Module-4: Balance Laws

Lecture-26: The Linear Momentum Balance and the Cauchy Stress Tensor

The concept of stress is central to the continuum mechanics. In this lecture, we show the existence of stress tensor. We also derive the differential form of linear momentum balance which is known as Cauchy's equation of motion-I (first Cauchy's equation of motion).

The Cauchy's theorem and the stress tensor:
Let the traction field $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ be a continuous function of $\boldsymbol{x}$ at any instant of time t for a given unit vector $\boldsymbol{n}$. Let the density $\rho(\boldsymbol{x}, t)$, acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$ and body force field per unit mass $\boldsymbol{b}(\boldsymbol{x}, t)$ be bounded for every $\boldsymbol{x}$ in $\mathscr{B}$ at any time t. Then the linear momentum balance implies the existence of stress tensor $\boldsymbol{\tau}$ such that

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\tau}(\boldsymbol{x}, t)\boldsymbol{n}. \tag{1}$$

Proof. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a set of orthonormal basis to the 3-D Cartesian space. Consider a tetrahedron shown in Fig. (1) with three faces lying on Cartesian coordinate planes pass through O and an inclined plane pass through A, B and C. Let $\boldsymbol{x}$ be a position vector of O. Let Γ_n, Γ_1, Γ_2 and Γ_3 be four bounding surfaces ΔABC, ΔOCB, ΔOAC ΔOBA, respectively. Outward normals to the surfaces ΔABC, ΔOCB, ΔOAC and ΔOBA are $\boldsymbol{n}$, $-\boldsymbol{e}_1$, $-\boldsymbol{e}_2$ and $-\boldsymbol{e}_3$, respectively.

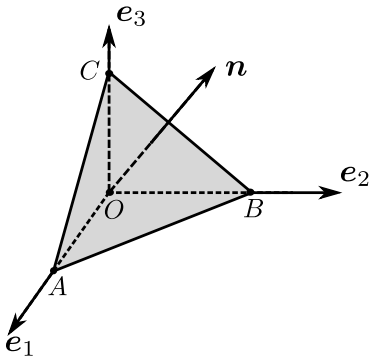


Figure 1: Tetrahedron used in Cauchy's theorem

Let Ω and Γ be domain and boundary of tetrahedron. Since Γ_n, Γ_1, Γ_2 and Γ_3 are boundary surface of tetrahedron, we have $\Gamma = \Gamma_n \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Let $\boldsymbol{v}(\boldsymbol{y}, t)$ be a velocity field where $\boldsymbol{y} \in \mathscr{B}$. Then using linear momentum balance, we get

$$\frac{d}{dt}\int_\Omega \rho \boldsymbol{v}\, \partial\Omega = \int_\Omega \rho \boldsymbol{b}\, \partial\Omega + \int_\Gamma \boldsymbol{t}(\boldsymbol{y}, t, \boldsymbol{n})\, \partial\Gamma.$$

Using the second transport theorem, we obtain

$$\int_\Omega \rho \frac{D\boldsymbol{v}}{Dt}\,\partial\Omega = \int_\Omega \rho\boldsymbol{b}\,\partial\Omega + \int_\Gamma \boldsymbol{t}(\boldsymbol{y},t,\boldsymbol{n})\,\partial\Gamma.$$

Applying the surface integral to four faces of tetrahedron (see Fig. 1), we can write

$$\int_\Omega \rho(\boldsymbol{a}-\boldsymbol{b})\,\partial\Omega = \int_{\Gamma_n} \boldsymbol{t}(\boldsymbol{y},t,\boldsymbol{n})\,\partial\Gamma + \int_{\Gamma_1} \boldsymbol{t}(\boldsymbol{y},t,\boldsymbol{n})\,\partial\Gamma + \int_{\Gamma_2} \boldsymbol{t}(\boldsymbol{y},t,\boldsymbol{n})\,\partial\Gamma + \int_{\Gamma_3} \boldsymbol{t}(\boldsymbol{y},t,\boldsymbol{n})\,\partial\Gamma.$$

