

Module- 2 : Fundamentals of Fluid Flow and Properties

Lecture – 5 :

Capillary Fluid (Incompressible) Flow & Hydrodynamic Lubrication

In this section we shall look into capillary flow and hydrodynamic lubrication relevant to fluid drive and oil hydraulics.

Capillary flow:

Let a flat plate is moving on another fixed flat plate (shoe) as shown in Fig. 2.5_1 (a) maintaining a parallel small gap (capillary) of h . We assume that plates are flat with good surface finish and the hydraulic fluids have very good adherence to the materials of the plates. The oil inside this capillary passage will develop a 'velocity gradient' of 0 to u . However, it will remain uniform unless the gap diverges or converges along the flow direction (Fig. 2.5_1 (b)). The oil is incompressible and therefore, for continuity in flow area of oab remains constant.

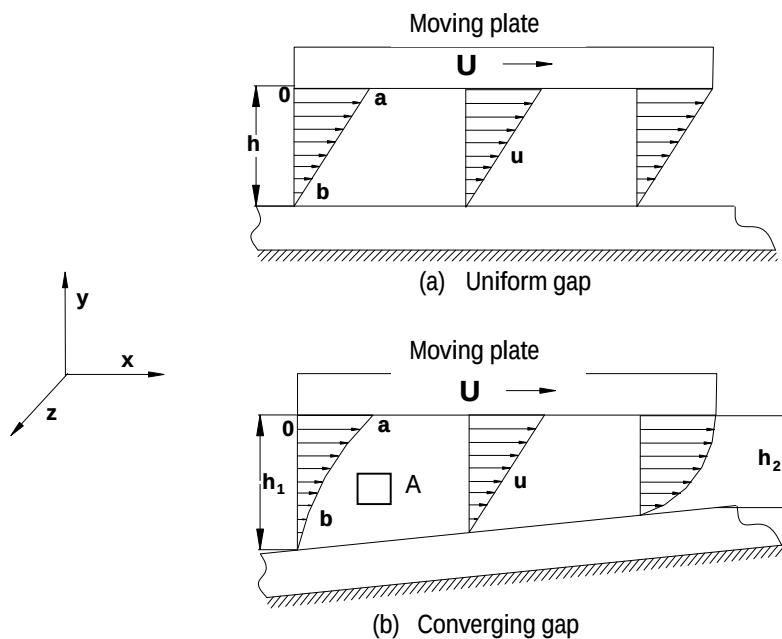


Fig. 2.5_1 : Variation of fluid velocity across film.

Consider an element of fluid as shown in Fig. 2.5_2 ,

For force equilibrium in x direction,

$$\left(\tau + \frac{\partial \tau}{\partial y} \delta y \right) \delta x \delta z - (\tau) \delta x \delta z = \left(p + \frac{\partial p}{\partial x} \delta x \right) \delta y \delta z - p \delta y \delta z \quad \dots \text{ 2.5}_1$$

The above expression reduces to:

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \quad \dots \text{ 2.5}_2$$

This means that change in shear stress in y direction is equal to the change of pressure in the x direction.

$$\text{Now, } \tau = \mu \frac{\partial u}{\partial y} \quad \dots \text{ 2.5}_3$$

[Unit verification: $N / m^2 = Pa \text{ Sec} \frac{m / \text{Sec}}{m}$]

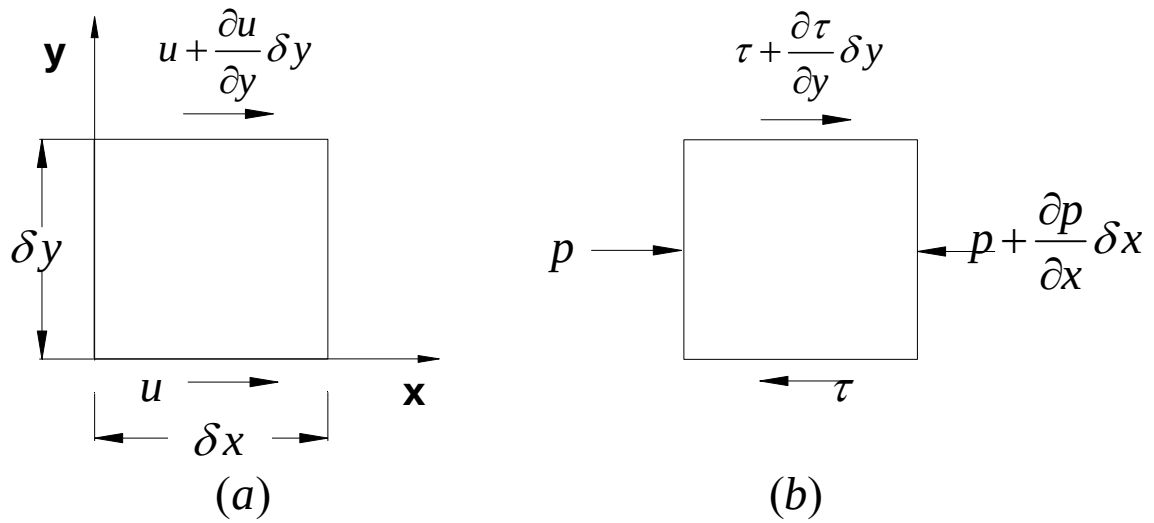


Fig. 2.5_2 : Element of fluid.

