
Module-5: Constitutive Relations

Lecture-37: Linear Elasticity

In this lecture, we discuss the theory of linear elasticity. Though there are strong assumptions in this theory, due to simplicity, it is widely used for predicting the behaviour of solids.

Kinematics:

We want to have a strain measure that is linear function of displacements. In order to obtain such a linear strain, we assume that the displacement gradient $\nabla_X \mathbf{u}$ is small. Therefore, the quadratic term that appears in the Green strain tensor (see Eq. (6) in Lecture-20) can be neglected. Consequently, we have a linear strain measure

$$\boldsymbol{\epsilon} = \frac{1}{2} \left(\nabla_X \mathbf{u} + (\nabla_X \mathbf{u})^T \right).$$

This linear strain is also known as small strain tensor. In addition to small displacement gradients, if the displacements are assumed to be small then the linearization of Almansi strain tensor also coincides with the small strain tensor. In other words, if displacement gradients and displacements are small then we have the linear strain

$$\boldsymbol{\epsilon} = \frac{1}{2} \left(\nabla_X \mathbf{u} + (\nabla_X \mathbf{u})^T \right) \approx \frac{1}{2} \left(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T \right)$$

We note that there is no difference between material gradient and spatial gradient in linear elasticity. Therefore, we can write the small strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right). \quad (1)$$

Characterization of rigid motion under small strain measure:

The displacement field $\mathbf{u}$ represents (infinitesimal) rigid motion (i.e., under the assumption of displacement gradients are small) if and only if

$$\mathbf{u} = \mathbf{W} \mathbf{X} + \mathbf{c}, \quad (2)$$

where $\mathbf{W}$ is constant skewsymmetric tensor and $\mathbf{c}$ is constant vector. An alternative form of rigid displacements can be represented by

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{X} + \mathbf{c}, \quad (3)$$

where $\boldsymbol{\omega}$ is the axial vector of $\mathbf{W}$.

We now show that Eq. (3) represents the rigid motion with the assumptions of small displacements and displacement gradients. Recall the following equation for rigid motion from Lecture-18.

$$\mathbf{x} = \mathbf{Q}(t) \mathbf{X} + \mathbf{c}(t). \quad (4)$$

We note that $\mathbf{Q}(0) = \mathbf{I}$. Using Taylor's expansion of $\mathbf{Q}(t)$ about $t = 0$, we get

$$\mathbf{Q}(t) = \mathbf{Q}(0) + \dot{\mathbf{Q}}|_{t=0}t + o(t).$$

It is easy to see $\dot{\mathbf{Q}}$ is skewsymmetric tensor as $\dot{\mathbf{Q}}\mathbf{Q}^T$ is skewsymmetric and $\mathbf{Q}(0) = \mathbf{I}$. Let us denote $\hat{\mathbf{W}} \equiv \dot{\mathbf{Q}}|_{t=0}$. Then $\mathbf{Q}(t)$ can be written as

$$\mathbf{Q}(t) = \mathbf{I} + \hat{\mathbf{W}}t + o(t). \quad (5)$$

Let $\mathbf{W} = \hat{\mathbf{W}}t$. Then for small t we have the following relation from Eqs. (4) and (5).

$$\mathbf{x} - \mathbf{X} = \mathbf{W}\mathbf{X} + \mathbf{c}. \quad (6)$$

Since displacement field $\mathbf{u} = \mathbf{x} - \mathbf{X}$, we get the result stated in Eq. (2). It is easy to see that $\mathbf{W}$ and $\mathbf{c}$ can be function of $\mathbf{X}$ but they are not function of time t .

As defined in Eq. (1), six components of strain tensor depends on three components of displacements. Hence, a unique strain field can be obtained for a give displacement field. But for a given strain components, the displacement field is not necessarily unique and compatible. In order to ensure a single valued displacement field, the given strain field should follow compatibility conditions discussed below.

Compatibility conditions:

The strain field ϵ should satisfy the following condition known as compatibility equations to ensure single valued displacement field.

$$\nabla \times (\nabla \times \epsilon) = \mathbf{O}. \quad (7)$$

In indicial notation

$$(\nabla \times (\nabla \times \epsilon))_{ij} = \epsilon_{imn}\epsilon_{jpq}\frac{\partial^2 \epsilon_{nq}}{\partial X_m \partial X_p} = 0. \quad (8)$$

Clearly, there are six equations as the strain tensor is symmetric. We now show that every displacement field satisfies the compatibility conditions.