Let A_S be surface area of Γ_n (i.e., ΔABC) and h be the distance from O to Γ_n. Let n_1, n_2 and n_3 be components of normal $\boldsymbol{n}$ along $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respectively. Then the surface area of Γ_1, Γ_2 and Γ_3 (i.e., area of ΔOCB, ΔOAC and ΔOBA) are given by n_1A_S, n_2A_S and n_3A_S, respectively. If h is small then, using the mean value theorem[1], we get

$$\rho(\boldsymbol{a}-\boldsymbol{b})|_{\boldsymbol{x}^{**}}\,\frac{1}{3}A_s h = (\boldsymbol{t}(\boldsymbol{x}^*,t,\boldsymbol{n}) + \boldsymbol{t}(\boldsymbol{x}_i^*,t,\boldsymbol{e}_i)n_i)\,A_s$$

where $\boldsymbol{x}^{**}$ is centroid of tetrahedron and $\boldsymbol{x}^*$ is centroid of Γ_n and $\boldsymbol{x}_i^*$ is centroid of Γ_i. Let h tends to zero such that the normal of Γ_n continues to be $\boldsymbol{n}$. Then the continuity of traction $\boldsymbol{t}$ and boundedness of $\rho(\boldsymbol{a}-\boldsymbol{b})$ implies

$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x},t,-\boldsymbol{e}_i)n_i.$$

Using the Cauchy's lemma (see Lecture-25), we get

$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_i)n_i. \tag{2}$$

Following similar steps one can show Eq. (2) for the tetrahedron lying in other octants of 3-D Cartesian space. We now define the Cauchy stress tensor $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau}(\boldsymbol{x},t) = \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_i)\otimes\boldsymbol{e}_i. \tag{3}$$

Equations (3) and (2) imply $\boldsymbol{\tau}(\boldsymbol{x},t)\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$. We note that one should use the tensor product definition given in Eq. (16) of Lecture-4 to see the equivalence between $\boldsymbol{\tau}(\boldsymbol{x},t)\boldsymbol{n}$ and $\boldsymbol{t}$. Thus, there is a spatial tensor field known as Cauchy stress tensor. □

The linear momentum balance: Cauchy's equation of motion-I:
Let $\mathscr{B}_0$ be a reference configuration of body. Let $\mathscr{B}$ be a deformed configuration at an instant of time t. Let Ω_0 be a arbitrary volume of $\mathscr{B}_0$ and Ω be a material volume corresponding to Ω_0. Let body force field per unit mass $\boldsymbol{b}(\boldsymbol{x},t)$ and acceleration field $\boldsymbol{a}(\boldsymbol{x},t)$ be continuous at every instant of time t. Then the Cauchy's hypothesis and linear momentum balance imply the following equation known as first Cauchy's equation of motion.

$$\rho\frac{D\boldsymbol{v}}{Dt} = \nabla_x\cdot\boldsymbol{\tau} + \rho\boldsymbol{b}, \tag{4}$$

where $\boldsymbol{\tau}$ is the Cauchy stress tensor and $\boldsymbol{v}$ is a spatial velocity field.

[1]If the domain of integral Ω_ϵ is sufficiently small then volume integral can be approximated by the product between volume of domain and integrand value at centroid, i.e., $\int_{\Omega_\epsilon} g(\boldsymbol{x})d\Omega_\epsilon = g(\boldsymbol{x}^*)\Omega_\epsilon$, where $\boldsymbol{x}^*$ is centroid of Ω_ϵ. Of course the integrand is assumed to be a bounded function in the domain. This result is applicable to scalar, vector and tensor valued function

Problem 1. *Show that linear momentum balance along with Cauchy's hypothesis imply the following governing differential equation.*

$$\rho \frac{D\boldsymbol{v}}{Dt} = \nabla_x \cdot \boldsymbol{\tau} + \rho \boldsymbol{b},$$

Proof. Let Γ be a boundary of material volume $\Omega \in \mathscr{B}$ and $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ be a traction field over Γ. Then the Cauchy's theorem implies the following relation (see Eq. (1)).