Differentiating wrt y,

$$\therefore \frac{\partial \tau}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \quad \dots \text{ 2.5}_4$$

Substituting eqn, ... 2.5_2

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$

... 2.5_5

It is the reduced form of **Navier-Stokes equation**.

Clearly if $\frac{\partial u}{\partial y}$ is constant in case of flow between two parallel plates (Fig. 2.5-1a). then, $\frac{\partial p}{\partial x} = 0$.

Hence, no pressure will be built up. Therefore, wedge (Fig. 2.5-1b). is essential for varying $\frac{\partial u}{\partial y}$ i.e.,

to generate pressure gradient and by thus the lubrication fluid film. In case of converging gap (Fig. 2.5-3) in the direction of motion an upward thrust will develop due to build up pressure. Thus the load carrying capacity of the plate will increase- which is the basic condition of hydrodynamic lubrication.

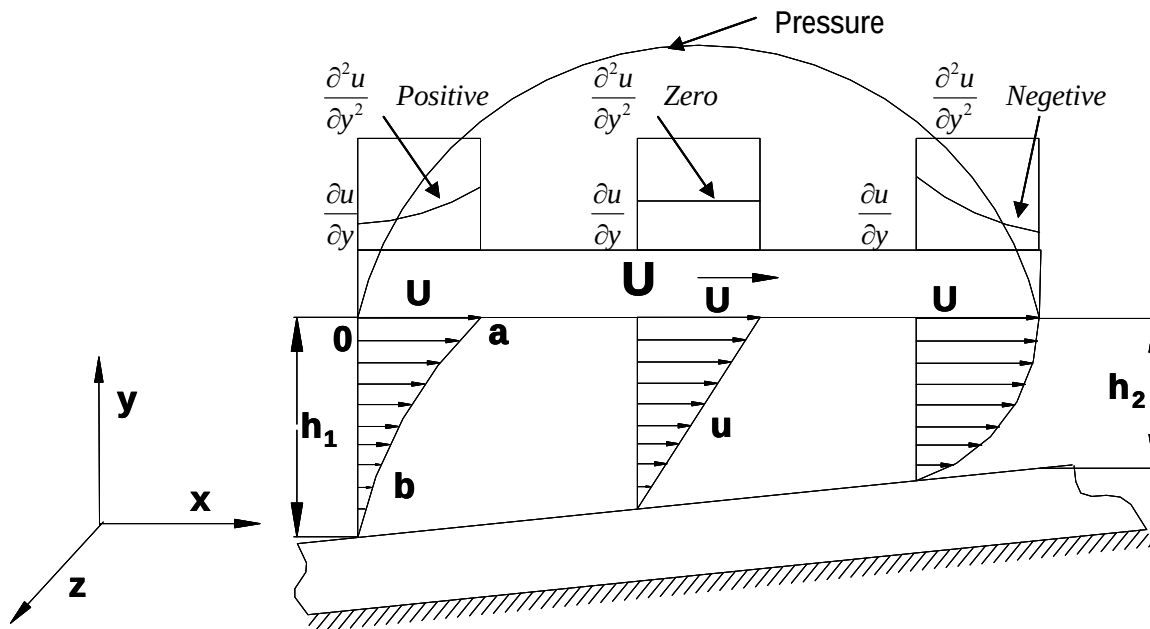


Fig. 2.5_3 : Pressure distribution in converging film.

The theory is applied for capillary flow as well as to develop hydrodynamic lubrication theory in sliding contact bearings which include the slipper pad and the journal bearing lubrication.

Leakage flow through two parallel plates:

In case of many fluid power machines and devices a parallel surface moves with respect to another fixed surface where one side of this capillary passage is exposed to high pressure fluid. For an example side plate of a gear pump, valve cum cover plate in piston pumps etc. Flow in such

capillary passages occur both due to the hydrostatic and the hydrodynamic pressure gradients, although in many cases hydrostatic pressure dominates.

