Mechanics

Module I: Force Systems

Lesson 1: Introduction

1 Force

Force in classical mechanics is the perceptible effect that characterizes electromagnetic and gravitational interaction between particles or bodies. Force, however, cannot be measured directly, but indirectly through the effect it produces (such as motion, or deflection/deformation).

Force is a vector quantity which

- has a magnitude and a direction, and
- combines according to the parallelogram law of addition (see Fig. 1).

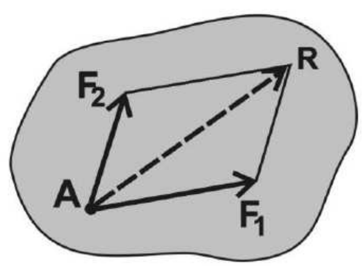

Figure 1: Parallelogram law of force addition

Force can be attractive or repulsive, and it follows the Newton's third law. The force that exists inside a body (between its parts) is known as internal force, as opposed to external force between two independent particles/bodies.

There are two kinds of forces (based on the range of their effective action):

- Body forces: act throughout the body (e.g., gravitational force)

- Surface forces: act only on the surface (e.g., pressure, drag)

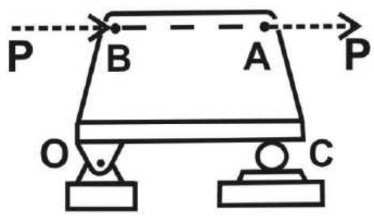

Figure 2: Translation of force vector along its line of action in a rigid body

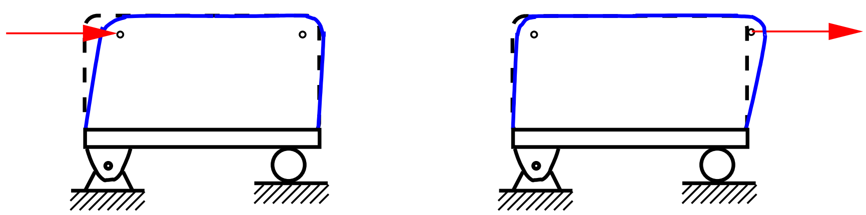

Figure 3: Translation of force vector along its line of action in a flexible body

A force on a rigid body can be translated along its line of action, as shown in Fig. 2. This is possible since we will be interested only in the reaction forces at the support/interaction points. On the other hand, in flexible bodies, we will be interested in the internal force distribution and the deformation, which will depend very much on the point of application of the force, as shown in

Fig. 3. In case of rigid bodies, this translation property allows one to define a net/total force on a body with a single line of action, as shown in Fig. 4.

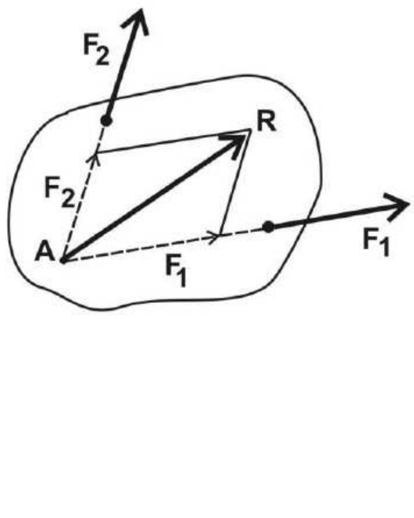

Figure 4: Addition of forces on a rigid body ($\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$)

2 Representation of Vectors

A vector can be represented using a coordinate system such as the Cartesian coordinate system, as shown in Fig. 5, with unit basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ with the following orthonormality properties for simplicity

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0, \quad \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0, \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1, \quad \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1, \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

The magnitude of the vector is independent of the choice of the coordinate system, though the representation of its direction will depend on it. A vector

