Module-4: Balance Laws

Lecture-27: The Angular Momentum Balance and the Properties of Cauchy Stress

We now show the implication of angular momentum balance on stress tensor. Let $\mathscr{B}_0$ be a reference configuration and $\mathscr{B}$ be a deformed configuration at an instant of time t. Let Ω_0 be a part of $\mathscr{B}_0$ and Ω be a corresponding material volume in $\mathscr{B}$. Let $\boldsymbol{x}_0$ be a fixed point in the space. Then the principle of angular momentum balance states that the rate of change of angular momentum of material volume Ω is equal to the net moment of forces acting on Ω with respect to $\boldsymbol{x}_0$, i.e.,

$$\frac{d}{dt}\int_{\Omega} \rho(\boldsymbol{x}-\boldsymbol{x}_0)\times\boldsymbol{v}\,\partial\Omega = \int_{\Omega}\rho(\boldsymbol{x}-\boldsymbol{x}_0)\times\boldsymbol{b}\,\partial\Omega + \int_{\Gamma}(\boldsymbol{x}-\boldsymbol{x}_0)\times\boldsymbol{t}\,\partial\Gamma, \tag{1}$$

where Γ is boundary of Ω, ρ is density field, $\boldsymbol{v}$ is spatial velocity field, $\boldsymbol{b}$ is body force field per unit mass, and $\boldsymbol{t}$ is traction over the surface Γ. Since the continuum is assumed to be non-polar, i.e., there are no body and surface couples, the moment is generated by the body forces and traction.

We now choose the fixed point $\boldsymbol{x}_0$ to be origin. Then the above equation reduces to

$$\frac{d}{dt}\int_{\Omega}\rho\boldsymbol{x}\times\boldsymbol{v}\,\partial\Omega = \int_{\Omega}\rho\boldsymbol{x}\times\boldsymbol{b}\,\partial\Omega + \int_{\Gamma}\boldsymbol{x}\times\boldsymbol{t}\,\partial\Gamma. \tag{2}$$

The angular momentum balance and symmetry of Cauchy stress tensor:
If classical continuum (i.e., non-polar continuum) follows the linear momentum balance, the angular momentum balance is equivalent to symmetry of the Cauchy stress tensor, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T. \tag{3}$$

Equation (3) is known as second Cauchy's equation of motion.

Proof. Applying the transport theorem-II to Eq. (2), we get

$$\begin{aligned}
\int_{\Omega}\rho\frac{D(\boldsymbol{x}\times\boldsymbol{v})}{Dt}\,\partial\Omega &= \int_{\Omega}\rho\boldsymbol{x}\times\boldsymbol{b}\,\partial\Omega + \int_{\Gamma}\boldsymbol{x}\times\boldsymbol{t}\,\partial\Gamma \\
\implies \int_{\Omega}\rho\left(\frac{D\boldsymbol{x}}{Dt}\times\boldsymbol{v}+\boldsymbol{x}\times\frac{D\boldsymbol{v}}{Dt}\right)\partial\Omega &= \int_{\Omega}\rho\boldsymbol{x}\times\boldsymbol{b}\,\partial\Omega + \int_{\Gamma}\boldsymbol{x}\times\boldsymbol{t}\,\partial\Gamma \\
\implies \int_{\Omega}\rho\boldsymbol{x}\times\frac{D\boldsymbol{v}}{Dt}\,\partial\Omega &= \int_{\Omega}\rho\boldsymbol{x}\times\boldsymbol{b}\,\partial\Omega + \int_{\Gamma}\boldsymbol{x}\times\boldsymbol{t}\,\partial\Gamma \quad (\text{Since } \boldsymbol{v} = \frac{D\boldsymbol{x}}{Dt}).
\end{aligned}$$

Upon substituting $\rho\frac{D\boldsymbol{v}}{Dt} = \nabla_x\cdot\boldsymbol{\tau}+\rho\boldsymbol{b}$ and $\boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n}$, we can write

$$\int_{\Omega}\boldsymbol{x}\times(\nabla_x\cdot\boldsymbol{\tau})\,\partial\Omega = \int_{\Gamma}\boldsymbol{x}\times(\boldsymbol{\tau}\boldsymbol{n})\,\partial\Gamma.$$