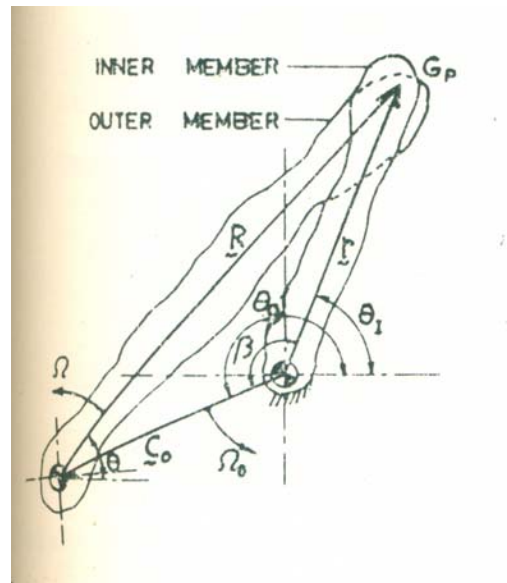
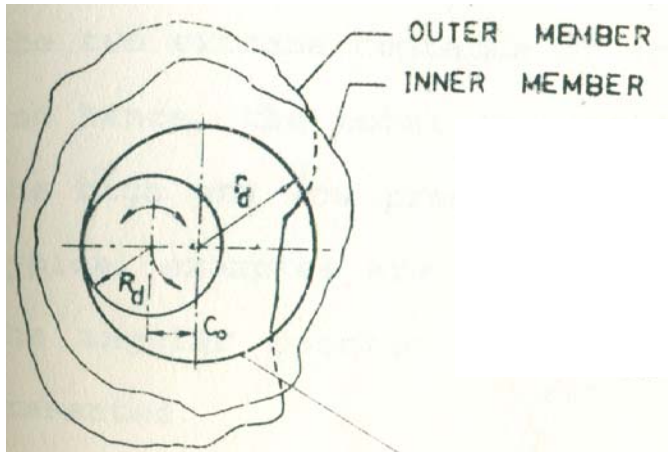
Referring to Fig. 2.5_4 the equation for pressure distribution can be developed as follows.

The equation 2.5_5 can be rearranged and written for two dimension flow for r, θ directions as,

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial r} \quad \dots 2.5_6$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu r} \frac{\partial p}{\partial \theta} \quad \dots 2.5_7$$

Where, u = fluid velocity in r direction, &
 v = fluid velocity in θ direction.



(a) (b)
 Fig. 2.5_4 : Motion of outer plate over inner plate.

From the bulk continuity,

$$\int_0^h \frac{\partial}{\partial r}(ru) \delta z + \int_0^h \frac{\partial u}{\partial \theta} \delta z = 0 \quad \dots 2.5_8$$

Integrating,

$$u = \frac{1}{2\mu} \cdot \left(\frac{\partial p}{\partial r} \right) \cdot z^2 + k_1 z + k_2 \quad \dots 2.5_9$$

$$v = \frac{1}{2\mu r} \cdot \left(\frac{\partial p}{\partial \theta} \right) \cdot z^2 + k_3 z + k_4 \quad \dots 2.5_10$$

Where, k_1 , k_2 , k_3 , & k_4 are integration constants.

Referring to [Fig.2.5-4](#)

Now the boundary conditions are, at $z = 0$, $u = 0$ $v = 0$ (i)

at $z = h$, the sliding velocities of an element in radial and tangential directions are derived as,

$$\begin{aligned} u &= (\Omega - \Omega_o) C_o \sin \beta \\ v &= (\Omega - \Omega_o) C_o \cos \beta - \Omega r \end{aligned} \quad \text{(ii)}$$

Substituting these boundary conditions in eqn. ... 2.5_9,

$$\therefore k_2 = 0$$

$$k_1 = \frac{1}{h} (\Omega - \Omega_o) C_o \sin \beta - \frac{1}{2\mu} \cdot \left(\frac{\partial p}{\partial r} \right) \cdot \frac{h^2}{h}$$

$$\text{Therefore, } u = \frac{1}{2\mu} \cdot \left(\frac{\partial p}{\partial r} \right) (z^2 - hz) + \frac{z}{h} (\Omega - \Omega_o) C_o \sin \beta \quad \dots$$

2.5_11

Similarly, from the boundary conditions and eqn. 2.5_10,

$$v = \frac{1}{2\mu r} \cdot \left(\frac{\partial p}{\partial \theta} \right) (z^2 - hz) + \frac{z}{h} \{ (\Omega - \Omega_o) C_o \cos \beta - \Omega r \} \quad \dots 2.5_12$$

Substituting in the continuity equation 2.5_8,

$$\begin{aligned} & \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \int_0^h (z^2 - hz) dz + \frac{\partial}{\partial r} (r) \frac{C_o}{h} (\Omega - \Omega_o) \sin \beta \int_0^h z dz + \frac{\partial}{\partial \theta} \left(\frac{\partial p}{\partial \theta} \right) \frac{1}{2\mu r} \int_0^h (z^2 - hz) dz \\ & + \frac{\partial}{\partial \theta} \frac{1}{h} \{ (\Omega - \Omega_o) C_o \cos \beta - \Omega r \} \int_0^h z dz = 0 \end{aligned} \quad \dots 2.5_{-13}$$

$$\begin{aligned} \text{Or, } & \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \left[\frac{z^3}{3} - \frac{hz^2}{2} \right]_0^h + \frac{C_o}{h} (\Omega - \Omega_o) \sin \beta \left[\frac{z^2}{2} \right]_0^h + \\ & \left(\frac{\partial^2 p}{\partial \theta^2} \right) \frac{1}{2\mu r} \left[\frac{z^3}{3} - \frac{hz^2}{2} \right]_0^h - \frac{1}{h} \frac{\partial}{\partial \theta} \{ (\Omega - \Omega_o) \cos \beta - \Omega r \} \left[\frac{z^2}{2} \right]_0^h = 0 \end{aligned}$$