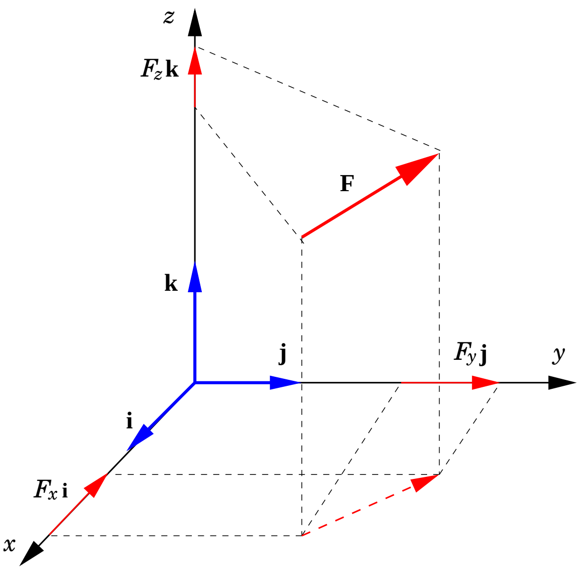

Figure 5: Representation of a vector in the Cartesian coordinate system

can be represented in the algebraic form as linear combination of the (unit) basis vectors as

$$\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}},$$

or as components in a matrix form as

$$\mathbf{F} = \left\{ \begin{array}{c} F_x \\ F_y \\ F_z \end{array} \right\}.$$

The magnitude of the vector is defined as $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$.

Unit vectors represent directions and are dimensionless. The unit vector in the direction of $\mathbf{F}$ is represented as $\hat{\mathbf{F}} = \mathbf{F}/F$. Using the orthonormality properties of the basis vectors, one can define various products of vectors.

3 Dot Product

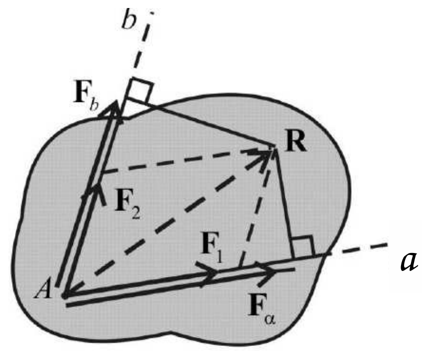

Figure 6: Difference between decomposition and projection in non-orthogonal coordinates

The dot product is a commutative product and results in a scalar quantity. It can be used to determine the angle between two directions represented by two unit vectors. For example, given two unit vectors $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, the angle θ between them is given by $\cos\theta = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = n_{1x}n_{2x} + n_{1y}n_{2y} + n_{1z}n_{2z}$. The projection of a vector $\mathbf{F}$ along a direction represented by a unit vector $\hat{\mathbf{n}}$ is given by $\mathbf{F}_n = (\mathbf{F}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}$. The projection of a vector onto itself gives the square of the magnitude of the vector, *i.e.*, $F^2 = \mathbf{F}\cdot\mathbf{F}$.

Vector projection and vector decomposition are different for non-orthogonal set of axes, as shown in Fig. 6. Unlike the projected vectors $\mathbf{F}_a$ and $\mathbf{F}_b$, the decomposed vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ add up vectorially (parallelogram law) to give back the original vector.

4 Cross Product

The cross product is a non-commutative vector product which yields a new vector which is orthogonal to both the multiplying vectors. The cross product

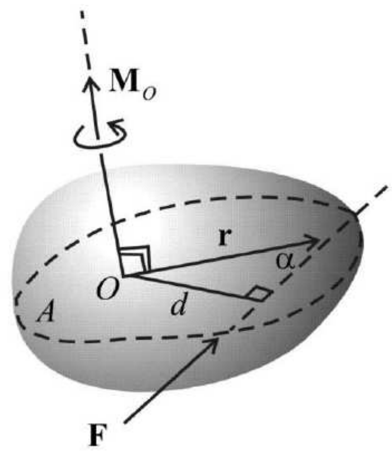

Figure 7: Moment of a force about a point

$\mathbf{c} = \mathbf{a} \times \mathbf{b}$ can be written in the determinant form as

$$\mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix},$$

or in the matrix product form

$$\left\{ \begin{array}{c} c_x \\ c_y \\ c_z \end{array} \right\} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \left\{ \begin{array}{c} b_x \\ b_y \\ b_z \end{array} \right\}.$$

The cross product is used to define the moment of a vector about a point (pole), as shown in Fig. 7. The moment of a force vector is sometimes simply termed as moment. Since cross-product is not commutative, it is important to note the definition of moment as

$$\begin{aligned} \text{Moment vector} &= \text{Arm vector} \times \text{Force vector} \\ \Rightarrow \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} \end{aligned}$$

where $\mathbf{r}$ is the arm vector from the pole to any convenient point on the line of force. The magnitude of the moment vector is given by $Fr \sin \alpha = Fd$,

