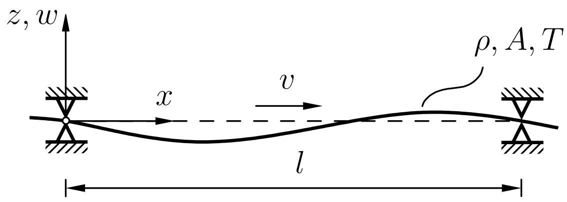

Fig. 1.4: A translating string

where κ is the stiffness per unit length of the foundation.

The equation of motion of the translating string shown in Fig. 1.4 can be written by replacing ∂_t in (1.3) by the material derivative $\partial_t + v\partial_x$ to obtain

$$w_{,tt} + 2vw_{,xt} - (c^2 - v^2)w_{,xx} = 0. \tag{1.5}$$

1.2 Axial Vibration of Bars

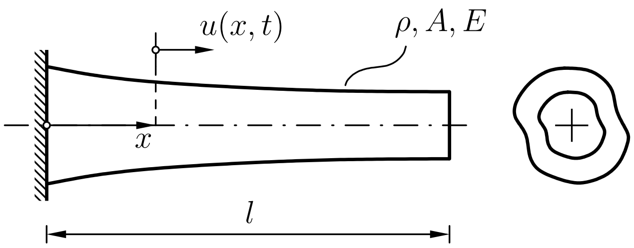

Fig. 1.5: Schematic representation of a bar

We consider the axial vibration of a bar, as shown in Fig. 1.5. An infinites-

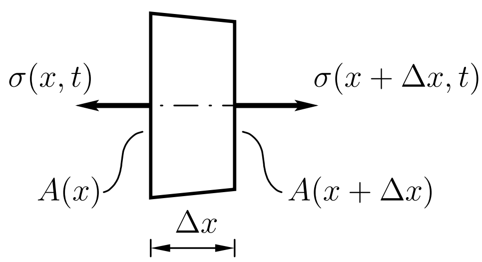

Fig. 1.6: Free body diagram of a bar element

imal element of the bar, is shown in Fig. 1.6. Using the field variable $u(x,t)$, the governing equations for the bar element can be written as

$$\text{Axial dynamics:} \qquad \rho A(x) u_{,tt}(x,t) = [\sigma_x(x,t)A(x)]_{,x} \tag{1.6}$$

$$\text{Material constitutive equation:} \qquad \sigma_x = E\epsilon_x = Eu_{,x}, \tag{1.7}$$

where E is the material's Young's modulus. Using the constitutive equation in the equation of axial dynamics, the equation of motion is obtained as

$$\rho A(x) u_{,tt} - [EA(x)u_{,x}]_{,x} = 0, \tag{1.8}$$

If the bar is homogeneous and uniform, then (1.8) simplifies to

$$u_{,tt} - c^2 u_{,xx} = 0, \tag{1.9}$$

where $c = \sqrt{E/\rho}$.

For a non-uniform bar, the equation of motion may be written as

$$a(x)w_{,tt} - [b(x)w_{,x}]_{,x} = 0, \tag{1.10}$$

consider the transformation of the space variable of the form $\mathrm{d}\xi = \sqrt{a(x)/b(x)}\mathrm{d}x$, where ξ is a new space variable. Let the transformation be represented by $x = s(\xi)$. Then, one can write $\partial_x = \sqrt{a(\xi)/b(\xi)}\partial_\xi$. Defining the transformation $w(\xi,t) = u(\xi,t)/h(\xi)$ (where $h^2(\xi) = \sqrt{a(\xi)b(\xi)}$) one can obtain the simplified transformed equation of motion

$$u_{,tt} - u_{,\xi\xi} + \alpha^2 u = 0. \tag{1.11}$$

for the special cases $h_{,\xi\xi}/h = \pm\alpha^2$, where α is a real constant including zero.

