Mechanics

Module V: Concept of Stress and Strain

Lesson 27: Generalized Hooke's Law - I

1 Lateral Strain: Poissons Ratio

All solid bodies undergo deformation in the direction of the applied force. In addition to this, a certain amount of lateral/transverse expansion (or contraction) is also observed. This remarkable property that is commonly observed is illustrated in Fig. 1 where the deformations are exaggerated. The lateral

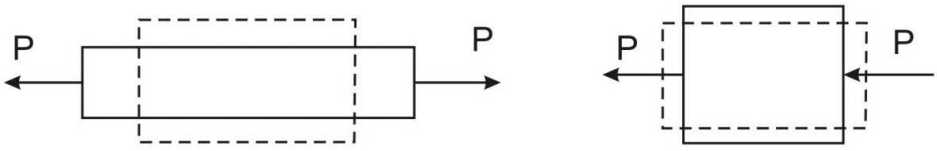


Figure 1:

expansion (or contraction) does not result in any additional stress unless it is inhibited/restricted. The lateral/transverse strain ϵ_t is defined as usual

$$\epsilon_t = \frac{l^t - l_0^t}{l_0^t}$$

where l_0^t and l^t are, respectively, the initial and final transverse dimensions. It is observed from experiments that, for homogeneous and isotropic materials

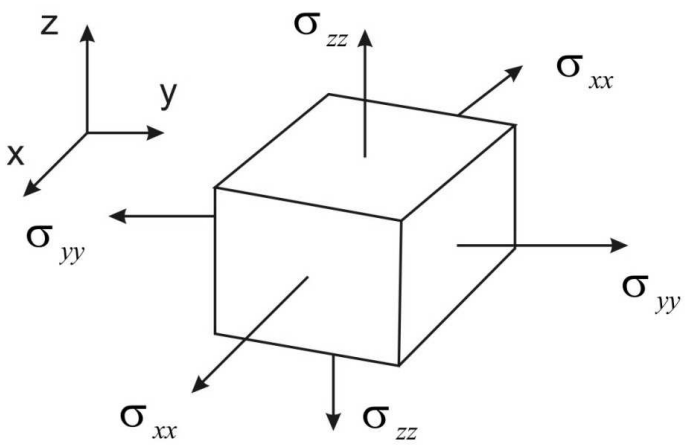


Figure 2:

within the elastic limit, the transverse strain bears a constant proportional relationship with the axial strain of the form

$$\epsilon_t = -\nu\epsilon_a \;\Rightarrow\; \nu = -\frac{\epsilon_t}{\epsilon_a}$$

where ϵ_a is the axial strain (produced by the uniaxial stress only), and the constant of proportionality ν is a material property known as the Poisson ratio. For most materials, the Poisson ratio varies between 0.25-0.35. For certain material such as concrete $\nu = 0.1$, while for incompressible material such as rubber $\nu = 0.5$, which is the largest value possible. In certain materials, known as auxetic materials, Poisson ratio is negative.

2 Normal Stress-Strain Relation

Consider an infinitesimal element of a solid body subject to triaxial normal stress, as shown in Fig. 2. The net normal strain in the x-axis direction

consists of the superposition of (a) the normal strain σ_{xx}/E, and (b) lateral/Poisson strains $-\nu\sigma_{yy}/E$ and $-\nu\sigma_{zz}/E$. It is to be noted here that the Poisson strains along x are calculated based on only the normal stresses along the y and z axes. Hence, the net normal strain in the x axes direction is

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) \qquad (1).$$

Similarly,

$$\epsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \qquad (2)$$

$$\epsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \qquad (3)$$

The relations (1)-(3) are the normal stress-strain relations of a linearly elastic body.

