Module 6

APPLICATIONS

NONLINEAR VIBRATION OF MECHANICAL SYSTEMS

6 Applications	**31-42**	**SDOF Free and Forced Vibration: Duffing Equation, van der pol's Equation: Simple or primary resonance, sub-super harmonic resonance.** **Parametrically excited system- Mathieu-Hill's equation, Floquet Theory, Instability regions; Multi-DOF nonlinear systems and Continuous system, System with internal resonances**

6 A Free Vibration of **nonlinear Systems**	1	**Single degree of freedom Nonlinear conservative systems with Cubic nonlinearities.**
	2	**Single degree of freedom nonlinear conservative systems with quadratic and Cubic and nonlinearities.**
	3	**Single degree of freedom non-conservative systems: viscous damping, quadratic and Coulomb damping**
	4	**non-conservative systems: Negative damping, van der Pol oscillator, simple pendulum with quadratic damping**

6 B Forced nonlinear Vibration	1	**Single degree of freedom Nonlinear systems with Cubic nonlinearities: Primary Resonance**
	2	**Single degree of freedom nonlinear systems with Cubic nonlinearities: Nonresonant Hard excitation**
	3	**Single degree of freedom Nonlinear systems with Cubic and quadratic nonlinearities and self sustained oscillations**
	4	**Multi-degree of freedom nonlinear systems**

6 C nonlinear Vibration of Parametrically excited system	1	Parametrically excited system: Floquet theory, Hill's infinite determinant
	2	Parametric Instability region: sandwich beam vibration
	3	Base excited magneto-elastic cantilever beam with tip mass
	4	System with internal resonance: Two-mode interaction: Base excited cantilever beam with tip mass at arbitrary position

Module 6 Lect 1

Free Vibration of Nonlinear Conservative system

In this lecture we will learn about the free vibration response of nonlinear conservative systems. Initially the qualitative analysis will be demonstrated and later by using one of the perturbation methods the free vibration response of the single degree of freedom system will be illustrated using different examples.

Qualitative Analysis of Nonlinear Systems

Consider the nonlinear conservative system given by the equation

$$\ddot{u} + f(u) = 0 \tag{6.1.1}$$

Multiplying $\dot{u}$ in Eq. (6.1.1) and integrating the resulting equation one can write as

$$\int (\dot{u}\ddot{u} + \dot{u}f(u))dt = h$$

$$\text{Or, } \int \dot{u}\frac{d(\dot{u})}{dt}dt + \int f(u)\frac{d(u)}{dt}dt = h \tag{6.1.2}$$

$$\text{Or, } \int \dot{u}d\dot{u} + \int f(u)du = h$$

$$\text{Or., } \frac{1}{2}\dot{u}^2 + F(u) = h, \quad \text{where, } F(u) = \int f(u)du \tag{6.1.3}$$

This represents that the sum of the kinetic energy and potential energy of the system is constant. Hence, for particular energy level *h*, the system will be under oscillation, if the potential energy $F(u)$ is less than the total energy *h*. From the above equation, one may plot the phase portrait or the trajectories for different energy level and study qualitatively about the response of the system using the following equation.

$$\dot{u} = \sqrt{2(h - F(u))} \tag{6.1.4}$$

It may be noted that velocity exists, or the body will move only when $h > F(u)$. One will obtain equilibrium points corresponding to $h = F(u)$ or when $F'(u) = f(u) = 0$. For minimum potential energy a center will be obtained and for maximum potential energy a saddle point will be obtained. The trajectory joining the two saddle points is known as homoclinic orbit. The response is periodic near the center.

Example 6.1.1 Perform qualitative analysis of spring-mass system with a soft spring. Take mass of the system as 1unit, linear stiffness 1 unit and stiffness corresponding to cubic nonlinear term as 0.1 unit.

Solution: In this case the equation of motion of the system can be given by

$$\ddot{x} + x - 0.1x^3 = 0 \tag{6.1.5}$$

For this system $F(x) = \int f(x)dx = \int (x - 0.1x^3)dx = \frac{1}{2}x^2 - \frac{1}{40}x^4$ (6.1.6)

Figure 6.1.1 shows the variation of potential energy $F(x)$ with x. It has its optimum values corresponding to $\left(F'(x) = f(x) = 0\right)$ $x = 0$ or $\pm\sqrt{10}$. While x equal to zero represents the system with minimum potential energy, the other two points represent the equilibrium points with maximum potential energy. Now by taking different energy level h, one may find the relation between the velocity v and displacement x as $v = \dot{x} = \sqrt{2(h - F(x))} = \sqrt{2(h - (0.5x^2 - 0.025x^4))}$

