
Module-3: Kinematics

Lecture-23: The Vorticity and Circulation

We now introduce vorticity vector and circulation that are important in the study of fluid mechanics. In addition, we discuss the Cauchy's vorticity formula and the Kelvin's circulation theorem.

Vorticity vector:

Let $\mathbf{v}$ be a spatial velocity field defined over the deformed body Ω at an instant of time t . Then the vorticity vector is defined by

$$\mathbf{w} = \nabla_x \times \mathbf{v}. \quad (1)$$

Since the spin tensor $\mathbf{W}$ is a skew-symmetric part of the velocity gradient $\mathbf{L}$, we can write

$$\mathbf{W} = \frac{1}{2}(\nabla_x \mathbf{v} - \nabla_x \mathbf{v}^T).$$

It is clear from the definition that $\mathbf{w}/2$ is an axial vector of the spin tensor $\mathbf{W}$. Therefore, the vorticity vector at a point gives twice the rate of rotation or angular velocity. We now define circulation and also the relation between circulation and vorticity.

Circulation:

Let Ω_0 be a reference configuration and Ω be a deformed configuration at an instant of time t . Let $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ be a mapping from Ω_0 to Ω . Let C be a closed curve in Ω_0 and C_t be a corresponding closed curve in Ω . Then the circulation $\mathfrak{C}$ is defined by

$$\mathfrak{C} = \oint_{C_t} \mathbf{v} \cdot d\mathbf{x}. \quad (2)$$

Let Γ be a surface that passes through the closed curve C_t . Then, using Stokes' theorem (see Lecture-16), we get

$$\oint_{C_t} \mathbf{v} \cdot d\mathbf{x} = \int_{\Gamma} (\nabla_x \times \mathbf{v}) \cdot \mathbf{n} \, \partial\Gamma, \quad (3)$$

where $\mathbf{n}$ is the outward unit normal to the surface Γ . Thus, the circulation at a point is directly related to the vorticity vector at that point.

Transport of circulation:

Let C be a closed curve in the reference configuration Ω_0 and C_t be a corresponding closed curve in deformed configuration Ω . Then the material time derivative of circulation is given by

$$\frac{d}{dt} \oint_{C_t} \mathbf{v} \cdot d\mathbf{x} = \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x}. \quad (4)$$

Proof. Let $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ be a map from Ω_0 to Ω . Let $\mathbf{F}(\mathbf{X}, t)$ be the deformation gradient. Let $\tilde{\mathbf{v}}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$ for $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, i.e., $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\boldsymbol{\chi}(\mathbf{X}, t), t) = \tilde{\mathbf{v}}(\mathbf{X}, t)$. Then

$$\begin{aligned} \frac{d}{dt} \oint_{C_t} \mathbf{v} \cdot d\mathbf{x} &= \frac{d}{dt} \oint_C \tilde{\mathbf{v}} \cdot \mathbf{F} d\mathbf{X} \\ &= \oint_C \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \mathbf{F} d\mathbf{X} + \tilde{\mathbf{v}} \cdot \frac{\partial \mathbf{F}}{\partial t} d\mathbf{X}. \end{aligned}$$

Let $\mathbf{L}$ be a velocity gradient. Then

$$\begin{aligned} \frac{d}{dt} \oint_{C_t} \mathbf{v} \cdot d\mathbf{x} &= \oint_C \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \mathbf{F} d\mathbf{X} + \tilde{\mathbf{v}} \cdot \mathbf{L} \mathbf{F} d\mathbf{X} \\ &= \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \oint_{C_t} \mathbf{v} \cdot \mathbf{L} d\mathbf{x} \\ &= \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \oint_{C_t} \mathbf{v} \cdot d\mathbf{v} \\ &= \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \oint_{C_t} \frac{1}{2} d(\mathbf{v} \cdot \mathbf{v}) \\ &= \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} \quad \left(\text{since } \oint_{C_t} d(\mathbf{v} \cdot \mathbf{v}) = 0 \right) \end{aligned}$$

This completes the proof. □

Kelvin's circulation theorem:

The motion preserves circulation if there exist a scalar function ϕ such that the acceleration $\frac{D\mathbf{v}}{Dt} = \nabla_x \phi$.