Let $\boldsymbol{u}$ be a constant arbitrary vector. Taking dot product on both sides of equation, we obtain

$$\begin{aligned}\left(\int_\Omega \boldsymbol{x}\times(\nabla_x\cdot\boldsymbol{\tau})\,\partial\Omega\right)\cdot\boldsymbol{u} &= \left(\int_\Gamma \boldsymbol{x}\times(\boldsymbol{\tau}\boldsymbol{n})\,\partial\Gamma\right)\cdot\boldsymbol{u}\\ &= \int_\Gamma (\boldsymbol{x}\times(\boldsymbol{\tau}\boldsymbol{n}))\cdot\boldsymbol{u}\,\partial\Gamma.\end{aligned}$$

Using a property of scalar triple product from Lecture-3, we can have $[\boldsymbol{x},\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{u}] = [\boldsymbol{u},\boldsymbol{x},\boldsymbol{\tau}\boldsymbol{n}] = \boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\cdot\boldsymbol{n}$. Substituting this relation in above equation, yields

$$\left(\int_\Omega \boldsymbol{x}\times(\nabla_x\cdot\boldsymbol{\tau})\,\partial\Omega\right)\cdot\boldsymbol{u} = \int_\Gamma \boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\cdot\boldsymbol{n}\,\partial\Gamma.$$

Applying the divergence theorem, we get

$$\begin{aligned}\int_\Omega (\boldsymbol{x}\times(\nabla_x\cdot\boldsymbol{\tau}))\cdot\boldsymbol{u}\,\partial\Omega &= \int_\Omega \nabla_x\cdot\left(\boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\right)\,\partial\Omega\\ &= \int_\Omega \left(\boldsymbol{\tau}:\nabla_x(\boldsymbol{u}\times\boldsymbol{x}) + (\boldsymbol{u}\times\boldsymbol{x})\cdot(\nabla_x\cdot\boldsymbol{\tau})\right)\,\partial\Omega\\ &= \int_\Omega \boldsymbol{\tau}:\nabla_x(\boldsymbol{u}\times\boldsymbol{x})\,\partial\Omega + \int_\Omega (\boldsymbol{x}\times(\nabla_x\cdot\boldsymbol{\tau}))\cdot\boldsymbol{u}\,\partial\Omega.\end{aligned}$$

Cancelling the second term in right hand side, above relation reduces to

$$\int_\Omega \boldsymbol{\tau}:\nabla_x(\boldsymbol{u}\times\boldsymbol{x})\,\partial\Omega = 0$$

It is easy to verify that $\nabla_x(\boldsymbol{u}\times\boldsymbol{x})$ is a skewsymmetric tensor with $\boldsymbol{u}$ as its axis, i.e., $(\nabla_x(\boldsymbol{u}\times\boldsymbol{x}))_{ij} = -\epsilon_{ijk}u_k$. Arbitrariness of $\boldsymbol{u}$ implies the symmetry of stress tensor $\boldsymbol{\tau}$. Thus, the Cauchy stress tensor is symmetric for non-polar continuous medium.

Converse can be easily showed by reversing the above steps. Thus, the angular momentum balance is equivalent to symmetry of the Cauchy stress. □

We now derive the angular momentum balance for control volume.

The angular momentum balance for control volume:
Let Ω_c be a fixed control volume in space and Γ_c be its boundary. Let $\boldsymbol{n}$ be unit outward normal to the surface Γ_c. Let $\boldsymbol{\tau}$ be Cauchy stress tensor field. Let $\boldsymbol{x}$ be a position in control volume. Let $\boldsymbol{u}$ be an arbitrary constant vector. Then considering the surface integral $\int_{\Gamma_c}(\boldsymbol{x}\times\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{u}\,\partial\Gamma$ with following the relation

$$(\boldsymbol{x}\times\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{u} = (\boldsymbol{\tau}\boldsymbol{n})\cdot(\boldsymbol{u}\times\boldsymbol{x}) = \boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\cdot\boldsymbol{n},$$

we can write

$$\begin{aligned}\int_{\Gamma_c}(\boldsymbol{x}\times\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{u}\,\partial\Gamma &= \int_{\Gamma_c}\left(\boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\right)\cdot\boldsymbol{n}\,\partial\Gamma\\ &= \int_{\Omega_c}\nabla_x\cdot\left(\boldsymbol{\tau}^T(\boldsymbol{u}\times\boldsymbol{x})\right)\,\partial\Omega\\ &= \int_{\Omega_c}\left(\boldsymbol{\tau}:\nabla_x(\boldsymbol{u}\times\boldsymbol{x}) + (\boldsymbol{u}\times\boldsymbol{x})\cdot(\nabla_x\cdot\boldsymbol{\tau})\right)\,\partial\Omega\end{aligned}$$