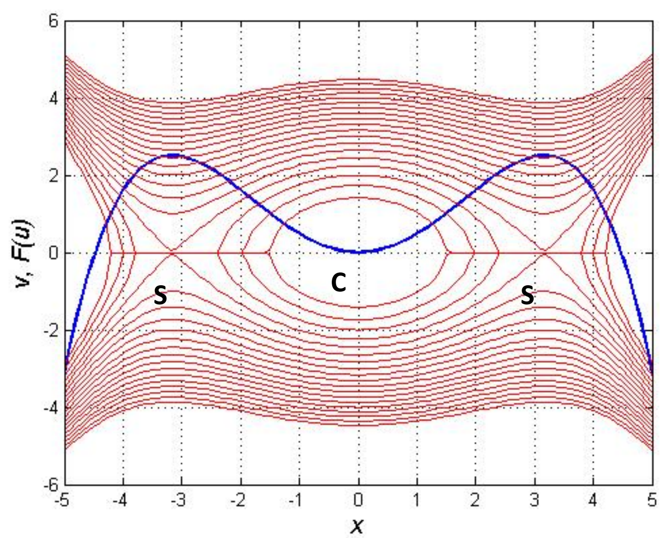
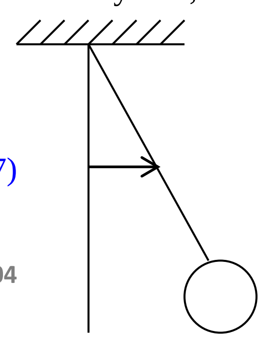


Fig. 6.1.1: Potential well (curve with blue colour) and phase portrait (red colour) showing saddle point (S) and center (C) corresponding to maximum and minimum potential energy.

Now by plotting the phase portrait one may find the trajectory which clearly depicts that the equilibrium point corresponding to maximum potential energy is a saddle point (marked by point S) where the equilibrium point is unstable and the equilibrium point corresponding to the minimum potential energy is stable center type (marked by point C). Clearly the orbit joining the points S and S is homoclinic orbit. Depending on different initial conditions i.e the total energy of the system, near the center one will obtain periodic orbits.



Example 6.1.2: Perform qualitative analysis for a simple pendulum.

Solution: The governing equation of motion of the system is given by (Eq. 2.1.7)

$$ml\ddot{\theta} + mg\sin\theta = 0 \text{ or } \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{6.1.7}$$

So, $F(\theta) = \int f(\theta)d\theta = \frac{g}{l}\int \sin\theta d\theta = -\frac{g}{l}\cos\theta$ (6.1.8)

Fig. 6.1.2: Simple pendulum

Hence, the potential function $F(\theta)$ has a minima corresponding to θ equal to zero and maxima corresponding to θ equal to odd multiple of π. So, the equilibrium point near $\theta = 0$ is a center and near $\theta = 180^\circ$ is a saddle point. The phase portrait for different energy level h is given in Figure 6.1.3 which is obtained by using the following relation.

$$v = \dot{\theta} = \sqrt{2(h - F(\theta))} = \sqrt{2\left(h + \frac{g}{l}\cos\theta\right)} \tag{6.1.9}$$