As $\beta = \theta_t + \text{Constant}$ vide Fig.- 2.5_4 (b), and $\frac{\partial}{\partial \theta} (\Omega r) = 0$, therefore the above expression is reduced to,

$$-\frac{h^3}{12\mu} r \frac{\partial^2 p}{\partial r^2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial r} - \frac{h^3}{12\mu r} \frac{\partial^2 p}{\partial \theta^2} + \frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta - \frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta = 0$$

$$\therefore r \frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2} = 0 \quad \dots 2.5_{-14}$$

$$[\text{Also can be written as } \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2} = 0]$$

The final equation of pressure distribution is expressed as,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = 0 \quad \dots 2.5_{-15}$$

It is independent of the velocity of moving plate. It is solved (mostly numerically) knowing the pressure distribution at boundaries.

The Hydrodynamic Theory - Mechanism of Fluid Lubrication:

We shall now consider the gradually varying capillary gap in the direction of motion. More specifically the converging passage in the direction of the motion of the body over the other, which have constructed the capillary passage, is considered. The most common example is the rotating journal in fixed bush (Fig. 2.5_6).

In a capillary passage the one dimension flow equation can be written as (ref eqn. 2.5_5),

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \cdot \frac{\partial p}{\partial x}$$

Integrating twice wrt y,

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{\mu} \cdot \left(\frac{\partial p}{\partial x} \right) \cdot y + k_1$$

Finally,

$$u = \frac{1}{2\mu} \cdot \left(\frac{\partial p}{\partial x} \right) \cdot y^2 + k_1 y + k_2 \quad \dots 2.5_16$$

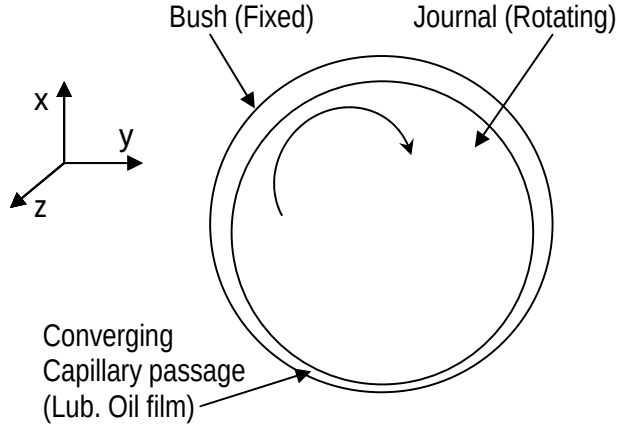


Fig.- 2.5_6 : Development of Converging capillary passage & (Lubrication oil film in Journal & Bush)

where, k_1 and k_2 are integration constant.

Applying boundary conditions,

At $y = 0$, $u = 0$... (i)

& at $y = h$, $u = U$... (ii)

Also, considering only one dimensional flow and using 1st. boundary condition in eqn. 2.5_16, we get, $k_2 = 0$,

Substituting again in eqn. ... 2.5_16

$$\therefore U = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) h^2 + k_1 h$$

Using 2nd. boundary condition

$$\therefore k_1 = \frac{U}{h} - \frac{h}{2\mu} \left(\frac{dp}{dx} \right)$$

$$u = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) y^2 + \left\{ \frac{U}{h} - \frac{h}{2\mu} \left(\frac{dp}{dx} \right) \right\} y \quad \dots 2.5_17$$

The eqn. 2.5_17 can be rewritten as:

$$u = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + \frac{Uy}{h} \quad \dots 2.5_18$$

The first part of the above equation presents pressure induced flow i.e., flow due to pressure difference and the second part is drag flow.

Now applying the continuity in flow (assuming no breakage in film), for an unit length in z direction, at any section,-

$$Q = \int_0^h u \, dy \quad \dots 2.5_19$$

Substituting the expression of u from eqn. ... 2.5_18 ,

$$\begin{aligned} Q &= \int_0^h \left[\frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + \frac{Uy}{h} \right] dy \quad \dots 2.5_20 \\ &= \frac{1}{2\mu} \frac{dp}{dx} \left[\frac{y^3}{3} - \frac{hy^2}{2} \right]_0^h + \frac{Uy^2}{2h} \Big|_0^h \\ &= -\frac{1}{12\mu} \frac{dp}{dx} h^3 + \frac{Uh}{2} \end{aligned}$$

Finally,

$$\therefore Q = \frac{Uh}{2} - \frac{h^3}{12\mu} \frac{dp}{dx} \quad \dots 2.5_21$$

Now for an incompressible fluid (say mineral oil) Q is constant. Therefore,

$$\frac{dQ}{dx} = 0 \quad \dots 2.5_22$$

Substituting eqn. 2.5_22 in eqn. 2.5_21 and then differentiating wrt x we get,

$$\frac{U}{2} \frac{dh}{dx} = \frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) \quad \dots 2.5_23$$

Now, considering flow in z direction (leakage flow) also (i.e., two dimensional flow in a capillary passage). It is to be noted that flow in y direction is negligibly small in such cases and ignored in this derivation.

$$\frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) + \frac{d}{dz} \left(\frac{h^3}{12\mu} \frac{dp}{dz} \right) = \frac{U}{2} \frac{dh}{dx} \quad \dots 2.5_{24}$$

It is Reynolds' differential equation for two dimensional flow with pressure gradient in a converging film (in the direction of motion) and considering end leakages.