Proof. Let Ω_0 be a reference configuration and C be a closed curve in Ω_0 . Let Ω be a deformed configuration at an instant of time t and C_t be a closed curve defined in Ω corresponding to C . Then the circulation $\mathfrak{L}$ is defined by

$$\mathfrak{L} = \oint_{C_t} \mathbf{v} \cdot d\mathbf{x}.$$

The material time derivative of circulation of the transport of circulation (see Eq. (4)),

$$\frac{d\mathfrak{L}}{dt} = \oint_{C_t} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x}.$$

Since the acceleration is gradient of potential function ϕ , we can write

$$\frac{D\mathbf{v}}{Dt} = \nabla_x \phi.$$

Combining above two equations, we get

$$\frac{d\mathfrak{L}}{dt} = \oint_{C_t} \nabla_x \phi \cdot d\mathbf{x} = \oint_{C_t} d\phi = 0.$$

Thus, the circulation $\mathfrak{L}$ is constant in C_t throughout the motion. □

The Kelvin's circulation theorem shows that the motion is irrotational at $t = 0$, i.e., $\nabla_x \times \mathbf{v}(\mathbf{x}, 0) = \mathbf{0}$, then the motion is irrotational forever, i.e., $\nabla_x \times \mathbf{v}(\mathbf{x}, t) = \mathbf{0}$, $\forall t \in [0, \infty)$.

Cauchy's vorticity formula:

Let Ω_0 be a reference configuration and Ω be a deformed configuration at an instant of time t . Let $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ be a mapping from Ω_0 to Ω . Let $\mathbf{F}$ be a deformation gradient and $\mathbf{L}$ be a velocity gradient. Let $\mathbf{v}$ be a velocity field and $\mathbf{a}$ be an acceleration field. Let $\boldsymbol{\omega}_0$ be a vorticity vector at $\mathbf{X} \in \Omega_0$ and $\boldsymbol{\omega}$ be a vorticity vector at $\mathbf{x} \in \Omega$. Then we show that the vorticity fields are related by

$$\boldsymbol{\omega} = J^{-1} \mathbf{F} \boldsymbol{\omega}_0, \quad (5)$$

where J is determinant of $\mathbf{F}$. This relationship is known as the Cauchy's vorticity formula.

Proof. The vorticity vector $\boldsymbol{\omega}(\mathbf{x}, t)$ in Ω is defined by

$$\boldsymbol{\omega} = \nabla_x \times \mathbf{v}.$$

Let $\tilde{\boldsymbol{\omega}}(\mathbf{X}, t)$ be Lagrangian description of vorticity vector, i.e., $\boldsymbol{\omega}(\mathbf{x}, t) = \boldsymbol{\omega}(\boldsymbol{\chi}(\mathbf{X}, t), t) = \tilde{\boldsymbol{\omega}}(\mathbf{X}, t)$. Since $\boldsymbol{\omega}_0$ is vorticity vector field at time $t = 0$, we have $\boldsymbol{\omega}_0 = \tilde{\boldsymbol{\omega}}(\mathbf{X}, 0)$.

Let C_t be a arbitrary closed curve in Ω and C be a corresponding curve in Ω_0 . Let Γ be surface in Ω bounded by C_t and Γ_0 be a surface corresponding to Γ in Ω_0 . Let $\mathbf{n}$ be a unit normal field to Γ and $\mathbf{n}^0$ be a unit normal field to Γ_0 . Then consider total time derivative of circulation,