Let $\mathbf{u}$ be a displacement field. Then the strain field is defined by

$$\epsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

Taking curl of curl of strain field, we get

$$\begin{aligned} \nabla \times (\nabla \times \epsilon) &= \frac{1}{2} \nabla \times (\nabla \times (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)) \\ &= \frac{1}{2} \nabla \times (\nabla \times (\nabla \mathbf{u})) + \frac{1}{2} \nabla \times (\nabla \times (\nabla \mathbf{u})^T) \\ &= \frac{1}{2} \nabla \times (\nabla \times (\nabla \mathbf{u})^T) \quad (\text{Since } \nabla \times (\nabla \mathbf{u}) = \mathbf{O}) \\ &= \frac{1}{2} \nabla \times (\nabla (\nabla \times \mathbf{u})) \quad (\text{Since } \nabla \times (\nabla \mathbf{u})^T = \nabla (\nabla \times \mathbf{u})) \\ &= \mathbf{O} \quad (\text{Since } \nabla \times (\nabla \mathbf{v}) = \mathbf{O}, \quad \forall \mathbf{v} \in \mathcal{V}) \end{aligned}$$

Thus, for a given displacement field the compatibility conditions are automatically satisfied. On the other hand, for a give strain field the compatibility ensures the single valued displacement field.

We now state two important properties of curl of curl of strain tensor which will be used for simplifying the compatibility conditions.

Properties of curl of curl of strain field:

- The tensor $\nabla \times (\nabla \times \epsilon)$ is symmetric as the strain field is symmetric, i.e.,

$$(\nabla \times (\nabla \times \epsilon))^T = \nabla \times (\nabla \times \epsilon). \quad (9)$$

- It follows from previous proof that the trace of tensor $\nabla \times (\nabla \times \epsilon)$ is zero, i.e.,

$$\text{tr} (\nabla \times (\nabla \times \epsilon)) = 0, \quad (10)$$

or, alternatively,

$$(\nabla \times (\nabla \times \epsilon))_{ii} = 0. \quad (11)$$

Using $\epsilon - \delta$ identity (see Lecture-3) and Eq. (8), we get

$$\nabla^2(\text{tr } \epsilon) - \nabla \cdot (\nabla \cdot \epsilon) = 0. \quad (12)$$

Problem 1. The compatibility stated in Eq. (7), i.e., $\nabla \times (\nabla \times \epsilon) = \mathbf{O}$, can be alternatively expressed as

$$\nabla^2 \epsilon + \nabla (\nabla (\text{tr } \epsilon)) - \nabla (\nabla \cdot \epsilon) - (\nabla (\nabla \cdot \epsilon))^T = \mathbf{O}. \quad (13)$$

or, alternatively in indicial notation,

$$\epsilon_{ij,kk} + \epsilon_{kk,ij} - \epsilon_{ik,kj} - \epsilon_{jk,ki} = 0. \quad (14)$$

It can be noted that comma represents partial derivative.

Proof. We use indicial notation to prove the result. Consider the ij^{th} component of curl of curl of strain field

$$(\nabla \times (\nabla \times \epsilon))_{ij} = \epsilon_{imn} \epsilon_{jpk} \frac{\partial^2 \epsilon_{nq}}{\partial X_m \partial X_p} \quad (15)$$

Recall the following general $\epsilon - \delta$ relation from Lecture-3.

$$\begin{aligned} \epsilon_{imn} \epsilon_{jpk} &= \det \begin{bmatrix} \delta_{ij} & \delta_{ip} & \delta_{iq} \\ \delta_{mj} & \delta_{mp} & \delta_{mq} \\ \delta_{nj} & \delta_{np} & \delta_{nq} \end{bmatrix} \\ &= \delta_{ij}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}) - \delta_{ip}(\delta_{mj}\delta_{nq} - \delta_{mq}\delta_{nj}) + \delta_{iq}(\delta_{mj}\delta_{np} - \delta_{mp}\delta_{nj}). \end{aligned}$$