No general solution for pressure p exists. Case to case it is solved applying boundary conditions.

Application in journal bearing design:

Journal bearing design using 'Sommerfeld number'

A solution of Reynolds equation (No. 2.5₂₄) with side leakage assumed to be zero was made by Sommerfeld. According to the solution:

$$\frac{R}{C} f = \phi \left[\left(\frac{R}{C} \right)^2 \frac{\mu n}{p} \right] = \phi S \quad \dots 2.5_{25}$$

In which,

$$S = \left(\frac{R}{C} \right)^2 \frac{\mu n}{p} \quad \dots 2.5_{26}$$

a dimensionless quantity, known as '**Sommerfeld Number**'.

[Note: It is dimensionless. $\left(\frac{m}{m} \right)^2 \frac{(N / m^2 \text{ sec}) \times (rev / \text{sec})}{N / m^2}$]

Where, S = Sommerfeld Number or bearing characteristics number,

R=Journal radius (m),

C=Radial clearance [(Exact diameter of bush-exact diameter of journal)/2] (m),

μ=Viscosity (Pa-s),

n=Rotating speed of journal (rps),

p=Average pressure on projected bearing area

[Load on a bearing/(length of bush x diameter of bush) = F/(LxD) Re. Fig. 2.5₅].

φ = A function,

f= Coefficient of Friction.

Raimondi and Boyd method of bearing design.

For solutions ***Raimondi and Boyd*** considered practical and particular cases. They considered both full (360°) and partial (180°) bearings, and infinite (L/D equal to 4 and above) to short (L/D below 1/4) bearings. Also, μ considered to be constant. They used computer techniques to translate the results of the *hydrodynamic* equation to various operating characteristics for various L/D ratios against a range of Sommerfeld numbers. Interpolation techniques are to be used for intermediate L/D ratios. They prepared charts (graphs) for various bearing parameters and characteristics. Those are adequate to select / design a journal bearing for general purpose applications.

Referring to the bearing nomenclatures as in **Fig. 2.5_6** those charts are listed below.

Graph/Chart: [1]

- | | |
|---|---|
| i) S vs $\frac{R}{C} f$ (Dimensionless) | vi) S vs $\frac{J\gamma C_H \Delta t}{p}$ (Dimensionless) |
| ii) S vs $\frac{h_o}{C}$ (Dimensionless) | vii) S vs $\frac{p}{p_{\max}}$ (Dimensionless) |
| iii) S vs θ_{h_o} (deg) | viii) S vs $\theta_{p_{\max}}$ (deg) |
| iv) S vs $\frac{Q}{RCnL}$ (Dimensionless) | ix) S vs θ_{p_o} (deg) |
| v) S vs $\frac{Q_s}{Q}$ (Dimensionless) | |

[Note: Only two charts- (ii) and (vi) are shown in this lecture note. For others ref. 1 is to be followed. Also, for further knowledge of bearing design any machine design book and bearing design book may be consulted.]

Bearing Nomenclature:

h_o = Minimum film thickness,

e = Journal displacement of eccentricity,

$\varepsilon = e / C$ = Eccentricity ratio,

J = Mechanical equivalent of heat (1 Nm/Joule)

γ = Density of oil = 863 kg / m³,

C_H = Specific heat of oil = 1755 Joule / kg / °C ,

Δt = Temperature rise in °C .

Q = Total flow in oil film, x direction.

Q_s = Side leakage flow from oil film, z direction.