1.3 Waves in a Fluid-filled Elastic Tube

Consider an elastic tube filled with an ideal incompressible fluid. We consider the propagation of axisymmetric deformations of the tube due to fluctuations in the fluid pressure. Assuming the problem to be one dimensional, the govering equations for this case are

$$\text{Continuity equation:} \qquad A_{,t} + (Au)_{,x} = 0 \tag{1.12}$$

$$\text{Momentum equation:} \qquad u_{,t} + uu_{,x} = -\frac{1}{\rho} p_{,x} \tag{1.13}$$

$$\text{Tube constitutive equation:} \qquad p = \mu h r_{,tt} + \frac{Eh}{a^2}(r - a), \tag{1.14}$$

where $A = \pi r^2$ is the area of cross-section of the tube, r is the radius of the tube, u, ρ and p are, respectively, the velocity, density and pressure of the fluid, μ is the density of the tube material, h is the thickness of the tube, E is the Young's modulus and a is the undeformed radius of the tube. Linearizing the equations around a basic state (U, ρ_0, p_0) the equation of motion for the radial deformation is obtained as

$$r_{,tt} + 2U r_{,xt} + \left(U^2 - \frac{Eh}{\rho_0 a}\right) r_{,xx} - \frac{\mu h a}{\rho_0} r_{,xxtt} = 0. \tag{1.15}$$

1.4 Torsional Vibration of Circular Bars

Consider the torsional vibrations of a circular bar, as shown in Fig. 1.7. A small sectional element of the bar is shown in Fig. 1.8. From Fig. 1.8,

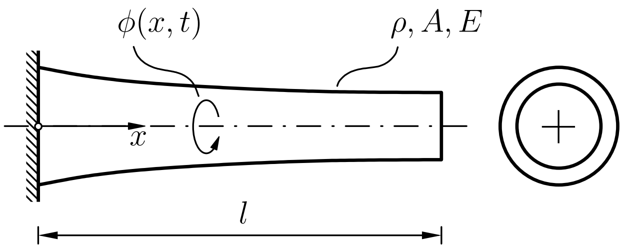

Fig. 1.7: Schematic representation of a circular bar

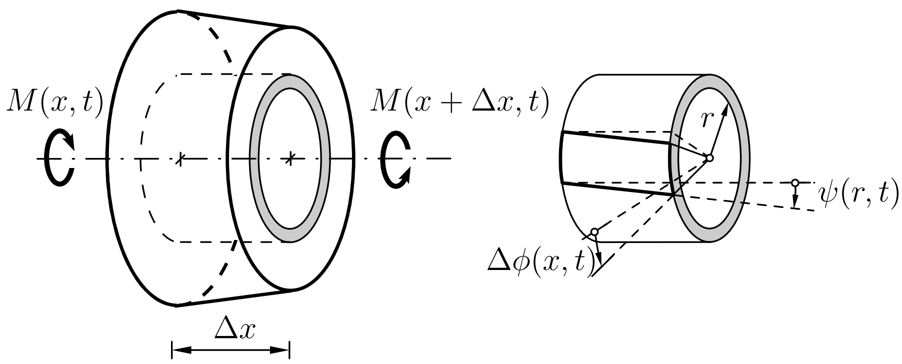

Fig. 1.8: Deformation of a bar element under torsion

we obtain the kinematic relation $r\Delta\phi(x,t) = \Delta x\psi(r,t)$, where $\psi(r,t)$ is the shear strain, and $\phi(x,t)$ is the angle of twist. Using Hooke's law, the shear stress $\tau_{x\phi}(r,t) = G\psi(r,t) = Gr\phi_{,x}$, where G is the shear modulus. Now, the torque at any cross-section x can be computed as

$$M(x,t) = \int_{A(x)} r\tau_{x\phi}(r,t)\,\mathrm{d}A = GI_p(x)\phi_{,x}, \tag{1.16}$$

where $A(x)$ represents the cross-sectional area, and $I_p(x)$ is the polar moment of the area. The moment of momentum balance for the element yields

$$\rho I_p\phi_{,tt} - (GI_p\phi_{,x})_{,x} = 0. \tag{1.17}$$

For a bar with uniform cross-section, we obtain the wave equation

$$\phi_{,tt} - c^2\phi_{,xx} = 0, \tag{1.18}$$

where $c = \sqrt{G/\rho}$.