Upon substituting the general $\epsilon - \delta$ relation in Eq. (15), yields

$$\begin{aligned} (\nabla \times (\nabla \times \epsilon))_{ij} &= \left(\frac{\partial^2 \epsilon_{nn}}{\partial X_m \partial X_m} - \frac{\partial^2 \epsilon_{mn}}{\partial X_m \partial X_n} \right) \delta_{ij} - \left(\frac{\partial^2 \epsilon_{nn}}{\partial X_i \partial X_j} - \frac{\partial^2 \epsilon_{jm}}{\partial X_i \partial X_m} \right) \\ &\quad + \left(\frac{\partial^2 \epsilon_{ni}}{\partial X_j \partial X_n} - \frac{\partial^2 \epsilon_{ji}}{\partial X_m \partial X_m} \right) \\ &= \left(\nabla^2(\text{tr } \epsilon) - \nabla \cdot (\nabla \cdot \epsilon) \right) \delta_{ij} \\ &\quad + \left[-\nabla(\nabla(\text{tr } \epsilon)) + \nabla(\nabla \cdot \epsilon) + (\nabla(\nabla \cdot \epsilon))^T - \nabla^2 \epsilon \right]_{ij}. \end{aligned} \quad (16)$$

Substitution of Eq. (12) in above equation give rise to

$$\nabla \times (\nabla \times \boldsymbol{\epsilon}) = -\nabla^2 \boldsymbol{\epsilon} - \nabla(\nabla(\text{tr } \boldsymbol{\epsilon})) + \nabla(\nabla \cdot \boldsymbol{\epsilon}) + (\nabla(\nabla \cdot \boldsymbol{\epsilon}))^T \quad (17)$$

Since $\nabla \times (\nabla \times \boldsymbol{\epsilon}) = \mathbf{O}$, we get

$$\nabla^2 \boldsymbol{\epsilon} + \nabla(\nabla(\text{tr } \boldsymbol{\epsilon})) - \nabla(\nabla \cdot \boldsymbol{\epsilon}) - (\nabla(\nabla \cdot \boldsymbol{\epsilon}))^T = \mathbf{O}.$$

The component form can be written as

$$\frac{\partial^2 \epsilon_{ij}}{\partial X_k \partial X_k} + \frac{\partial^2 \epsilon_{kk}}{\partial X_j \partial X_j} - \frac{\partial^2 \epsilon_{ik}}{\partial X_j \partial X_k} - \frac{\partial^2 \epsilon_{jk}}{\partial X_i \partial X_k} = 0. \quad (18)$$

Alternatively, it can be represented by

$$\epsilon_{ij, kk} + \epsilon_{kk, ij} - \epsilon_{ik, kj} - \epsilon_{jk, ki} = 0.$$

This is a convenient component form of compatibility. □

The stress measure:

In linear elasticity, we assume that the displacement gradient is small and displacements are small. Therefore, all the stress measures, i.e., Cauchy stress $\boldsymbol{\tau}$, first Piola-Kirchhoff stress $\mathbf{T}$ and second Piola-Kirchhoff stress $\mathbf{S}$, are approximately equal. Recall the following relations between Cauchy stress and Piola-Kirchhoff stresses from Lecture-28.

$$\mathbf{T} = J\boldsymbol{\tau}\mathbf{F}^{-1} \quad \text{and} \quad \mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}, \quad (19)$$

where $\mathbf{F}$ and J are deformation gradient and its determinant, respectively. Since $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$ (see Lecture-17) and the displacement gradient $\nabla \mathbf{u}$ is small, $\mathbf{F} \approx \mathbf{I}$. Hence, the relations shown in Eq. (19) imply

$$\mathbf{T} \approx \boldsymbol{\tau} \quad \text{and} \quad \mathbf{S} \approx \boldsymbol{\tau}. \quad (20)$$

Thus, there is no difference between various stress measures in linear elasticity.

The governing equations:

From the above discussion, the Cauchy stress, first Piola-Kirchhoff stress, and second Piola-Kirchhoff stress are identical. Thus, we have the following governing equation for linear elasticity.

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \cdot \boldsymbol{\tau} + \rho_0 \mathbf{b}_0. \quad (21)$$

Constitutive relation:

We now use the linear constitutive relation that is similar to St venant-Kirchhoff hyperelastic model. Of course the material is also assumed to be isotropic. The only difference is that the small strain measure $\boldsymbol{\epsilon}$ is used in linear elasticity whereas Green strain is used as strain measure in St venant-Kirchhoff model. Therefore, the constitutive relation is given by

$$\boldsymbol{\tau} = \lambda(\text{tr } \boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon}, \quad (22)$$

where λ and μ are Lamé constants.