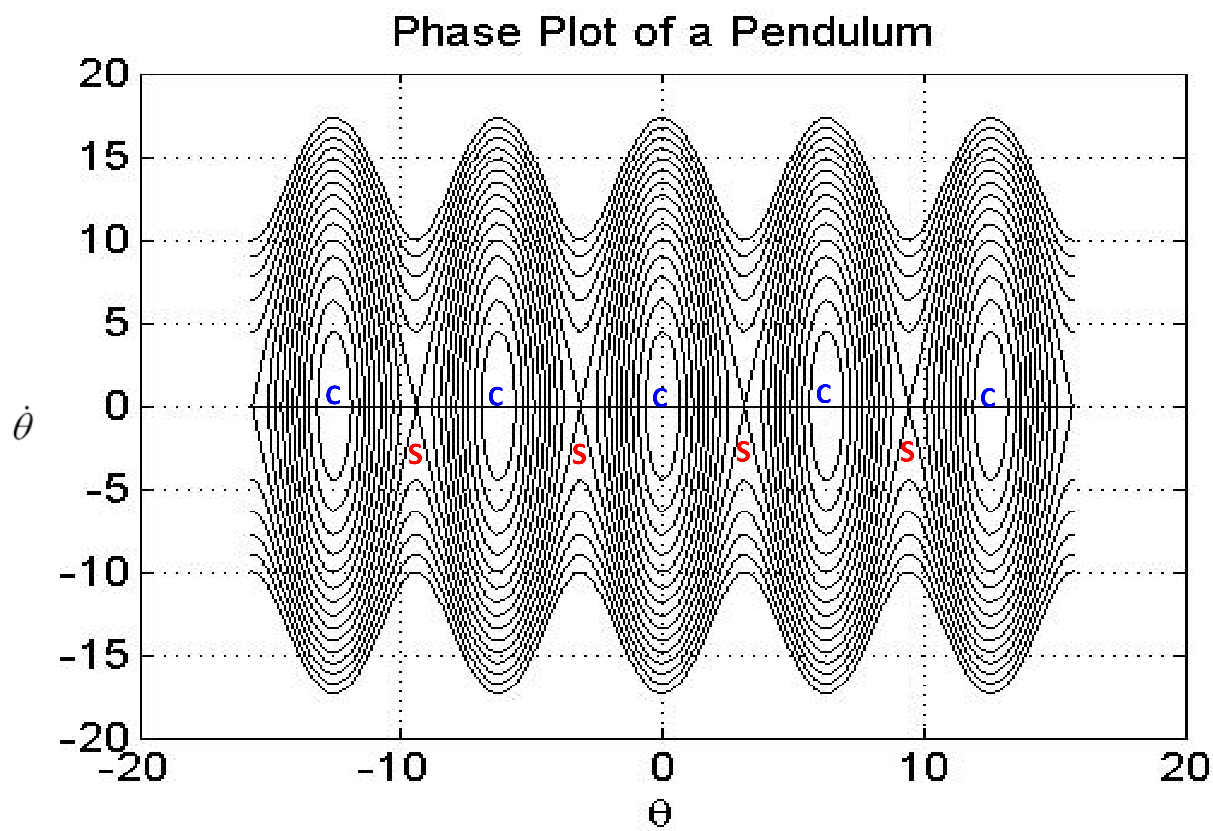


Figure 6.1.3: Phase portrait for the motion of a simple pendulum

Approximate solution method

Let us consider the Duffing equation with cubic nonlinear term for the free vibration study of a nonlinear system.

$$\frac{d^2u}{dt^2}+\omega_0^2 u+\varepsilon u^3=0 \qquad (6.1.10)$$

Using method of multiple scales, the solution of this equation can be given by

$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2+o(\varepsilon^3) \qquad (6.1.11)$$

Taking the time scale $T_n=\varepsilon^n t$ (6.1.12)

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2+\ldots\ldots\ldots\ldots\ldots,\quad D_n=\frac{\partial}{\partial T_n} \qquad (6.1.13)$$

$$\frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\varepsilon^2\left(D_1^2+2D_0D_2\right)+\ldots\ldots\ldots\ldots\ldots \qquad (6.1.14)$$

Substituting Eqs. (6.1.11-6.1.14) in Eq. (6.1.10) and separating terms with different order of ε one obtains the following equations.

Order of ε^0

$$D_0^2 u_0+\omega_0^2 u_0=0 \qquad (6.1.15)$$

Order of ε^1

$$D_0^2 u_1+\omega_0^2 u_1=-2D_0D_1u_0-u_0^3 \qquad (6.1.16)$$

Order of ε^2

$$D_0^2 u_2+\omega_0^2 u_2=-2D_0D_1u_1-2D_0D_2u_0-D_1^2u_0-3u_0^2u_1 \qquad (6.1.17)$$

Solution of Eq. (6.1.15) can be given by

$$u_0=A(T_1,T_2,)e^{i\omega_0 T_0}+\bar{A}(T_1,T_2,)e^{-i\omega_0 T_0} \qquad (6.1.18)$$

Substituting Eq.(6.1.18) in Eq. (6.1.16) one obtains

$$D_0^2 u_1+\omega_0^2 u_1=-[2i\omega_0 D_1 A+3A^2\bar{A}]e^{i\omega_0 T_0}-A^3 e^{3i\omega_0 T_0}+cc \qquad (6.1.19)$$

To eliminate the secular term (coefficient of $e^{i\omega_0 T_0}$) term marked in pink colour should be equated to zero.

Hence, $2i\omega_0 D_1 A+3A^2\bar{A}=0$ (6.1.20)

From Eq. (6.1.19), the solution of u_1 can be written as

$$u_1 = \frac{A^3}{8\omega_0^2} e^{3i\omega_0 T_0} + cc \tag{6.1.21}$$

Now Substituting $A = \frac{1}{2} a \exp(i\beta)$ (where a and β are real number) in Eq. (6.1.20) and separating the real and imaginary parts one obtains

$$\frac{\partial a}{\partial T_1} = 0, \text{ and } -\omega_0 a \frac{\partial \beta}{\partial T_1} + \frac{3}{8} a^3 = 0 \tag{6.1.22}$$

Hence, a is not a function of T_0 and T_1. So up on integration Eq. (6.1.22) can be written as

$$a = a(T_2), \text{ and } \beta = \frac{3}{8\omega_0^2} a^2 T_1 + \beta_0(T_2) \tag{6.1.23}$$