Since Cauchy stress $\boldsymbol{\tau}$ is a symmetric tensor and $\nabla_x(\boldsymbol{u}\times\boldsymbol{x})$ is a skewsymmetric tensor, we have $\boldsymbol{\tau}:\nabla_x(\boldsymbol{u}\times\boldsymbol{x}) = 0$. Therefore, above equation reduces to

$$\begin{aligned}\int_{\Gamma_c}(\boldsymbol{x}\times\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{u}\,\partial\Gamma &= \int_{\Omega_c}(\boldsymbol{u}\times\boldsymbol{x})\cdot(\nabla_x\cdot\boldsymbol{\tau})\,\partial\Omega\\ &= \int_{\Omega_c}(\boldsymbol{x}\times\nabla_x\cdot\boldsymbol{\tau})\cdot\boldsymbol{u}\,\partial\Omega.\end{aligned}$$

Since $\boldsymbol{u}$ is an arbitrary constant vector, we have

$$\int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{\tau n}\, \partial\Gamma = \int_{\Omega_c} \boldsymbol{x} \times \nabla_x \cdot \boldsymbol{\tau}\, \partial\Omega.$$

Let $\boldsymbol{t}$ be a traction vector and $\boldsymbol{v}$ be a velocity field. Then the linear momentum balance implies

$$\begin{aligned}\int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma &= \int_{\Omega_c} \boldsymbol{x} \times \left(\rho \frac{D\boldsymbol{v}}{Dt} - \rho \boldsymbol{b}\right) \partial\Omega \\ &= \int_{\Omega_c} \rho \boldsymbol{x} \times \frac{D\boldsymbol{v}}{Dt}\, \partial\Omega - \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega.\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}\int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma + \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega &= \int_{\Omega_c} \rho \boldsymbol{x} \times \frac{D\boldsymbol{v}}{Dt}\, \partial\Omega \\ &= \int_{\Omega_c} \rho \frac{D}{Dt}(\boldsymbol{x} \times \boldsymbol{v})\, \partial\Omega \\ &= \int_{\Omega_c} \left(\frac{D}{Dt}(\rho \boldsymbol{x} \times \boldsymbol{v}) - \frac{D\rho}{Dt}(\boldsymbol{x} \times \boldsymbol{v})\right) \partial\Omega.\end{aligned}$$

Recall the relation between material and spatial time derivative from Lecture-20. The relation between material time derivative and spatial time derivative along with mass conservation $\dfrac{D\rho}{Dt} = \rho(\nabla_x \cdot \boldsymbol{v})$, give rise to

$$\begin{aligned}\int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma + \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega &= \int_{\Omega_c} \frac{\partial}{\partial t}(\rho \boldsymbol{x} \times \boldsymbol{v}) + \nabla_x(\rho \boldsymbol{x} \times \boldsymbol{v})\boldsymbol{v} + \rho(\nabla_x \cdot \boldsymbol{v})(\boldsymbol{x} \times \boldsymbol{v})\, \partial\Omega \\ &= \int_{\Omega_c} \frac{\partial}{\partial t}(\rho \boldsymbol{x} \times \boldsymbol{v})\, \partial\Omega + \int_{\Omega_c} \nabla_x \cdot (\rho(\boldsymbol{x} \times \boldsymbol{v}) \otimes \boldsymbol{v})\, \partial\Omega\end{aligned}$$

Using divergence theorem to the second term on right hand side of equation, we get

$$\int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma + \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega = \int_{\Omega_c} \frac{\partial}{\partial t}(\rho \boldsymbol{x} \times \boldsymbol{v})\, \partial\Omega + \int_{\Gamma_c} (\rho(\boldsymbol{x} \times \boldsymbol{v}) \otimes \boldsymbol{v})\, \boldsymbol{n}\, \partial\Gamma$$

We obtain the following form of the conservation of angular momentum for control volume using the definition of tensor product between two vectors.

$$\int_{\Omega_c} \frac{\partial}{\partial t}(\rho \boldsymbol{x} \times \boldsymbol{v})\, \partial\Omega + \int_{\Gamma_c} \rho(\boldsymbol{x} \times \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n})\, \partial\Gamma = \int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma + \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega. \tag{4}$$

Since Ω_c is fixed region in space, the conservation of angular momentum for control volume can also be written in the following form.

$$\frac{d}{dt} \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{v}\, \partial\Omega + \int_{\Gamma_c} \rho(\boldsymbol{x} \times \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{n})\, \partial\Gamma = \int_{\Gamma_c} \boldsymbol{x} \times \boldsymbol{t}\, \partial\Gamma + \int_{\Omega_c} \rho \boldsymbol{x} \times \boldsymbol{b}\, \partial\Omega. \tag{5}$$

Properties of the Cauchy stress tensor:
We now state some important properties of Cauchy stress tensor.