We will show the relation between engineering constants (i.e., Young's modulus E and Poisson's ratio ν) and Lamé constants using thought experiments. For isotropic material, the strains can also be represented in terms of stress tensor

$$\boldsymbol{\epsilon} = \frac{1 + \nu}{E} \boldsymbol{\tau} - \frac{\nu}{E} \text{tr}(\boldsymbol{\tau}) \mathbf{I}. \quad (23)$$

We now present relation between engineering constants (Young's modulus E and Poisson's ratio ν) and Lamé constants along with bounds on these constants.

Bounds on Lamé constants:

We conduct three thought experiments to find bounds on Lamé constants and corresponding Young's modulus and Poisson's ratio. We assume that the reference configuration is stress free (i.e., in natural state) and material is isotropic. Since there is no difference between Cauchy stress and second Piola-Kirchhoff stress in linear elasticity, we use the Cauchy stress as stress measure.

Experiment of simple shear:

The following equation of motion (or mapping) represents the simple shear (see Fig. 1)

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad \text{and} \quad x_3 = X_3.$$

Evaluating $\boldsymbol{\epsilon}$ and substituting in Eq. (22), we get

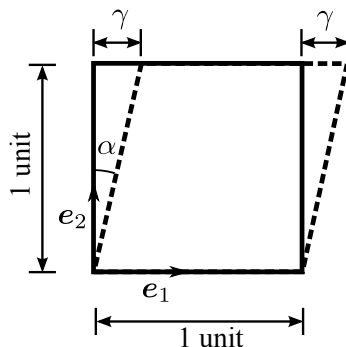


Figure 1: Simple shear

$$\tau_{12} = \mu \gamma.$$

Since $\tau_{12} > 0$ for $\gamma > 0$, we have

$$\mu > 0. \quad (24)$$

Experiment of uniform compression of a sphere:

The equation of motion for the uniform compression (hydrostatic compression) is given by

$$\mathbf{x} = (1 - \epsilon) \mathbf{X},$$

where the constant $\epsilon \in (0, 1)$.

The Cauchy stress tensor

$$\boldsymbol{\tau} = -\epsilon(3\lambda + 2\mu) \mathbf{I}.$$

Since the Cauchy stress is hydrostatic pressure, i.e., $\boldsymbol{\tau} = -p\mathbf{I}$, and ϵ and p are positive constants, we have

$$3\lambda + 2\mu > 0. \quad (25)$$

Experiment of uniform extension of a bar:

The following motion describes uniform extension (accounting Poisson's contraction).

$$x_1 = (1 + \epsilon)X_1, \quad x_2 = (1 - \nu\epsilon)X_2, \quad \text{and} \quad x_3 = (1 - \nu\epsilon)X_3.$$

Substitution of computed strain tensor $\boldsymbol{\epsilon}$ in Eq. (22) yields

$$\tau_{22} = \tau_{33} = \epsilon [\lambda(1 - 2\nu) - 2\mu\nu].$$

Since there are no lateral stresses, i.e., $\tau_{22} = \tau_{33} = 0$, we get

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad (26)$$

The stress along $\mathbf{e}_1$ direction on the $\mathbf{e}_1$ -plane is given by

$$\tau_{11} = \epsilon [\lambda(1 - 2\nu) + 2\mu].$$

Since the axial stress $\tau_{11} = E\epsilon$, we get

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}. \quad (27)$$

The engineering constants E and ν are known as *Young's modulus* and *Poisson's ratio*, respectively. In summary, we have

$$\begin{aligned} E &= \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \\ \lambda &= \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}. \end{aligned} \quad (28)$$

The inequalities shown in Eq. (24) and (25) are equivalent to $E > 0$ and $-1 < \nu < 0.5$, i.e.,

$$\mu > 0 \text{ and } 3\lambda + 2\mu > 0 \iff E > 0 \text{ and } -1 < \nu < 0.5.$$

Approach for solving linear elasticity problems:

In linear elasticity, our aim is to find the stress and displacement fields for a given traction and displacement boundary conditions. In order to find the stress and displacement fields, there are two distinct approaches called displacement approach and stress approach.

In displacement approach, the governing equation shown in Eq. (21) is expressed in terms of displacements known as *Navier equation of elasticity*. The solution is obtained by solving the Navier equation subjected to the given boundary conditions.