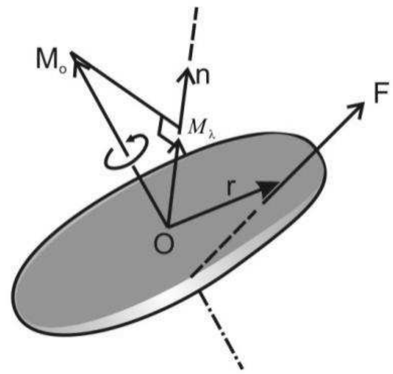

Figure 8: Moment of a force about an axis

where d is the perpendicular distance between the pole and the line of the force.

The projection of the moment of a vector about a pole O along a direction λ, as shown in Fig. 8, is known as the moment of the vector about an axis. The axis is defined by the unit vector $\hat{\mathbf{n}}$ representing the direction and passing through the pole. Moment about an axis is a scalar quantity represented by the vector triple product

$$M_\lambda = \mathbf{r} \times \mathbf{F} \cdot \hat{\mathbf{n}}.$$

5 Problems

Problem 1

A plate is subjected to the two forces, as shown in the Fig. 9. (a) Determine the resultant force. (b) Decompose the resultant force in the x-a (non-orthogonal) coordinate system.

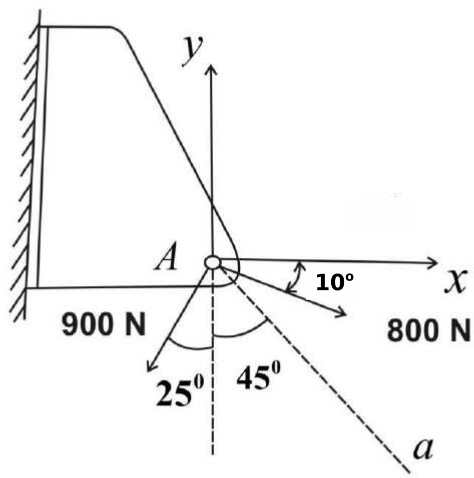

Figure 9: Problem 1

Solution

Representation of the given forces:

$$\mathbf{F}_1 = 800(\cos 10^\circ \hat{\imath} - \sin 10^\circ \hat{\jmath})\ \text{N}$$

$$\mathbf{F}_2 = 900(-\sin 25^\circ \hat{\imath} - \cos 25^\circ \hat{\jmath})\ \text{N}$$

Resultant force:

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= (800\cos 10^\circ - 900\sin 25^\circ)\hat{\imath} + (-800\sin 10^\circ - 900\cos 25^\circ)\hat{\jmath}\ \text{N}\end{aligned}$$

One can also represent

$$\mathbf{F} = F_x\hat{\imath} + F_a\hat{a}.$$

Hence,

$$\mathbf{F}\cdot\hat{\jmath} = F_a\hat{a}\cdot j$$

$$\Rightarrow F_a = 1350\ \text{N}.$$

Using the relation $\mathbf{F} \cdot \hat{\imath} = F_x + F_a \hat{a} \cdot \hat{\imath}$, one can easily obtain $F_x = -547$ N. Thus, the representation of $\mathbf{F}$ in the x-a coordinate system is given by

$$\mathbf{F} = -547\hat{\imath} + 1350\hat{a} \text{ N}.$$

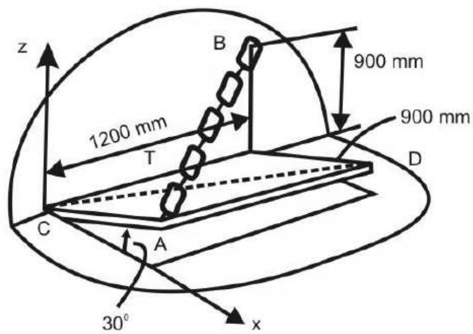

Figure 10: Problem 2

Problem 2

The tension in the chain AB supporting the door shown in Fig. 10 is 100 N. Determine the component of the tension force along CD. Also, determine the moment of the tension force about the axis CD.