$$\begin{aligned} \frac{d}{dt} \oint_{C_t} \mathbf{v} \cdot d\mathbf{x} &= \frac{d}{dt} \int_{\Gamma} (\nabla_x \times \mathbf{v}) \cdot \mathbf{n} \partial\Gamma \\ &= \frac{d}{dt} \int_{\Gamma} \boldsymbol{\omega} \cdot \mathbf{n} \partial\Gamma \\ &= \frac{d}{dt} \int_{\Gamma_0} \tilde{\boldsymbol{\omega}} \cdot J \mathbf{F}^{-T} \mathbf{n}^0 \partial\Gamma_0 \\ &= \int_{\Gamma_0} \frac{\partial(\tilde{\boldsymbol{\omega}} \cdot J \mathbf{F}^{-T} \mathbf{n}^0)}{\partial t} \partial\Gamma_0 \\ &= \int_{\Gamma_0} \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} \cdot J \mathbf{F}^{-T} \mathbf{n}^0 + \tilde{\boldsymbol{\omega}} \cdot \frac{\partial J}{\partial t} \mathbf{F}^{-T} \mathbf{n}^0 + \tilde{\boldsymbol{\omega}} \cdot J \frac{\partial \mathbf{F}^{-T}}{\partial t} \mathbf{n}^0 \partial\Gamma_0 \\ &= \int_{\Gamma_0} \frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} \cdot J \mathbf{F}^{-T} \mathbf{n}^0 + \tilde{\boldsymbol{\omega}} \cdot (\text{tr } \mathbf{L}) J \mathbf{F}^{-T} \mathbf{n}^0 + \tilde{\boldsymbol{\omega}} \cdot J (-\mathbf{L}^T \mathbf{F}^{-T}) \mathbf{n}^0 \partial\Gamma_0 \\ &= \int_{\Gamma_0} \left(\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + \tilde{\boldsymbol{\omega}} (\text{tr } \mathbf{L}) - \mathbf{L} \tilde{\boldsymbol{\omega}} \right) \cdot J (-\mathbf{L}^T \mathbf{F}^{-T}) \mathbf{n}^0 \partial\Gamma_0 \\ &= \int_{\Gamma} \left(\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega} (\text{tr } \mathbf{L}) - \mathbf{L} \boldsymbol{\omega} \right) \cdot \mathbf{n} \partial\Gamma. \end{aligned}$$

Applying Stokes' theorem to Eq. (4), we get

$$\frac{d}{dt} \oint_{C_t} \mathbf{v} \cdot d\mathbf{x} = \oint_{C_t} \mathbf{a} \cdot d\mathbf{x} = \int_{\Gamma} (\nabla_x \times \mathbf{a}) \cdot \mathbf{n} \partial\Gamma.$$

Comparing above two equations, we get

$$\int_{\Gamma} (\nabla_x \times \mathbf{a}) \cdot \mathbf{n} \partial\Gamma = \int_{\Gamma} \left(\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega} (\text{tr } \mathbf{L}) - \mathbf{L} \boldsymbol{\omega} \right) \cdot \mathbf{n} \partial\Gamma. \quad (6)$$

The arbitrariness of curve C and surface Γ implies

$$\nabla_x \times \mathbf{a} = \frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega}(\text{tr } \mathbf{L}) - \mathbf{L}\boldsymbol{\omega}.$$

Since the motion is circulation preserving, we have $\nabla_x \times \mathbf{a} = \mathbf{0}$. Therefore, Eq. (6) can be written as

$$\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega}(\text{tr } \mathbf{L}) - \mathbf{L}\boldsymbol{\omega} = \mathbf{0}.$$

We can express the above equation using total derivative

$$\frac{D(J\mathbf{F}^{-1}\boldsymbol{\omega})}{Dt} = 0.$$

Since Ω_0 is the configuration at time $t = 0$, we have $\mathbf{F} = \mathbf{I}$, $J = 1$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$. We get following relation by integrating above differential equation and substituting initial conditions $\mathbf{F} = \mathbf{I}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$.

$$\boldsymbol{\omega}_0 = J\mathbf{F}^{-1}\boldsymbol{\omega}.$$

This relation is known as Cauchy's vorticity formula. □

Similar to previous result of Kelvin's circulation theorem, we can conclude from the Cauchy's vorticity formula that the motion is irrotational once then it is irrotational forever.

References

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