In stress approach, the compatibility conditions shown in Eq. (7) is expressed in terms of stress components known as *Beltrami-Michell compatibility conditions*. The stress field is guessed such that it satisfies the Beltrami-Michell compatibility conditions and the boundary conditions.

We now present both the Navier equation of elasticity and Beltrami-Michell compatibility conditions.

Navier equation of elasticity:

Substitution of the strain displacement relation, i.e., Eq. (1) in constitutive relation (i.e., in Eq. (22)), we get

$$\boldsymbol{\tau} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad (29)$$

Upon substituting Eq. (29) in Eq. (21), we get the following Navier equation of elasticity.

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho_0 \mathbf{b}, \quad (30)$$

where ρ_0 is density, λ and μ are Lamé constants.

Beltrami-Michell compatibility conditions:

In theory of elasticity problems, it is easy to guess stresses rather than solving for displacements. Therefore, it is necessary to verify the compatibility conditions. We now derive the compatibility conditions in terms of stress components known as Beltrami-Michell compatibility conditions.

We consider the static problem in linear elasticity. Hence, the governing equation Eq. (21) for static case reduces to

$$\nabla \cdot \boldsymbol{\tau} + \rho_0 \mathbf{b} = 0. \quad (31)$$

Substituting the constitutive relation shown in Eq. (23) in compatibility conditions (i.e., in Eq. (13)), we get

$$\left(\frac{1+\nu}{E} \nabla^2 \boldsymbol{\tau} - \frac{\nu}{E} \nabla^2 (\text{tr } \boldsymbol{\tau}) \mathbf{I} \right) + \left(\frac{1-2\nu}{E} \nabla (\nabla (\text{tr } \boldsymbol{\tau})) \right) - \left(\frac{1+\nu}{E} \nabla (\nabla \cdot \boldsymbol{\tau}) - \frac{\nu}{E} \nabla (\nabla (\text{tr } \boldsymbol{\tau})) \right) - \left(\frac{1+\nu}{E} [\nabla (\nabla \cdot \boldsymbol{\tau})]^T - \frac{\nu}{E} \nabla (\nabla (\text{tr } \boldsymbol{\tau})) \right) = 0.$$

Rearrangement of terms yields

$$(1+\nu) \nabla^2 \boldsymbol{\tau} + \nabla (\nabla (\text{tr } \boldsymbol{\tau})) = \nu \nabla^2 (\text{tr } \boldsymbol{\tau}) \mathbf{I} + (1+\nu) \left(\nabla (\nabla \cdot \boldsymbol{\tau}) + [\nabla (\nabla \cdot \boldsymbol{\tau})]^T \right).$$

Substituting the $\nabla \cdot \boldsymbol{\tau} = -\rho_0 \mathbf{b}$ from equilibrium equation (see Eq. (31)), we get

$$\nabla^2 \boldsymbol{\tau} + \frac{1}{1+\nu} \nabla (\nabla (\text{tr } \boldsymbol{\tau})) = \frac{\nu}{1+\nu} \nabla^2 (\text{tr } \boldsymbol{\tau}) \mathbf{I} - \rho_0 \left(\nabla \mathbf{b} + (\nabla \mathbf{b})^T \right).$$

Taking trace on both sides of equation, we obtain

$$\nabla^2 (\text{tr } \boldsymbol{\tau}) = -\frac{1+\nu}{1-\nu} \rho_0 (\nabla \cdot \mathbf{b}). \quad (32)$$

Substituting this relation in above equation we get the following Beltrami-Michell compatibility equation.

$$\nabla^2 \boldsymbol{\tau} + \frac{1}{1+\nu} \nabla (\nabla (\text{tr } \boldsymbol{\tau})) = -\frac{\nu}{1-\nu} \rho_0 (\nabla \cdot \mathbf{b}) \mathbf{I} - \rho_0 \nabla \mathbf{b} - \rho_0 (\nabla \mathbf{b})^T. \quad (33)$$

In any linear elasticity problem, the stress field should be guessed such that it satisfies the Beltrami-Michell compatibility condition, i.e., Eq. (33).

Reference

1. C. S. Jog, *Foundations and Applications of Mechanics: Continuum Mechanics*, Volume-I, 2007, Narosa Publishing House Pvt. Ltd., New Delhi.