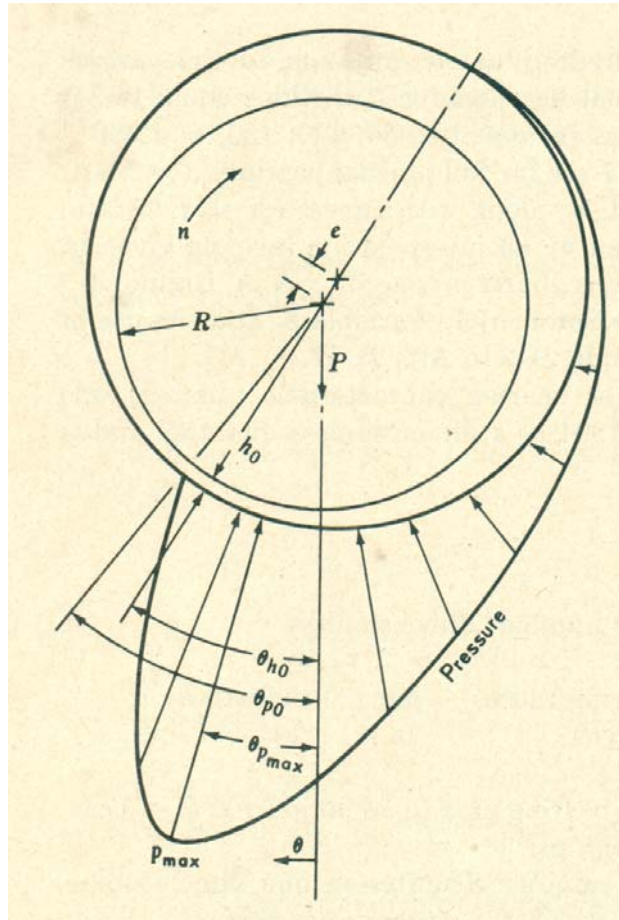


Fig. 2.5_6 : Journal Bearing- Nomenclature and pressure distribution.

A typical design problem:

In a gear pump (ref. Fig. 2.5_7) each journal bearing has the following specifications:

$F = 2600 \text{ N}$, $R = 30 \text{ mm}$, Clearance ratio, i.e., $C / R = 0.001$, $L = 84 \text{ mm}$, $\mu = 0.015 \text{ Pas} - \text{sec}$ and $n = 30 \text{ rps}$.

Examine whether the bearing will perform with mineral based hydraulic oil satisfactorily or not.

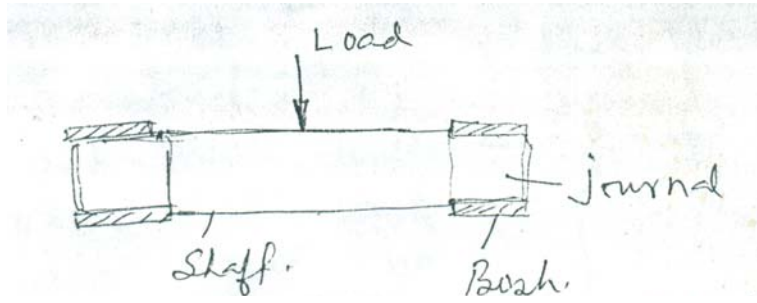


Fig. 2.5_7 : Journal Bearings- in a gear pump.

Soln.:

$$p = \frac{F}{LD} = \frac{2600}{0.084 \times 0.060} = 0.516 \times 10^6 \text{ Pas}$$

$$\frac{L}{D} = \frac{84}{60} = 1.4$$

$$S = \left(\frac{R}{C} \right)^2 \frac{\mu n}{p} = (0.001)^2 \times \frac{0.015 \times 30}{0.516 \times 10^6} = 0.87$$

From Chart (ii) as indicate above and Fig. 2.5_8 , it is found that $\frac{h_o}{C} = 0.9$.

$$\therefore C = 0.001 \times 30 = 0.03 \text{ mm}$$

$$\text{And, } \therefore h_o = 0.9 \times 0.03 = 0.027 \text{ mm}$$

[Note: Acceptable range of h_o is $2 \times 10^{-6} \text{ mm}$ to $8 \times 10^{-6} \text{ mm}$ per 2.5 mm of journal diameter.]

Similarly from charts other parameters are as follows:

$$\theta_{h_o} = 75^\circ, \frac{p}{p_{\max}} = 0.57, \theta_{p_{\max}} \cong 4^\circ$$

$$\text{And from Fig. 2.5_9, } \frac{J\gamma C_H \Delta t}{p} \cong 70$$

The temperature rise can be calculated as:

$$\Delta t = 70 \frac{p}{J\gamma C_H} = \frac{70 \times 0.516 \times 10^6}{1 \times 863 \times 1755} = 23.84^\circ$$

J = Mechanical equivalent of heat (1 Nm/Joule)

γ = Density of oil = 863 kg / m^3 ,

C_H = Specific heat of oil = $1755 \text{ Joule / kg / } ^\circ \text{C}$,

Δt = Temperature rise in $^\circ \text{C}$.

It is to be noted that actual temperature rise will be different as the same well is also the fluid being pumped.

However, the calculated data it is concluded that the design is satisfactory.