Substituting Eq. (6.1.18) and Eq. (6.1.21) in Eq. (6.1.17), one can write

$$D_0^2 u_2 + \omega_0^2 u_2 = -\underbrace{\left(2i\omega_0 D_2 A - \frac{15A^3\bar{A}^2}{8\omega_0^2} \right)}_{\text{Secular term}} e^{i\omega_0 T_0} + \frac{21}{8\omega_0^2} A^4 \bar{A} e^{3i\omega_0 T_0} - \frac{3}{8\omega_0^2} A^5 e^{5i\omega_0 T_0} + cc \tag{6.1.24}$$

For the secular term to be zero one can write $2i\omega_0 D_2 A - \frac{15A^3\bar{A}^2}{8\omega_0^2}$ (6.1.25)

The solution of remaining part of Eq. (6.1.24) can be written as

$$u_2 = -\frac{A^5}{64\omega_0^4} e^{5i\omega_0 T_0} + \frac{21}{64\omega_0^4} A^4 \bar{A} e^{3i\omega_0 T_0} + cc \tag{6.1.26}$$

Now Substituting $A = \frac{1}{2} a \exp(i\beta)$ (where a and β are real number) in Eq. (6.1.25) and separating the real and imaginary parts one obtains

$$\frac{\partial a}{\partial T_2} = 0, \text{ and } -\omega_0 \frac{\partial \beta}{\partial T_2} = \frac{15}{256\omega_0^2} a^4 \tag{6.1.27}$$

So, a is a constant. Now using Eqs. (6.1.23, 6.1.27) one can write

$$\beta_0 = -\frac{15}{256\omega_0^3} a^4 T_2 + \gamma \tag{6.1.28}$$

Hence, from Eq. (6.1.23)

$$\beta = \frac{3}{8\omega_0} a^2 T_1 - \frac{15}{256\omega_0^3} a^4 T_2 + \gamma \tag{6.1.29}$$

Hence the solution of the system can be written as

$$u = a\cos(\omega t + \gamma) + \frac{\varepsilon a^3}{32\omega_0^2}\left(1 - \varepsilon \frac{21a^2}{32\omega_0^2}\right)\cos 3(\omega t + \gamma) + \frac{\varepsilon^2 a^5}{1024\omega_0^4}\cos 5(\omega t + \gamma) + o(\varepsilon^3) \tag{6.1.30}$$

where, $\omega = \omega_0 + \varepsilon \frac{3a^2}{8\omega_0} - \varepsilon^2 \frac{15a^4}{256\omega_0^3} + o(\varepsilon^3)$ (6.1.31)

From Eq. (6.1.31) it may be noted that the frequency of oscillation is a function of amplitude oscillation. It may be recalled that in case of linear system frequency is independent of amplitude of oscillation.

For, the simple pendulum taking $\omega_0 = \sqrt{g/L}$, where g is the acceleration due to gravity and L is the length of the pendulum (= 1 m) , the variation of frequency with amplitude is shown in Fig. 6.1.4. The phase portrait for the periodic response considering only the first order solution is shown in Fig. 6.1.5.

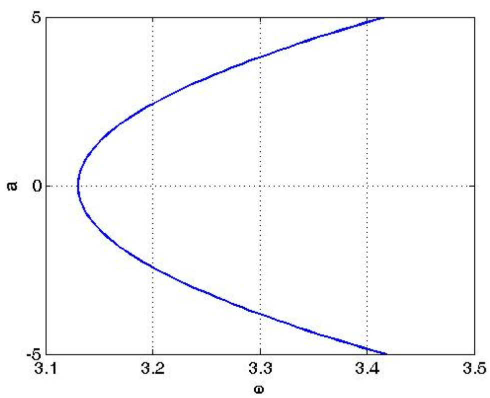


Fig. 6.1.4: Frequency amplitude relation for a simple pendulum

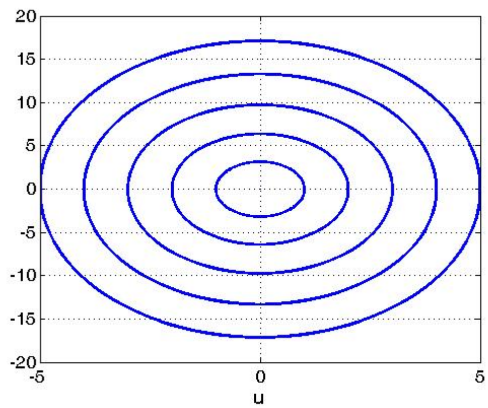


Fig. 6.1.5: Phase portrait showing the periodic motion for a simple pendulum

Exercise Problems:

1. Perform qualitative analysis for the following nonlinear systems. Using equation (6.1.30) plot the response of the system and compare both the results.