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\tau}(\boldsymbol{x}, t)\boldsymbol{n},$$

where $\boldsymbol{n}$ is normal to the boundary Γ and $\boldsymbol{\tau}$ is Cauchy stress tensor. Since $\boldsymbol{b}(\boldsymbol{x}, t)$ is body force, $\boldsymbol{v}(\boldsymbol{x}, t)$ is velocity, $\rho(\boldsymbol{x}, t)$ is density and $\boldsymbol{t}(\boldsymbol{x}, t)$ is traction, we can write the linear momentum balance for material volume Ω as

$$\frac{d}{dt}\int_\Omega \rho \boldsymbol{v}\, \partial\Omega = \int_\Omega \rho \boldsymbol{b}\, \partial\Omega + \int_\Gamma \boldsymbol{t}\, \partial\Gamma.$$

Substituting stress tensor and applying second transport theorem to the above equation, we get

$$\int_\Omega \rho \frac{D\boldsymbol{v}}{Dt}\, \partial\Omega = \int_\Omega \rho \boldsymbol{b}\, \partial\Omega + \int_\Gamma \boldsymbol{\tau}\boldsymbol{n}\, \partial\Gamma.$$

Applying the divergence theorem to the surface integral and grouping terms, yields

$$\int_\Omega \left(\rho \frac{D\boldsymbol{v}}{Dt} - \nabla_x \cdot \boldsymbol{\tau} - \rho \boldsymbol{b}\right) \partial\Omega = \boldsymbol{0}.$$

Since the material volume Ω is arbitrary, application of the localization theorem give rise to the first Cauchy's equation of motion shown in Eq. (4). We now show that the Cauchy stress and differential form of equation of motion (first Cauchy's equation of motion) also imply the linear momentum balance. Considering the linear momentum of material volume Ω, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \rho \boldsymbol{v}\, \partial\Omega &= \int_\Omega \rho \frac{D\boldsymbol{v}}{Dt}\, \partial\Omega && \text{(Using transport theorem-II)} \\
&= \int_\Omega (\nabla_x \cdot \boldsymbol{\tau} + \rho \boldsymbol{b})\, \partial\Omega && \text{(Using first Cauchy's equation of motion)} \\
&= \int_\Gamma \boldsymbol{\tau}\boldsymbol{n}\, \partial\Gamma + \int_\Omega \rho \boldsymbol{b}\, \partial\Omega && \text{(Using divergence theorem)} \\
&= \int_\Gamma \boldsymbol{t}\, \partial\Gamma + \int_\Omega \rho \boldsymbol{b}\, \partial\Omega && \text{(Since } \boldsymbol{\tau}\boldsymbol{n} = \boldsymbol{t}\text{)}.
\end{aligned}$$

Thus, the linear momentum balance along with Cauchy's hypothesis imply the existence of Cauchy stress tensor. Conversely, the Cauchy stress tensor with first Cauchy's equation of motion imply the linear momentum balance. □

Cauchy stress and differential equation of motion in indicial notation:
The index represention of Eqs. (1) and (4) can be written as

$$\begin{aligned}
t_i &= \tau_{ij} n_j, \\
\rho\left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j\right) &= \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i.
\end{aligned}$$

Equation (3) implies

$$\tau_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{\tau} \boldsymbol{e}_j = t_i(\boldsymbol{x}, t, \boldsymbol{e}_j).$$

Thus, the component of stress τ_{ij} represents the i'th component of the traction $\boldsymbol{t}$ (i.e., the component of $\boldsymbol{t}$ along $\boldsymbol{e}_i$) on the surface whose normal is in the j'th direction (i.e., the surface with $\boldsymbol{e}_j$ as its normal). It can be observed that τ_{11}, τ_{22} and τ_{33} are normal stresses as they act normal to the surface whereas other components (τ_{12}, τ_{13}, τ_{21}, τ_{23}, τ_{31}, τ_{32}) represent shear stresses as they act tangential to the surface. The stress components on infinitesimal cube are shown in Fig. (2).

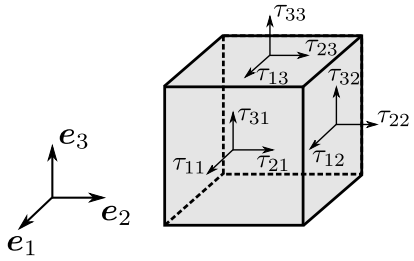


Figure 2: Cauchy stress components on infinitesimal cube

Linear momentum balance in control volume:
The control volume approach is suitable for fluid mechanics. So, we now present the integral form of linear momentum balance in control volume. Let Ω_c be a fixed control volume, i.e., Ω_c is fixed volume in space, and Γ_c be a boundary of Ω_c. Using the first Cauchy's equation of motion (differential form of linear momentum balance in Eulerian framework), we can write

$$\int_{\Omega_c} \left(\rho \frac{D\boldsymbol{v}}{Dt} - \rho \boldsymbol{b} - \nabla_x \cdot \boldsymbol{\tau} \right) \partial\Omega = 0.$$

