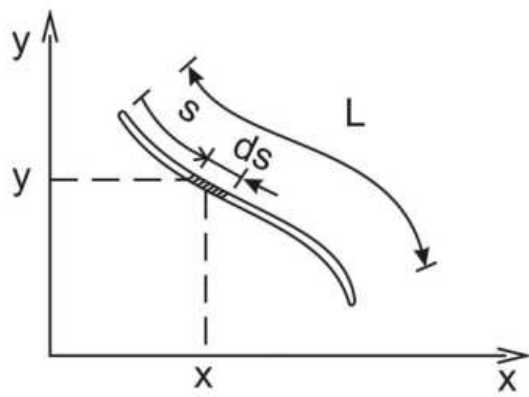


Mechanics

Module IV: Properties of Curves and Areas

Lesson 20: Second Moment of Curves and Areas

The second moment of a distribution gives an idea of its spread and extent. It defines an important property known as the radius of gyration of the distribution.



$$\begin{aligned} I_x &= \int_L y^2 ds \\ I_y &= \int_L x^2 ds \\ I_z &= \int_L (x^2 + y^2) ds \\ &= I_x + I_y \end{aligned}$$

Figure 1:

The second moments I_x and I_y about x and y axes, respectively, for curves and areas are defined as shown in Figs. 1 and 2. Here, $I_z = I_x + I_y$ is also known as the polar moment of the area, and denoted by I_p . The radius of gyration is defined as

$$k_x^2 = \frac{I_x}{A}, \quad k_y^2 = \frac{I_y}{A}, \quad k_z^2 = \frac{I_z}{A} = k_x^2 + k_y^2.$$

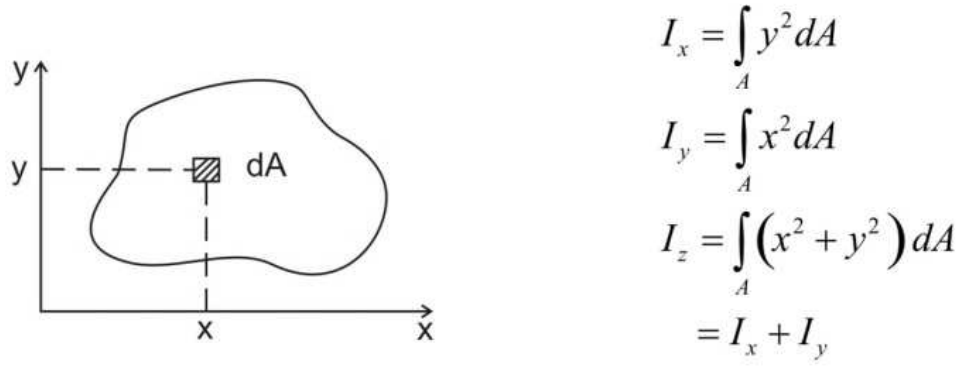


Figure 2:

It may be noted that the second moments of area are always positive quantities with dimension L^4 .

1 Parallel Axis Theorem

Consider Fig. 3 and assume that the second moments of the area about the centroidal axes $\bar{x}$ - $\bar{y}$ are given by I_x and I_y , and defined through

$$\bar{I}_x = \int_A \bar{y}^2 dA, \quad \bar{I}_y = \int_A \bar{x}^2 dA.$$

Now, consider the second moment of the area A about the axes xy shown in Fig. 3.

$$\begin{aligned} I_x &= \int_A y^2 dA = \int_A (\bar{y} + d_y)^2 dA \\ &= \int_A \bar{y}^2 dA + 2d_y \int_A \bar{y} dA + d_y^2 \int_A dA. \end{aligned}$$