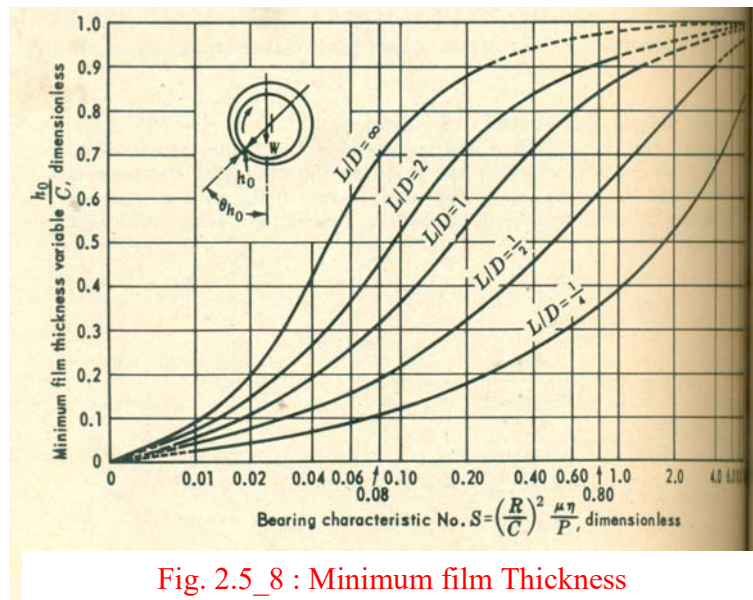


Fig. 2.5_8 : Minimum film Thickness

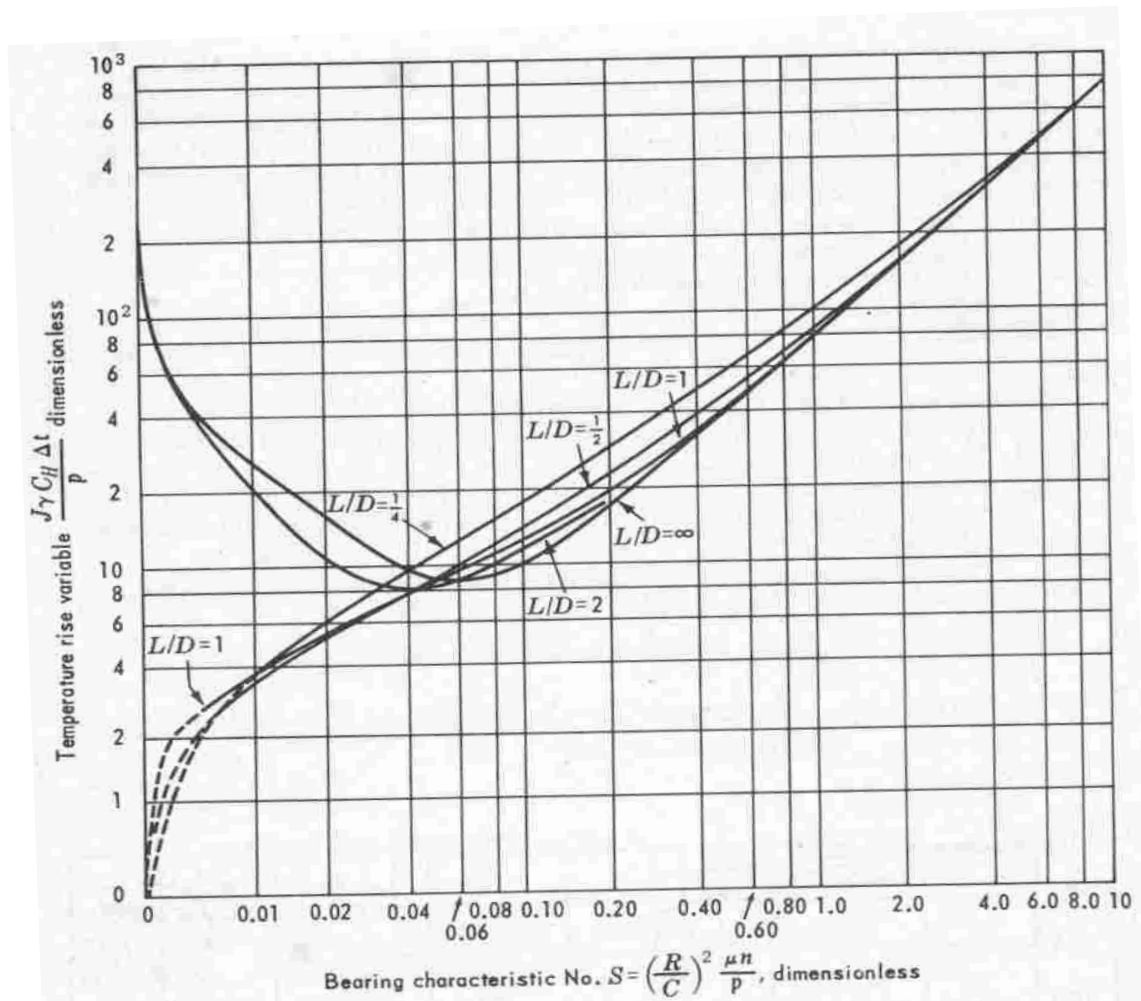


Fig. 2.5_9 : Film temperature rise (Ref. 1).

In this section flow rate (Q), velocity (u or v) and pressure (p) equations are presented in terms of the passage geometry such as diameter (d), length (L), capillary gap (h) etc., viscosity (μ) are presented for incompressible fluids.

Some standard equations are as follows.