Solution

Representation of chain tension:

$$\mathbf{T} = 100\frac{\overrightarrow{AB}}{AB} \text{ N}.$$

Here,

$$\overrightarrow{AB} = \overrightarrow{CB} - \overrightarrow{CA}$$

$$\overrightarrow{CB} = 1.2\hat{\jmath} + 0.9\hat{k} \text{ m} \qquad \overrightarrow{CA} = 0.9(\cos 30°\hat{\imath} + \sin 30°\hat{k}) \text{ m}$$

Thus,

$$\overrightarrow{AB} = -0.779\hat{\imath} + 1.2\hat{\jmath} + 0.45\hat{k} \text{ m}$$

and $AB = 1.5$ m. Hence,

$$\mathbf{T} = 100(-0.5193\hat{\imath} + 0.8\hat{\jmath} + 0.3\hat{k}) \text{ N}.$$

Now, $T_{CD} = \mathbf{T} \cdot \widehat{CD}$, where

$$\widehat{CD} = \frac{1}{1.5}(0.9\cos 30°\hat{\imath} + 1.2\hat{\jmath} + 0.9\sin 30°\hat{k}).$$

Carrying out the dot product, $T_{CD} = 46$ N.

Moment of the tension force about CD can be easily obtained from $M_{CD} = \overrightarrow{CA} \times \mathbf{T} \cdot \widehat{CD}$. This yields

$$M_{CD} = \begin{vmatrix} 0.5193 & 0.8 & 0.3 \\ 0.45\sqrt{3} & 0 & 0.45 \\ -51.93 & 80 & 30 \end{vmatrix} = -37.392 \text{ Nm}$$

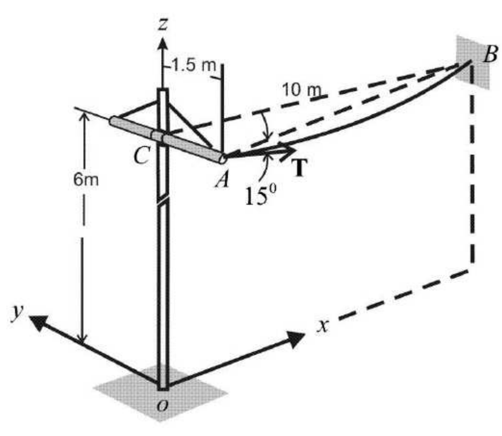

Figure 11: Problem 3

Problem 3

A cable joins two points A and B at the same height from the ground. Given that T= 800 N, represent the cable tension vector in the coordinate system shown. Also, calculate the bending moment about the x-axis and the torsional moment about the z-axis due to the cable tension.

Solution

Representation of cable tension:

$$\mathbf{T} = 800(\cos 15^\circ \widehat{AB} - \sin 15^\circ \hat{k})\ \mathrm{N}.$$

Now,

$$\overrightarrow{AB} = \overrightarrow{CB} - \overrightarrow{CA}$$

where

$$\overrightarrow{CB} = 10\hat{\imath}\ \mathrm{m} \qquad \overrightarrow{CA} = -1.5\ \hat{\jmath}\ \mathrm{m}.$$

Thus, $\overrightarrow{AB} = 10\hat{\imath}+1.5\hat{\jmath}$ m, and $\widehat{AB} = \frac{1}{\sqrt{102.25}}(10\hat{\imath}+1.5\hat{\jmath})$. Using this expression, we obtain

$$\mathbf{T} = 764\hat{\imath} + 114.6\hat{\jmath} - 207\hat{k}\ \mathrm{N}.$$

Moment of $\mathbf{T}$ about the x-axis:

$$M_x = \overrightarrow{OA} \times \mathbf{T} \cdot \hat{\imath}$$

where $\overrightarrow{OA} = 6\hat{k} - 1.5\hat{\jmath}$ m. Thus, the bending moment about the x-axis

$$M_x = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1.5 & 6 \\ 746 & 114.6 & -207 \end{vmatrix} = 998\ \mathrm{Nm}$$

Similarly, the torsional moment about the z-axis

$$M_z = \overrightarrow{OA} \times \mathbf{T} \cdot \hat{k} = 1146.3 \text{ Nm}.$$