Since $\int_A \bar{y} dA = 0$ ($\bar{x}$ being the centroidal axis),

$$I_x = \bar{I}_x + Ad_y^2.$$

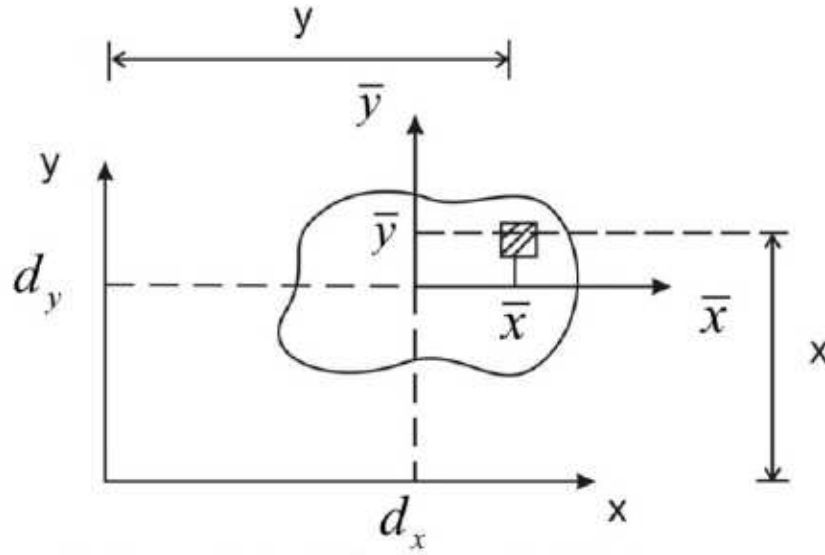


Figure 3:

Similarly, one can show that

$$I_y = \bar{I}_y + Ad_x^2.$$

This is the parallel axis theorem.

It may be noted that the second moment of area is linear over area. For composite areas, the second moment of the elemental areas about a single axis are additive, and yield the second moment of the composite area about that axis.

Problem 1

Determine the second moment of the area of a rectangular section shown in Fig. 4 about the centroidal axes and about the edges.

Solution

The centroid of a rectangular area is at its geometric center G, as shown in

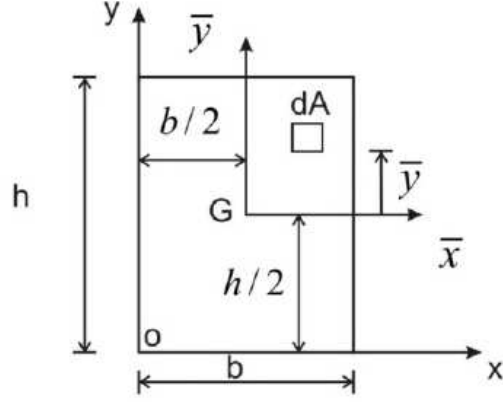


Figure 4:

Fig. 4. From the definition of second moment of area

$$\bar{I}_x = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} \bar{y}^2 d\bar{x} d\bar{y} = \frac{bh^3}{12}.$$

Similarly,

$$\bar{I}_y = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} \bar{x}^2 d\bar{x} d\bar{y} = \frac{hb^3}{12}.$$

The second moment of the area about the edges (x and y axes) is obtained using the parallel axis theorem

$$I_x = \bar{I}_x + Ad_y^2 = \frac{bh^3}{12} + Ah^2 = \frac{bh^3}{3}, \quad I_y = \bar{I}_y + Ad_x^2 = \frac{hb^3}{12} + Ab^2 = \frac{hb^3}{3}$$

The polar moment of the area about z -axis is obtained as

$$I_p = I_x + I_y = \frac{bh}{3}(b^2 + h^2).$$

Problem 2

Determine the second moment of the triangular area about the x and x' axes shown in Fig. 5. Also determine the radius of gyration about x axis.

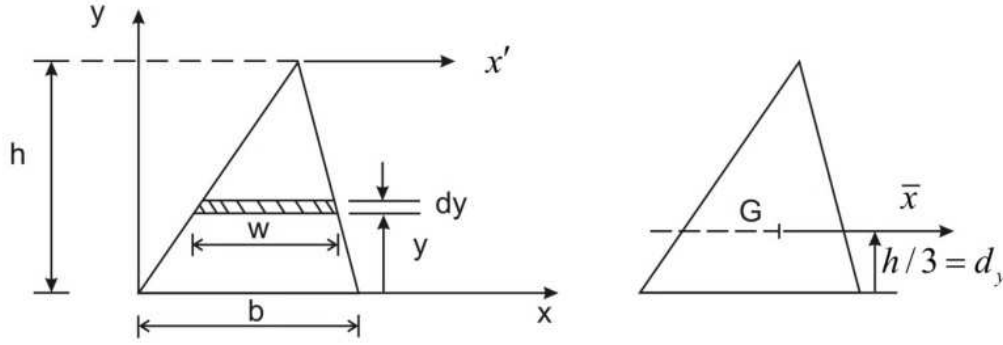


Figure 5:

Solution

Consider the shaded area. The width of the area is given by $w = b(1 - y/h)$, where $y \in [0, h]$. From the definition and Fig. 5

$$I_x = \int_0^h y^2 w dy = \frac{bh^3}{12}.$$

It is known that the centroidal height from the base is $d_y = h/3$. Hence, using parallel axis theorem between the parallel axes x and x' yields

$$I'_x = I_x + A \left(\frac{2h}{3} \right)^2 = \frac{bh^3}{4}.$$

Radius of gyration k_x and k'_x are obtained from the definitions

$$Ak_x^2 = I_x \Rightarrow k_x = \frac{h}{\sqrt{6}}, \quad Ak'^2_x = I'_x \Rightarrow k'_x = \frac{h}{\sqrt{2}}.$$

Problem 3

Determine the second moments of the area for the sector of a circle shown in Fig. 6. Also determine the radius of gyration.

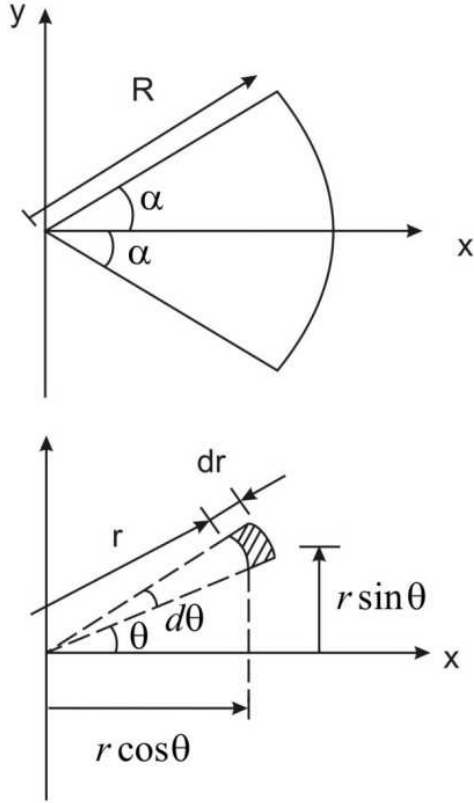


Figure 6:

Solution

From the definitions

$$I_x = \int_0^R \int_{-\alpha}^{\alpha} r^2 \sin^2 \theta r d\theta dr = \frac{R^4}{4} \left(\alpha - \frac{1}{2} \sin 2\alpha \right)$$

$$I_y = \int_0^R \int_{-\alpha}^{\alpha} r^2 \cos^2 \theta r d\theta dr = \frac{R^4}{4} \left(\alpha + \frac{1}{2} \sin 2\alpha \right)$$

The polar moment of area

$$I_p = I_x + I_y = \frac{R^4 \alpha}{2}.$$

Form the definition of the radius of gyration

$$Ak_x^2 = I_x \Rightarrow k_x = \frac{R}{2\sqrt{\alpha}} \left(\alpha - \frac{1}{2} \sin 2\alpha \right)^{1/2}$$

$$Ak_y^2 = I_y \Rightarrow k_y = \frac{R}{2\sqrt{\alpha}} \left(\alpha + \frac{1}{2} \sin 2\alpha \right)^{1/2}$$