Recall the relation between material and spatial time derivative of vector field from Lecture-20. Upon substituting the relation between material and spatial time derivative for velocity field yields

$$\int_{\Omega_c} \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla_x \boldsymbol{v})\boldsymbol{v} \right) \partial\Omega = \int_{\Omega_c} \rho \boldsymbol{b} \, \partial\Omega + \int_{\Omega_c} \nabla_x \cdot \boldsymbol{\tau} \, \partial\Omega.$$

Applying divergence theorem to the second term on the right hand side, we get

$$\int_{\Omega_c} \left(\frac{\partial(\rho \boldsymbol{v})}{\partial t} - \frac{\partial \rho}{\partial t}\boldsymbol{v} + (\nabla_x \boldsymbol{v})\rho \boldsymbol{v} \right) \partial\Omega = \int_{\Omega_c} \rho \boldsymbol{b} \, \partial\Omega + \int_{\Gamma_c} \boldsymbol{\tau} \boldsymbol{n} \, \partial\Gamma,$$

where $\boldsymbol{n}$ is normal to the control surface Γ_c.

Using the conservation of mass in Eulerian framework, i.e., $\dfrac{\partial \rho}{\partial t} = -\nabla_x \cdot (\rho \boldsymbol{v})$, we can write

$$\int_{\Omega_c} \frac{\partial(\rho \boldsymbol{v})}{\partial t} \, \partial\Omega + \int_{\Omega_c} \left(\nabla_x \cdot (\rho \boldsymbol{v}) \right) \boldsymbol{v} + (\nabla_x \boldsymbol{v})\rho \boldsymbol{v} \, \partial\Omega = \int_{\Omega_c} \rho \boldsymbol{b} \, \partial\Omega + \int_{\Gamma_c} \boldsymbol{\tau} \boldsymbol{n} \, \partial\Gamma$$

By substituting the relations $\nabla_x \cdot (\boldsymbol{v} \otimes (\rho \boldsymbol{v})) = \left(\nabla_x \cdot (\rho \boldsymbol{v}) \right) \boldsymbol{v} + (\nabla_x \boldsymbol{v})\rho \boldsymbol{v}$ and $\boldsymbol{t} = \boldsymbol{\tau} \boldsymbol{n}$, we can obtain

$$\int_{\Omega_c} \frac{\partial(\rho \boldsymbol{v})}{\partial t} \, \partial\Omega + \int_{\Omega_c} \nabla_x \cdot (\boldsymbol{v} \otimes (\rho \boldsymbol{v})) \, \partial\Omega = \int_{\Omega_c} \rho \boldsymbol{b} \, \partial\Omega + \int_{\Gamma_c} \boldsymbol{t} \, \partial\Gamma.$$

Applying the divergence theorem to the second term on the left hand side of equation, we get the following linear momentum balance for the control volume.

$$\int_{\Omega_c} \frac{\partial(\rho \boldsymbol{v})}{\partial t}\, \partial\Omega + \int_{\Gamma_c} \rho \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})\, \partial\Gamma = \int_{\Omega_c} \rho \boldsymbol{b}\, \partial\Omega + \int_{\Gamma_c} \boldsymbol{t}\, \partial\Gamma. \tag{5}$$

Since Ω_c is fixed in space, we can also write the linear momentum balance as

$$\frac{d}{dt}\int_{\Omega_c} \rho \boldsymbol{v}\, \partial\Omega + \int_{\Gamma_c} \rho \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})\, \partial\Gamma = \int_{\Omega_c} \rho \boldsymbol{b}\, \partial\Omega + \int_{\Gamma_c} \boldsymbol{t}\, \partial\Gamma. \tag{6}$$

The term $\int_{\Gamma_c} \rho \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})\, \partial\Gamma$ represents the rate of change of linear momentum due to the flux across the control surface Γ_c.

Reference

1. C. S. Jog, *Foundations and Applications of Mechanics: Continuum Mechanics*, Volume-I, 2007, Narosa Publishing House Pvt. Ltd., New Delhi.