3 Dilation and Bulk Modulus

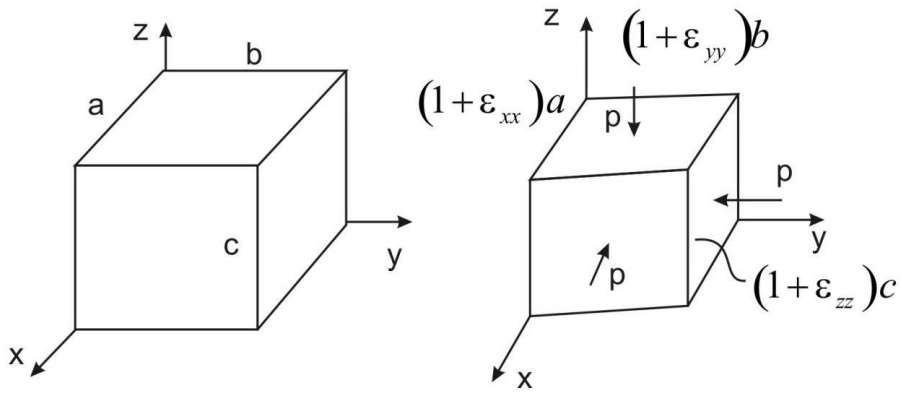


Figure 3:

Consider a solid block of initial volume $V = abc$ subjected to a uniform triaxial hydrostatic state of stress, as shown in Fig. 3. The stresses are given

by $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ where the negative sign implies a compressive stress. The corresponding normal strains are obtained from (1)-(3) as $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = -\frac{1-2\nu}{E}p$ The volume of the block after compression is given by (assuming small strains)

$$V' = a(1+\epsilon_{xx})a(1+\epsilon_{yy})a(1+\epsilon_{zz}) \approx abc(1+\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}).$$

Hence, the volume strain is given by

$$\frac{\Delta V}{V} = \frac{V'-V}{V} = \epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz} = -\frac{3(1-2\nu)}{E}p$$

and the bulk modulus is obtained as

$$K = \frac{-p}{\Delta V/V} = \frac{E}{3(1-2\nu)}.$$

It may be noted from the expression of the volume strain that, if $\nu = 0.5$, $\Delta V/V = 0$ (linear incompressibility).

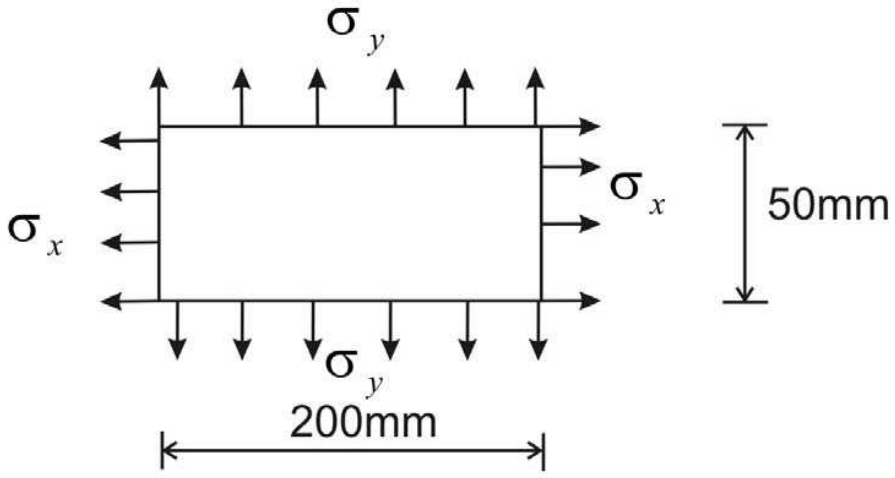


Figure 4:

Problem 1

A 10 mm thick steel plate, shown in Fig. 4, is subjected to uniform edge stresses σ_x and σ_y. If the thickness reduces by 2×10^{-4} mm, estimate the change in the area of the plate. Take $E = 200$ GPa and $\nu = 0.25$.

Solution

In this case, the normal strains are related to the stresses by

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}\sigma_x$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \qquad (1)$$

Here, the strain in the thickness $\epsilon_z = -2 \times 10^{-4}/10$. Hence, from the expression of ϵ_z above $\sigma_x + \sigma_y = 16$ MPa. Change in area

$$\Delta A = (\epsilon_x + \epsilon_y)A = \frac{1}{E}[(\sigma_x + \sigma_y) - \nu(\sigma_x + \sigma_y)]A = 0.6 \text{ mm}^2.$$