$$Ak_p^2 = I_p \Rightarrow k_p = \frac{R}{\sqrt{2}}$$

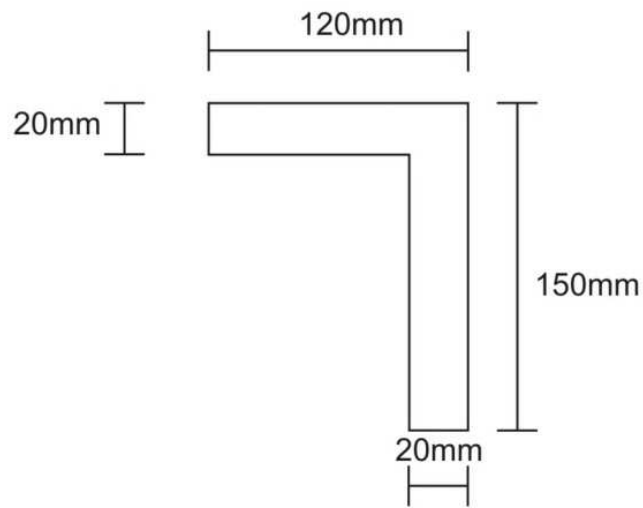


Figure 7:

Problem 4

For the crosssection of an angle shown in Fig. 7, calculate the second moment of the area about centroidal $\bar{x}$ -axis.

Solution

Consider the two elemental areas I and II and the centroidal x -axis of area I as the reference line shown in Fig. 8. The centroidal $\bar{x}$ -axis of the composite

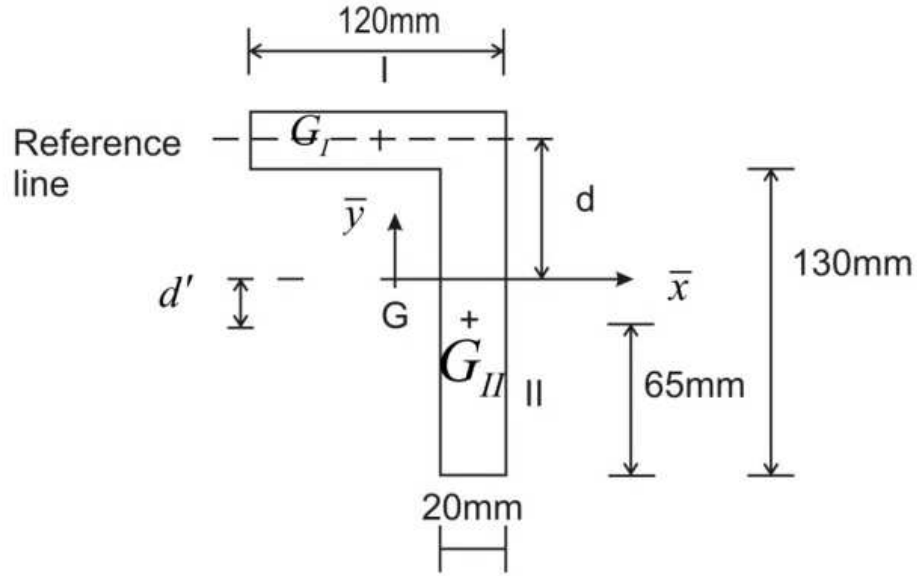


Figure 8:

area can be located with respect to this reference line as

$$d = \frac{A_I y_c^I + A_{II} y_c^{II}}{A_I + A_{II}}$$

$$= \frac{(20 \times 120) \times 0 + (20 \times 130) \times 75}{(20 \times 120) + (20 \times 130)} = 39 \text{ mm.}$$

Second moment of area about $\bar{x}$ is given by $\bar{I}_x = \bar{I}_x^I + \bar{I}_x^{II}$ where I_x^I and I_x^{II} are, respectively, the second moments of the elemental areas I and II about the $\bar{x}$ -axis. These are obtained as follows.

$$\bar{I}_x^I = I_x^{G_I} + A_I d_I^2$$

$$= \frac{120 \times 20^3}{12} + (20 \times 120) \times 39^2 \text{ mm}^4 = 3.73 \times 10^6 \text{ mm}^4$$

$$\bar{I}_x^{II} = I_x^{G_{II}} + A_{II} d_{II}^2$$

$$= \frac{20 \times 130^3}{12} + (20 \times 130) \times 36^2 \text{ mm}^4 = 7.03 \times 10^6 \text{ mm}^4$$

Thus, $\bar{I}_x = 10.76 \times 10^6 \text{ mm}^4$.