(a) $\ddot{\mu}$ +4μ+μ^3 =0

(b) $\ddot{\mu}$ +μ+μ^2 =0

(c) $\ddot{\mu}$ +9μ+0.5 μ^2+ 0.1μ^3 =0

(d) $\ddot{\mu}$ +100μ+ 10μ^3 =0

(e) $\ddot{\mu}$ +μ+0.1 μ^3+ 0.05μ^3 =0

Module 6 Lecture 2

Free vibration of nonlinear single degree of freedom conservative systems with quadratic and cubic nonlinearities.

In this lecture the free vibration response of a nonlinear single degree of freedom system with quadratic and cubic nonlinearities will be discussed with numerical examples. As studied in module 3, the equation of motion of a nonlinear single degree of freedom system with quadratic and cubic nonlinearities can be given by

$$\ddot{x} + \omega_0^2 x + \varepsilon \alpha_2 x^2 + \varepsilon \alpha_3 x^3 = 0 \tag{6.2.1}$$

Here ω_0 is the natural frequency of the system α_2 and α_3 are the coefficient of the quadratic and cubic nonlinear terms. Also ε is the book-keeping parameter which is less than 1. Using method of multiple scales the solution of this equation can be written as

$$x(t;\varepsilon) = \varepsilon x_1(T_0, T_1, T_2, ...) + \varepsilon^2 x_2(T_0, T_1, T_2, ...) + \varepsilon^3 x_3(T_0, T_1, T_2, ...) + ... \tag{6.2.2}$$

Using different time scales T_0, T_1, and T_2 where $T_n = \varepsilon^n t$ and Eq. (6.2.2) in Eq. (6.2.1) and separating the terms with different order of ε one can write the following equations.

Order of ε^1

$$D_0^2 x_1 + \omega_0^2 x_1 = 0 \tag{6.2.3}$$

Order of ε^2

$$D_0^2 x_2 + \omega_0^2 x_2 = -2D_0 D_1 x_1 - \alpha_2 x_1^2 \tag{6.2.4}$$

Order of ε^3

$$D_0^2 x_3 + \omega_0^2 x_3 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3 \tag{6.2.5}$$

The solution of (6.2.3) can be written as

$$x_1 = A(T_1, T_2)\exp(i\omega_0 T_0) + \bar{A}\exp(-i\omega_0 T_0)\,. \tag{6.2.6}$$

Here A is an unknown complex function and $\bar{A}$ is the complex conjugate of A. Substituting (6.2.3) into (6.2.4) leads to

$$D_0^2 x_2 + \omega_0^2 x_2 = -\underbrace{2i\omega_0 D_1 A \exp(i\omega_0 T_0)}_{\text{Secular term}} - \alpha_2 \left[A^2 \exp(2i\omega_0 T_0) + A\bar{A} \right] + cc \tag{6.2.7}$$

Here cc denotes the complex conjugate of the preceding terms. To have a bounded solution one should eliminate the secular term and hence

$$D_1 A = \frac{dA}{dT_1} = 0 \tag{6.2.8}$$

Therefore A must be independent of T_1. With $D_1 A = 0$ the particular solution of (6.2.7) can be written as

$$x_2 = \frac{\alpha_2 A^2}{3\omega_0^2} \exp(2i\omega_0 T_0) - \frac{\alpha_2}{\omega_0^2} A\bar{A} + cc \tag{6.2.9}$$

Substituting the expression for x_1 and x_2 from equation (6.2.6) and (6.2.9) into (6.2.5) and recalling that $D_1 A = 0$ we obtain

$$\begin{aligned} D_0^2 x_3 + \omega_0^2 x_3 = & -\underbrace{\left[2i\omega_0 D_2 A - \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{3\omega_0^2} A^2 \bar{A} \right] \exp(i\omega_0 T_0)}_{\text{Secular Term}} \\ & - \frac{3\alpha_3\omega_0^2 + 2\alpha_2^2}{3\omega_0^2} A^3 \exp(3i\omega_0 T_0) + cc \end{aligned} \tag{6.2.10}$$

To eliminate the secular terms from x_3, we must put

$$2i\omega_0 D_2 A + \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{3\omega_0^2} A^2 \bar{A} = 0 \tag{6.2.11}$$

Using $A = \frac{1}{2} a \exp(i\beta)$ where a and β are real function of T_2 in Eq. (6.2.11) and separating the result into real and imaginary parts, one obtains

$$\omega a' = 0 \text{ and } \omega_0 a \beta' + \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^2} a^3 = 0 \tag{6.2.12}$$

where the prime denotes the derivative with respect to T_2. As $a'=0$, a is a constant and

$$\beta' = -\frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^2\omega_0 a}a^3 \quad \text{or} \quad \beta = \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^3}a^2 T_2 + \beta_0 \qquad (6.2.13)$$

