Mechanics

Module V: Concept of Stress and Strain

Lesson 28: Generalized Hooke's Law - II

Apart from normal stress, a body/element may be subjected to shear stress. This produces a volume preserving distortion characterized by a change in angle between two planes. This deformation is quantified in terms of the shear strain $\gamma_{xy} = 2\phi$ (rad), as shown in Fig. 1

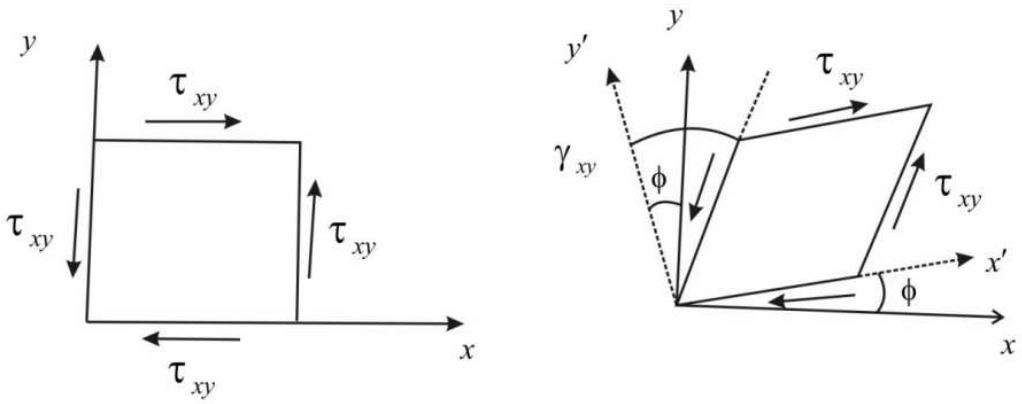


Figure 1:

The relation between the shear stress and the corresponding shear strain is obtained from torsion experiments of circular tubes. For low values of shear stress (much lower than the yield shear stress) the relationship is linear of the form $\tau_{xy} = G\gamma_{xy}$, where τ_{xy} (γ_{xy}) is the shear stress (strain), and G is the shear modulus of elasticity, or the modulus of rigidity.

1 Generalized Hooke's Law

The normal stressstrain and the shear stressstrain relations taken together for a homogeneous and isotropic elastic body is referred to as the generalized Hooke's law. These are written as

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz})$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz})$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

These relations involve 3 material constants namely E, ν and G. However, only 2 of them are independent, as explained in the following.

Consider a square element in a body under pure shear stress, as shown in Fig. 2(a). The stress tensor for this state of stress is given by

$$[\sigma] = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$$

Consider the representation of this state of stress on a square element rotated at 45°, as shown in Fig. 2(b). Then, the stress tensor can be constructed by determining the normal and shear stresses on each surface of the rotated element. For example, for the plane with unit normal $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2})$ and

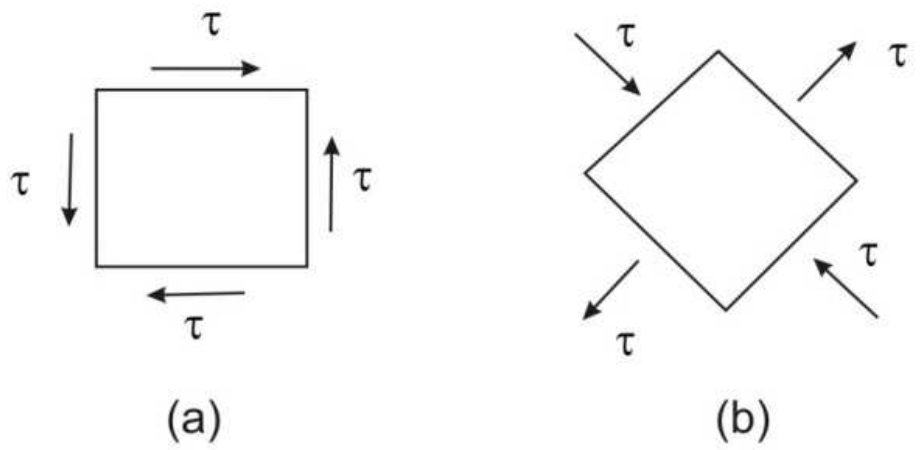


Figure 2:

unit tangent $\hat{\mathbf{s}} = (-1/\sqrt{2}, 1/\sqrt{2})$, the normal and shear stresses are obtained as

$$\sigma_n = \hat{\mathbf{n}}^T[\sigma]\hat{\mathbf{n}} = \tau, \qquad \sigma_s = \hat{\mathbf{s}}^T[\sigma]\hat{\mathbf{n}} = 0$$