1) *Steady flow through circular pipes* (ref. Fig. 2.5_10)

$$\frac{dp}{dx} = \frac{128\mu Q}{\pi d^4} = \frac{32\mu u_{av}}{d^2}$$

$$u_{av} = \frac{4Q}{\pi d^2}$$

$$Q = \frac{\pi d^4}{128\mu L} (p_u - p_d)$$

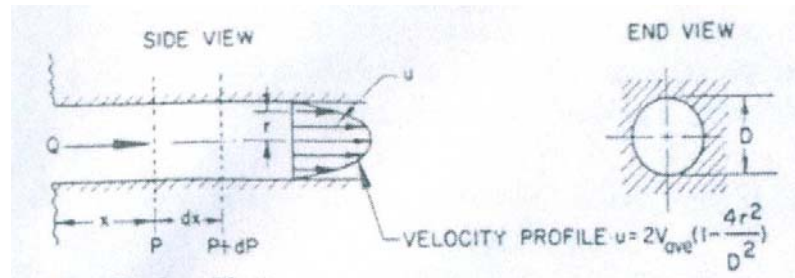


Fig. 2.5_10 : Steady flow through circular pipes

2) *Steady flow between stationary flat plates* (ref. Fig. 2.5_11)

Pressure drop in x direction:

$$\frac{dp}{dx} = \frac{12\mu Q}{bh^3}$$

$$Q = -\frac{bh^3}{12\mu L} (p_u - p_d)$$

$$Q = V_{ave} \times bh$$

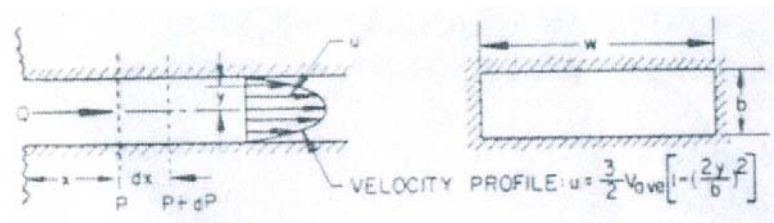


Fig. 2.5_11 : Steady flow between stationary flat plates

$$\therefore \frac{Q}{bh^3} = \frac{V_{ave}bh}{bh^3}$$

$$V_{ave} = \frac{3}{2} V_{ave} \left[1 - \left(\frac{2y}{b} \right)^2 \right]$$

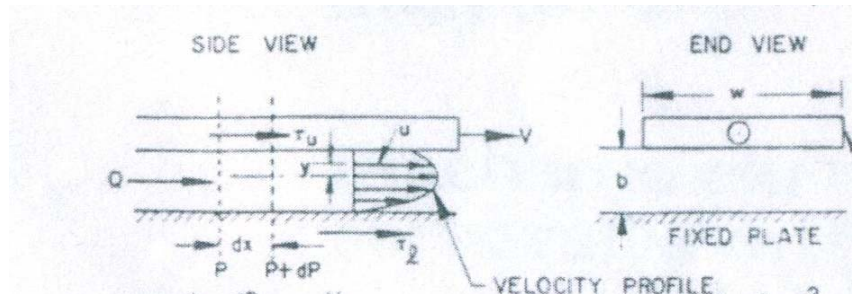
3) **Steady flow between stationary and moving flat parallel plates.** (Ref. Fig. 2.5_12)

$$Q = \left[-\frac{bh^3}{12\mu} \frac{dp}{dx} + \frac{v h b}{2} \right] \quad b \gg h$$

$$\frac{dp}{dx} = \frac{p_u - p_d}{L}$$

$$\tau_u = -\frac{h}{2} \frac{dp}{dx} - \mu \frac{U}{h}$$

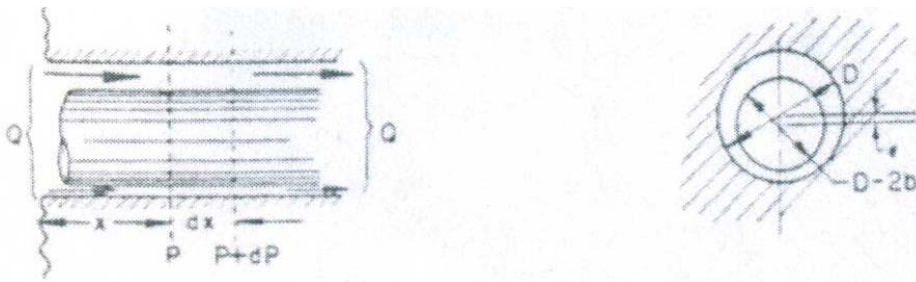
$$\tau_l = -\frac{h}{2} \frac{dp}{dx} + \mu \frac{U}{h}$$



$$u = \left(\frac{y}{h} + \frac{1}{2} \right) U + \frac{1}{2\mu} \frac{dp}{dx} \left(y^2 - \frac{h^2}{4} \right)$$

Fig. 2.5_12 : Steady flow between stationary and moving flat parallel plates

4) **Steady flow in annulus between circular shaft and cylinder.** (Ref. Fig. 2.5_13)



$$Q = \frac{\pi D b^3 \left[1 + 1.5 \left(\frac{e}{b} \right)^3 \right] (p_u - p_d)}{12\mu L}$$

$$\frac{dp}{dx} = \frac{12\mu Q}{\pi D b^3 \left[1 + 1.5 \left(\frac{e}{b} \right)^3 \right]}$$

Fig. 2.5_13 : Steady flow in annulus between circular shaft and cylinder.

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