Here β_0 is a constant. Now using $T_2 = \varepsilon^2 t$ one may write

$$A = \frac{1}{2}a\exp\left[i\frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^3}\varepsilon^2 a^2 t + i\beta_0\right] \qquad (6.2.14)$$

Substituting Eq. (6.2.14) in the expressions for x_1 and x_2 in Eqs. (6.2.6), (6.2.9) and (6.2.2), one obtains

$$x = \varepsilon a\cos(\omega t + \beta_0) - \frac{\varepsilon^2 a^2 \alpha_2}{2\omega_0^2}\left[1 - \frac{1}{3}\cos(2\omega t + 2\beta_0)\right] + O(\varepsilon^3) \qquad (6.2.15)$$

Here $\omega = \omega_0\left[1 + \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^4}\varepsilon^2 a^2\right] + O(\varepsilon^3)$ (6.2.16)

This solution is in good agreement with the solution obtained using the Lindstedt-poincare' procedure. [Nayfeh and Mook, 1979].

Example 6.2.1: Taking $\alpha_1 = \omega_0^0$ = k=100, $\alpha_2 = 1$ and $\alpha_3 = 1.5$ in eq^n 6.2.1find the response of the system.
Using Eq. (6.2.16) the variation of frequency with amplitude is shown in figure (6.2.1). Taking two values of a (viz., *a*=0.009 and *a*=.029) the time response has been plotted in figure (6.2.2). It may be noted

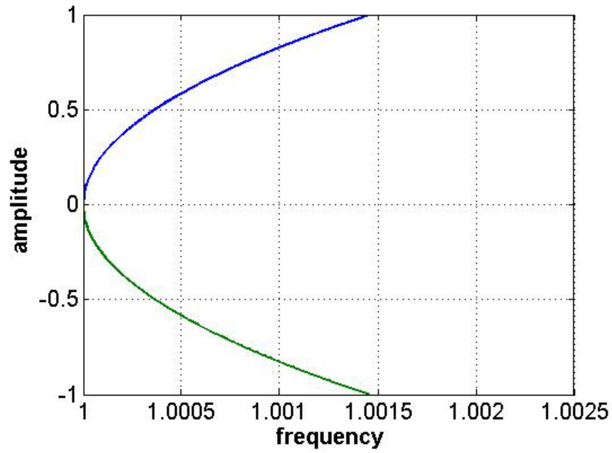
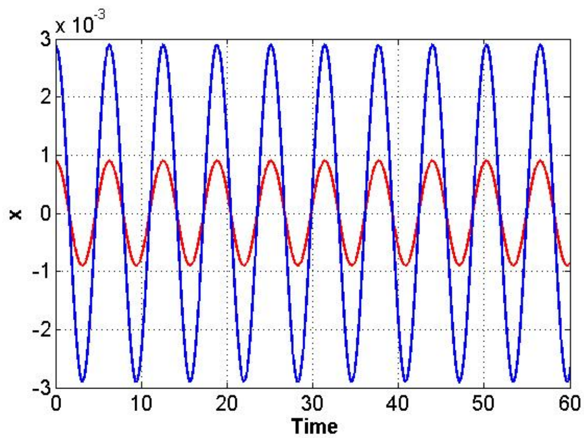


Figure 6.2.1: Variation of amplitude with frequency for



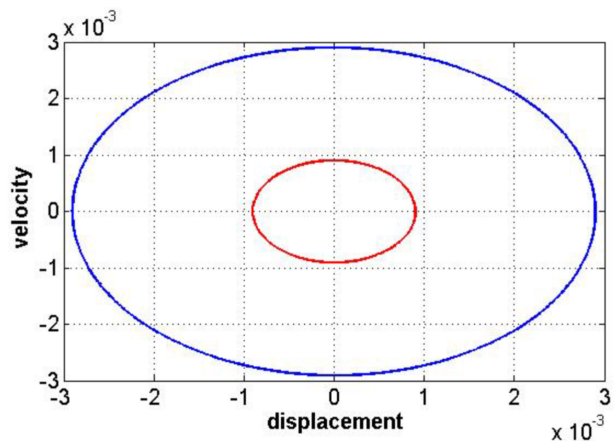


Figure 6.2.2.(a): Time response (b) Phase portrait corresponding to initial amplitude a=0.009 and a=.029

By changing the quadratic nonlinear terms α_2 from .5 to 2.5 and keeping all other parameter same figure (6.2.3) shows the variation of the frequency with amplitude of oscillation.