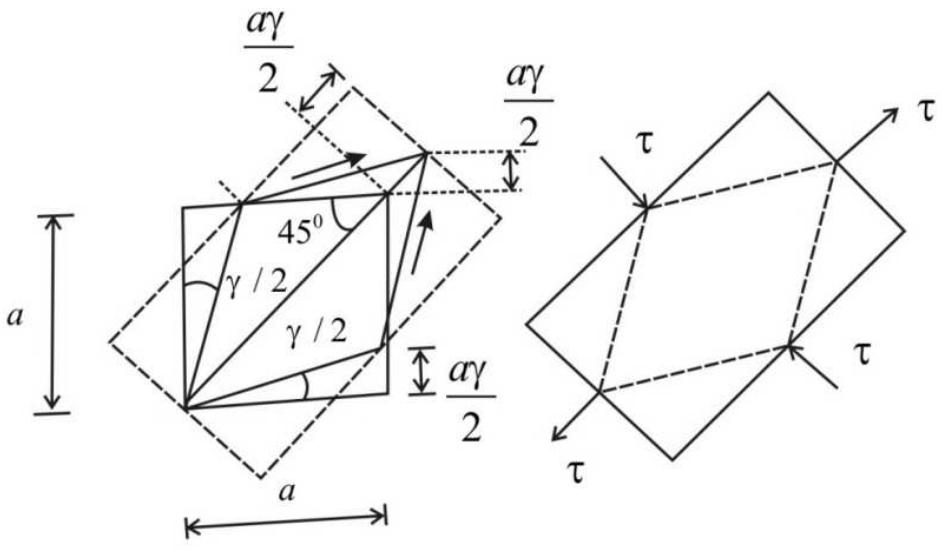
Repeating the calculation for a plane perpendicular to the above plane (with unit normal $\hat{\mathbf{n}} = (-1/\sqrt{2}, 1/\sqrt{2})$ and unit tangent $\hat{\mathbf{s}} = (-1/\sqrt{2}, -1/\sqrt{2})$), we can write the stress tensor in the rotated element (rotated by $\pi/4$ rad) as

$$[\sigma]_{\pi/4} = \begin{bmatrix} \tau & 0 \\ 0 & \tau \end{bmatrix}$$

Consider now the corresponding strains in the elements, as shown in Fig. 3. The extensional strain along the diagonal in Fig. 3(a) is given by (upto linear order)

$$\epsilon = \frac{a\gamma/\sqrt{2}}{a\sqrt{2}} = \frac{\gamma}{2} = \frac{\tau}{2G}.$$

This is also the longitudinal extensional strain in the element shown in



(a) (b)

Figure 3:

Fig. 3(b), and may be expressed as

$$\epsilon = \frac{\tau}{E} - \frac{\nu}{E}(-\tau) = \frac{1+\nu}{E}\tau$$

$$\Rightarrow\ G = \frac{E}{2(1+\nu)}.$$

This relates the 3 material constants. In the case of steel, $E = 200$ GPa and $\nu = 0.3$, implying $G = 76.9$ GPa.

2 Plane Stress and Plane Strain

The state of stress at a point of a body comprising nonzero stresses only on two mutually orthogonal planes is known as plane stress, or biaxial state of stress. A body subjected to plane stress at each point represents a plane stress problem. Usually, stress distributions in pressurized/stretched membranes, thin plates and shells are approximated as plane stress problems.

Consider a plane stress state represented by the stress elements $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$. Then, using the generalized Hooke's law

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy}$$

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx}$$

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

It may be noted that in plane stress problems, the strain is triaxial in general (due to the Poisson strain). Thus, thickness of a membrane, thin plate/shell can change under plane stress.

Similar to a plane stress state, a plane strain state is one in which the strains are restricted to one plane. In this case, the strain components $\gamma_{zz} = \gamma_{yz} = \epsilon_{zx} = 0$. Hence,

$$\frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = 0 \;\Rightarrow\; \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

Using this expression in the first other two normal strain expressions yields

$$\epsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$$

$$\epsilon_{yy} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$$

Thus, in a plane strain problem, the stress state is triaxial in general. The shear strain relation remains unchanged as $\gamma_{xy} = \tau_{xy}/G$.