1.5 Transverse Vibration of Beams

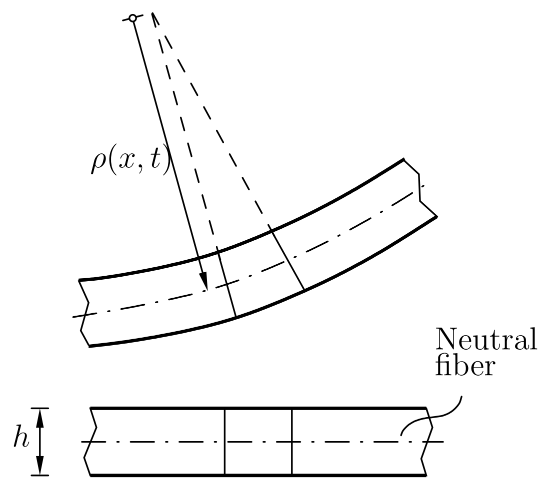

Fig. 1.9: Schematic representation of a beam under planar deflection

Consider a straight beam undergoing a planar deflection under uniaxial bending as shown in Fig. 1.9. Referring to Fig. 1.10, the strain-displacement relation at any height z from the neutral plane is given by

$$\epsilon_x(x,z,t) = -\frac{z}{\rho(x,t)} \approx -zw_{,xx}(x,t) \qquad \text{(assuming } w_{,x} \ll 1\text{)}, \tag{1.19}$$

where $w(x,t)$ is the transverse deflection field of the neutral plane. From Hooke's law as $\sigma_x(x,z,t) = E\epsilon_x(x,z,t) = -Ezw_{,xx}(x,t)$, where E is Young's modulus. The bending moment at any section can then be written as

$$M(x,t) = -\int_{-h/2}^{h/2} z\sigma_x(x,z,t)\,\mathrm{d}A = EI(x)w_{,xx}(x,t), \tag{1.20}$$

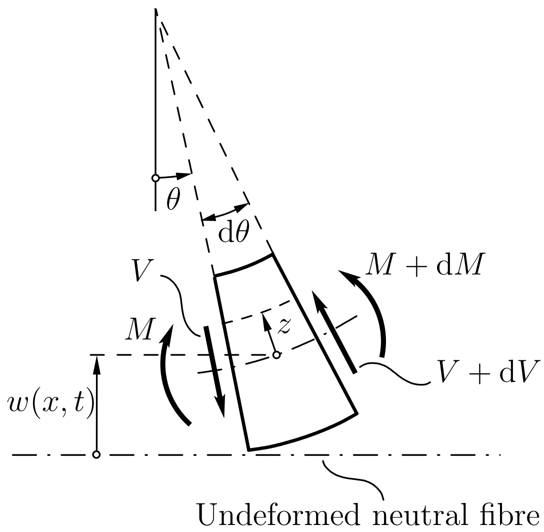

Fig. 1.10: Infinitesimal element of a deflected beam

where $I(x)$ is the second moment of area of cross-section of the beam about the neutral axis. The equations governing the dynamics the infinitesimal element are given by

$$\text{Transverse dynamics:} \qquad \rho A w_{,tt} = V_{,x} \tag{1.21}$$

$$\text{Rotational dynamics:} \qquad \rho I(x)\theta_{,tt} = M_{,x} + V. \tag{1.22}$$

Simplifying the above equations and using $\theta_{,tt} \approx w_{,xtt}$, we can obtain the *Rayleigh beam model*

$$\rho A w_{,tt} + [EIw_{,xx}]_{,xx} - [\rho I w_{,xtt}]_{,x} = p(x,t). \tag{1.23}$$

When the rotary inertia term $(\rho I w_{,xtt})_{,x}$ is neglected, we obtain the *Euler-Bernoulli beam model*

$$\rho A w_{,tt} + [EIw_{,xx}]_{,xx} = p(x,t). \tag{1.24}$$