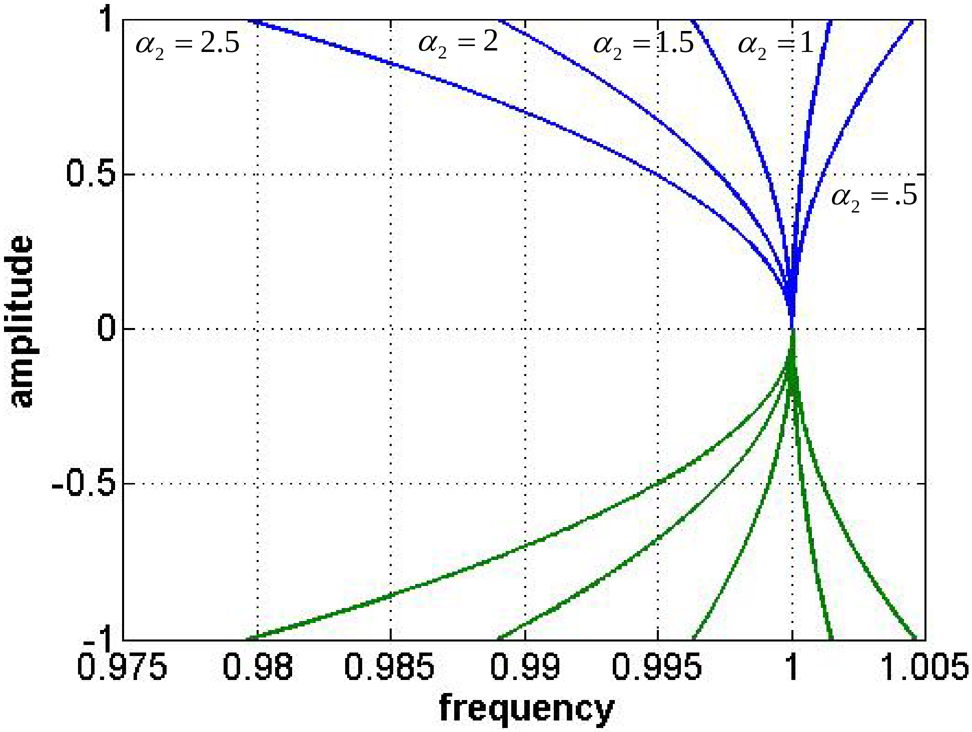


Figure 6.2.3: Variation of frequency with amplitude for different values of α_2

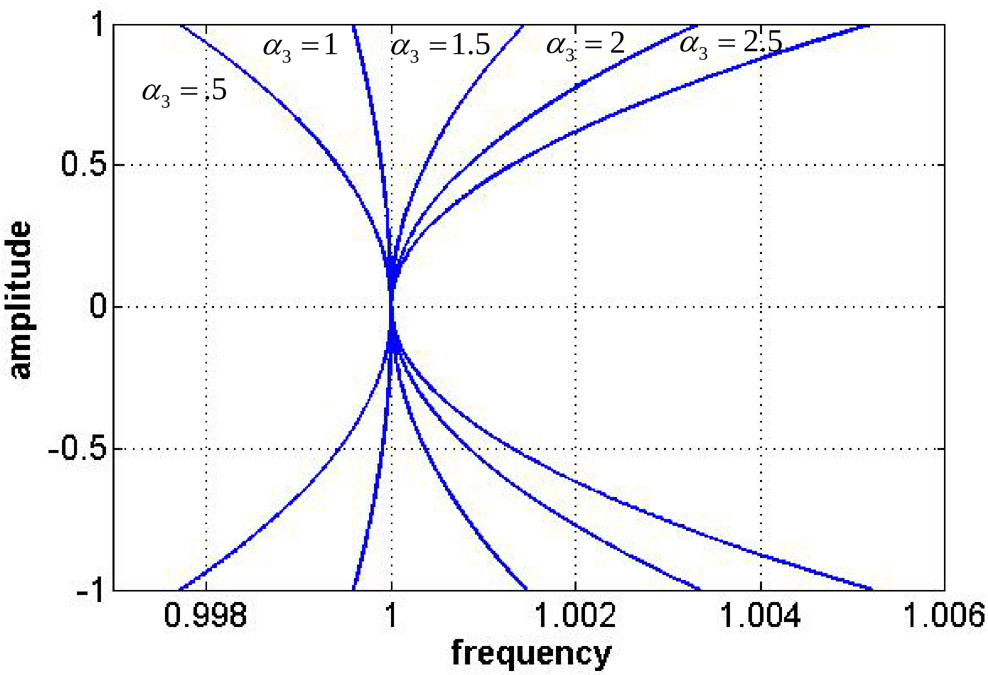


Figure 6.2.4: Variation of frequency with amplitude for different values of α_3

In Figure 6.2.3, by varying the amplitude form -1 to +1 , the frequency of the system changes and it decreases from 1.005 to .98 by increasing α_2 from .5 to 2.5. One may observe the reverse phenomenon by increasing α_3. Also the orientation of the frequency response curve changes from left to right when α_3 is greater than 1 or $\alpha_2 \leq 1$. It may be noted that for a linear system the response frequency of the system does not depend on the amplitude of the response. But in case of the nonlinear system it depends on the response amplitude. A Matlab code is given below which may be used to obtain the frequency response, time response, and phase portrait for different system parameters.

```
%NPTEL WEB MODULE6 L2--Free Vibration: Duffing Oscillator
% Response plot using the method of multiple scales
clear all
clc
omega=1;
alpha1=omega^2;
alpha2=1;
alpha3=1.5;
ep=0.1;
p1=(9*alpha3*alpha1-10*alpha2^2)/(24*alpha1^2)
i=1
for a=0:0.001:1
om=sqrt(alpha1)*(1+p1*ep^2*a.^2)
s(i,1)=a;
s(i,2)=om;
i=i+1;
end
figure(1)
plot(s(:,2),s(:,1),s(:,2),-s(:,1),'linewidth',2)
grid on
set(gca,'FontSize',15) % For changing fontsize of tick no
xlabel('\bf frequency','Fontsize',15)
ylabel('\bf amplitude','Fontsize',15)
n1=length(s)
bt0=0;
for ii=10:20:40
   om1=s(ii,2)
   a1=s(ii,1)
   T=2*pi/om1;
   jj=1;
for t=0:T/1000:10*T
p2=ep^2*a1^2*alpha2/(2*alpha1);
p3=1-(1/3)*cos(2*om1*t+2*bt0);
x(jj)=ep*a1*cos(om1*t+bt0)-p2*p3;
tt(jj)=t;
jj=jj+1;
end
n2=jj-1
for k=2:n2
xt(k-1)=(x(k)-x(k-1))/(tt(k)-tt(k-1));
```

```
end
figure(2)
plot(tt,x,'linewidth',2)
hold on
grid on
set(gca,'FontSize',15) % For changing fontsize of tick no
xlabel('\bf Time','Fontsize',15)
ylabel('\bfx','Fontsize',15)
figure(3)
plot(x(1:n2-1),xt,'linewidth',2)
hold on
end
grid on
set(gca,'FontSize',15) % For changing fontsize of tick no
xlabel('\bf displacement','Fontsize',15)
ylabel('\bf velocity','Fontsize',15)
```