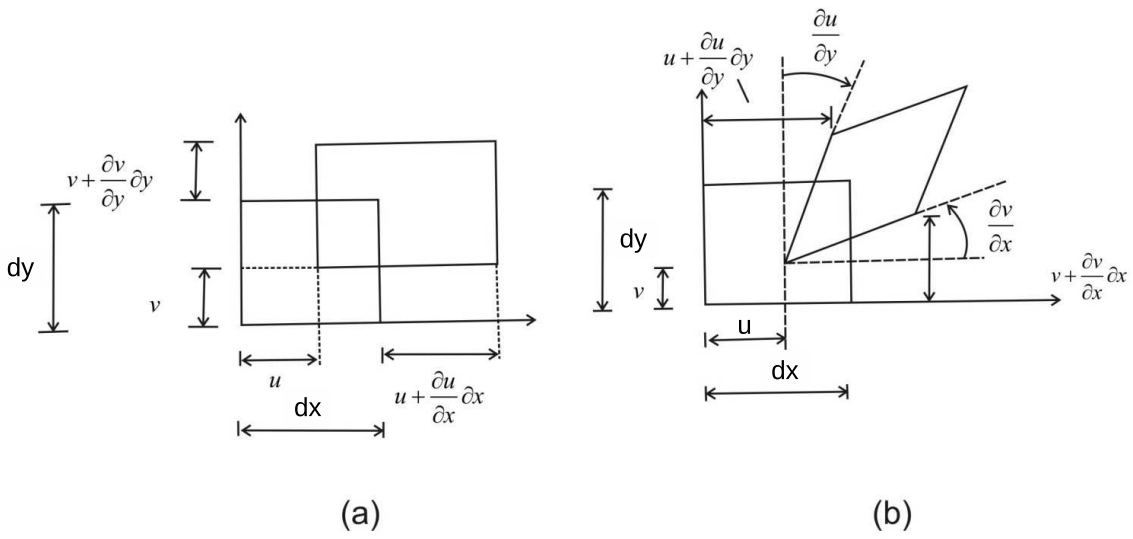


Figure 4:

3 Strain-displacement relations

Consider the deformations of an infinitesimal element, as shown in Fig. 4. The normal strain in the $x$axis direction in Fig. 4(a) can be written as

$$\epsilon_{xx} = \frac{u + \frac{\partial u}{\partial x}\mathrm{d}x - u}{\mathrm{d}x} = \frac{\partial u}{\partial x}$$

Similarly,

$$\epsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

The shear strain in the x-y plane in Fig. 4(a) can be written as

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Similarly,

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

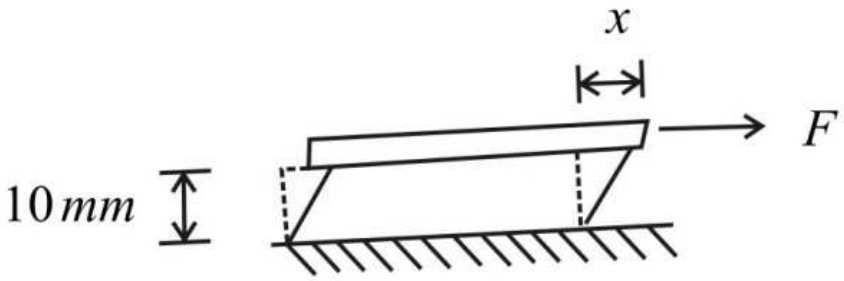


Figure 5:

The strains are arranged as a matrix/tensor (referred to as the strain tensor) in the form

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_{zz} \end{bmatrix}$$

since they obey the coordinate transformation (rotational) properties of matrices/tensors in this form. Note that, as tensor elements, the shear strain components are $\epsilon_{xy} = \gamma_{xy}/2$, $\epsilon_{yz} = \gamma_{yz}/2$ and $\epsilon_{zx} = \gamma_{zx}/2$.

Problem 1

A shear mount consists of a 10 mm thick rubber block with a metal plate fixed on the top, as shown in Fig. 5. If the shear modulus of rubber is 0.65 MPa, determine the required area of the block if the desired tangential stiffness $k = F/x = 100$ kN/m.

Solution

Here, shear strain $\gamma = x/10 \times 10^{-3} = 100x$. Hence, shear stress

$$\tau = \frac{F}{A} = G\gamma \;\Rightarrow\; A = \frac{F/x}{100G} = 1.54 \times 10^{-3}\ \text{m}$$

Problem 2

A body is subjected to a displacement field $\mathbf{u} = (x^2+yz)\hat{i}+(2y+xz)\hat{j}-xy\hat{k}$.

Determine the strain tensor field, the volume strain field and the stress field.

Solution

The strain tensor components are calculated as

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = 2x, \qquad \epsilon_{yy} = \frac{\partial u_x}{\partial y} = 2, \qquad \epsilon_{zz} = \frac{\partial u_z}{\partial z} = 0$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = z$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = 0$$

Hence, the strain tensor field reads

$$[\epsilon] = \begin{bmatrix} 2x & z & 0 \\ z & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The volume strain is obtained as

$$\frac{\Delta V}{V} = \text{Trace}[\epsilon] = 2(x+1)$$

The stress field is obtained from the generalized Hooke's law as

$$\sigma_{xx} = \frac{2E}{1-3\nu^2-2\nu^3}[(1-\nu^2)x + \nu(1+\nu)]$$

$$\sigma_{yy} = \frac{2E}{1-3\nu^2-2\nu^3}[(1-\nu^2) + \nu(1+\nu)x]$$

$$\sigma_{zz} = \frac{2E\nu(1+\nu)}{1-3\nu^2-2\nu^3}(1+x)$$

$\tau_{xy} = G\gamma_{xy} = 2G\epsilon_{xy} = 2Gz$, and $\tau_{yz} = \tau_{zx} = 0$.