1. The stress tensor $\boldsymbol{\tau}$ is symmetric.
   Proof follows from balance of angular momentum as discussed previously.

2. At any point in continuum body at least a set of three orthogonal planes exist such that the traction vector is collinear with the normal $\boldsymbol{n}$, i.e., $\boldsymbol{t} = \lambda\boldsymbol{n}$.
   Proof follows from the symmetry of stress tensor and the relation between traction and stress tensor, i.e., $\boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n}$.

3. The plane where unit normal $\boldsymbol{n}$ and traction $\boldsymbol{t}$ are collinear is called principal plane and the direction $\boldsymbol{n}$ is called principal direction. Since $\boldsymbol{t} = \lambda\boldsymbol{n}$, the scalar λ is called *principal stress*. We note that all principal stresses are real. The principal directions corresponding to distinct principal stresses are mutually orthogonal.
   Proof follows from the symmetry of stress tensor (see Lecture-10).

4. The principal directions are the directions corresponding to extremum normal stress.

   Proof. Let $\boldsymbol{n}$ be a unit vector, i.e., $\boldsymbol{n} \cdot \boldsymbol{n} = 1$. Let $\boldsymbol{\tau}$ be a stress tensor which is symmetric. The normal stress T_n acting on plane with normal $\boldsymbol{n}$ is given by

$$T_n = \boldsymbol{n} \cdot \boldsymbol{\tau}\boldsymbol{n}. \tag{6}$$

   Then the condition for optimality can be written as

$$\frac{\partial(\boldsymbol{n}\boldsymbol{\tau} \cdot \boldsymbol{n})}{\partial n_k} + \lambda\frac{\partial(1-(\boldsymbol{n}\cdot\boldsymbol{n}))}{\partial n_k} = 0, \tag{7}$$

   where λ is the Lagrange multiplier and n_k is k'th component of $\boldsymbol{n}$. We can get the following relations from tensor calculus.

$$\begin{aligned}
\frac{\partial(\boldsymbol{n}\cdot\boldsymbol{\tau}\boldsymbol{n})}{\partial n_k} &= \frac{\partial(n_i(\boldsymbol{\tau}\boldsymbol{n})_i)}{\partial n_k} = \frac{\partial(n_i\tau_{ij}n_j)}{\partial n_k} = \frac{\partial n_i}{\partial n_k}\tau_{ij}n_j + n_i\tau_{ij}\frac{\partial n_j}{\partial n_k} \\
\frac{\partial(\boldsymbol{n}\cdot\boldsymbol{n})}{\partial n_k} &= \frac{\partial(n_i n_i)}{\partial n_k} = \frac{\partial n_i}{\partial n_k}n_i + n_i\frac{\partial n_i}{\partial n_k} \\
\frac{\partial n_i}{\partial n_k} &= \delta_{ik}
\end{aligned}$$

   Upon substituting these three relations in above optimality condition, i.e., in Eq. (7), we get

$$\begin{aligned}
\delta_{ik}\tau_{ij}n_j + n_i\tau_{ij}\delta_{jk} - \lambda\delta_{ik}n_i - \lambda n_i\delta_{ik} &= 0 \\
\implies \tau_{kj}n_j + \tau_{ik}n_i - 2\lambda n_k &= 0 \\
\implies 2\tau_{kj}n_j - 2\lambda n_k &= 0 \qquad (\text{Since } \tau_{ij} = \tau_{ji}) \\
\implies (\boldsymbol{\tau}\boldsymbol{n})_k &= \lambda n_k \\
\implies \boldsymbol{\tau}\boldsymbol{n} &= \lambda\boldsymbol{n}.
\end{aligned}$$

   It is easy to see that the extremum of normal stress $\boldsymbol{n}\cdot\boldsymbol{\tau}\boldsymbol{n}$ is given by the Lagrange multiplier as $\lambda = \boldsymbol{n}\cdot\boldsymbol{\tau}\boldsymbol{n}$. Thus, extremum of the quantity is equal to eigenvalue or principal stress and the extremum directions are eigenvectors or principal directions. □

5. The angle between maximum shear stress plane and maximum normal stress is 45° and the angle between maximum shear stress plane and minimum normal stress is also 45°. In other words, let λ_1, λ_2 and λ_3 be principal stresses such that $\lambda_1 \geq \lambda_2 \geq$

λ_3 and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ be corresponding principal directions. Then the normal to maximum shear stress plane makes 45° angle with both $\boldsymbol{n}_1$ and $\boldsymbol{n}_3$. The maximum shear stress is given by $(\lambda_1 - \lambda_3)/2$.

Proof. Let $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ be a set of orthonormal eigenvectors of stress tensor $\boldsymbol{\tau}$. Then the set forms a basis to the 3-D Euclidean space. Let us consider a plane with a unit normal $\tilde{\boldsymbol{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$. Then the components of traction vector $\boldsymbol{t}$ on the plane is given by $(\lambda_1\tilde{n}_1, \lambda_2\tilde{n}_2, \lambda_3\tilde{n}_3)$. Therefore, the square of magnitude of shear stress is given by

$$\begin{aligned} T_s^2 &= \boldsymbol{t}\cdot\boldsymbol{t} - T_n^2 \qquad \text{(where } T_n \text{ is normal stress)} \\ &= \boldsymbol{t}\cdot\boldsymbol{t} - (\boldsymbol{n}\cdot\boldsymbol{t})^2 \\ &= (\lambda_1\tilde{n}_1)^2 + (\lambda_2\tilde{n}_2)^2 + (\lambda_3\tilde{n}_3)^2 - \left(\lambda_1(\tilde{n}_1)^2 + \lambda_2(\tilde{n}_2)^2 + \lambda_3(\tilde{n}_3)^2\right)^2 \\ &= (\tilde{n}_1\tilde{n}_2)^2(\lambda_1-\lambda_2)^2 + (\tilde{n}_2\tilde{n}_3)^2(\lambda_2-\lambda_3)^2 + (\tilde{n}_3\tilde{n}_1)^2(\lambda_3-\lambda_1)^2 \end{aligned}$$

Similar to previous proof, the condition for extremization of shear stress can be stated as

$$\frac{\partial T_s^2}{\partial \tilde{n}_k} + \gamma\frac{\partial(1-\tilde{n}_1^2-\tilde{n}_2^2-\tilde{n}_3^2)}{\partial \tilde{n}_k} = 0.$$

Taking partial derivative, we get

$$\begin{aligned} \left((\lambda_1-\lambda_2)^2\tilde{n}_2^2 + (\lambda_3-\lambda_1)^2\tilde{n}_3^2 - \gamma\right)\tilde{n}_1 &= 0 \\ \left((\lambda_1-\lambda_2)^2\tilde{n}_1^2 + (\lambda_2-\lambda_3)^2\tilde{n}_3^2 - \gamma\right)\tilde{n}_2 &= 0 \\ \left((\lambda_1-\lambda_3)^2\tilde{n}_1^2 + (\lambda_2-\lambda_3)^2\tilde{n}_2^2 - \gamma\right)\tilde{n}_3 &= 0. \end{aligned}$$

In order to get maximum shear stress, we need to solve above equations along with constraint equation $\tilde{n}_1^2 + \tilde{n}_2^2 + \tilde{n}_3^2 = 1$. It is easy to verify that the following sets are solutions to above equations (Chandrasekharaiah and Debnath, 1994):

(i) $\tilde{n}_1 = \pm 1$, $\tilde{n}_2 = 0$, $\tilde{n}_3 = 0$.
(ii) $\tilde{n}_1 = 0$, $\tilde{n}_2 = \pm 1$, $\tilde{n}_3 = 0$.
(iii) $\tilde{n}_1 = 0$, $\tilde{n}_2 = 0$, $\tilde{n}_3 = \pm 1$
(iv) $\tilde{n}_1 = \tilde{n}_2 = \pm\frac{1}{\sqrt{2}}$, $\tilde{n}_3 = 0$.
(v) $\tilde{n}_1 = \tilde{n}_3 = \pm\frac{1}{\sqrt{2}}$, $\tilde{n}_2 = 0$.

Among all sets of solutions the maximum shear stress planes are given by solution set (v), i.e., $\tilde{n}_1 = \tilde{n}_3 = \pm\frac{1}{\sqrt{2}}$, $\tilde{n}_2 = 0$. Furthermore, the maximum shear stress is given by

$$T_s = \frac{1}{2}(\lambda_1 - \lambda_3).$$

□

References

1. C. S. Jog, *Foundations and Applications of Mechanics: Continuum Mechanics*, Volume-I, 2007, Narosa Publishing House Pvt. Ltd., New Delhi.

2. D. S. Chandrasekharaiah and L. Debnath, *Continuum Mechanics*, 1994, Academic Press Inc., London.