Exercise problem 6.2.1:

Find the frequency response of a single degree of freedom system with mass=1 kg, stiffness=100 N/m, nonlinear quadratic and cubic stiffness parameter equal to 20 N/m^2 and 10 N/m^3 respectively. Vary the book-keeping parameter and study the variation of frequency with amplitude.

Module 6 Lecture 3

FREE VIBRATION OF NONLINEAR SINGLE DEGREE OF FREEDOM NONCONSERVATIVE SYSTEMS

In this lecture discussion on the vibration of a linear single degree of freedom system with viscous, Coulomb damping, quadratic dumping and will be carried out using method of averaging.

System with viscous damping

Let us consider a single degree of freedom system with viscous damping. The equation of motion of this system with mass m, stiffness k and damping c can be written as

$$m\ddot{u} + ku + c\dot{u} = 0\,. \tag{6.3.1}$$

The same system can be written using the term natural frequency ω_n, damping ratio ζ as

$$\ddot{u} + \omega_n^2 u + 2\zeta\omega_n\dot{u} = 0 \tag{6.3.2}$$

$$\text{Or, } \ddot{u} + \omega_n^2 u = f\left(u, \dot{u}\right) = -2\zeta\omega_n\dot{u} = -2\varepsilon\mu\dot{u} \tag{6.3.3}$$

By using Krylov-Bogoliubov method of averaging for an under damped system ($\zeta < 1$) the solution can be written as

$$u = a\sin(\omega_n t + \beta) \tag{6.3.4}$$

where

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega_n}\int_0^{2\pi}\sin\phi\left(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\right)d\phi \tag{6.3.5}$$

$$\dot{\beta} = -\frac{\varepsilon}{2\pi\omega_n a}\int_0^{2\pi}\cos\phi\left(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\right)d\phi \tag{6.3.6}$$

Substituting expression for $f(u,\dot{u})$ from Eq. (6.3.3) in Eq. (6.3.5) and Eq. (6.3.6), one obtains

$$\dot{a} = -\frac{\varepsilon\mu a}{\pi}\int_0^{2\pi}\sin^2\phi\, d\phi = -\varepsilon\mu a \tag{6.3.7}$$

and $\dot{\beta} = -\frac{\varepsilon\mu}{\pi}\int\sin\phi\cos\phi\, d\phi = 0$. (6.3.8)

Solving Eq. (6.3.7) and Eq. (6.3.8) yields

$$a = a_0\exp(-\varepsilon\mu t) = a_0\exp(-\zeta\omega_n t), \quad \beta = \beta_0 \tag{6.3.9}$$

Here a_0 and β_0 are the initial displacement and phase of the response. Substituting Eq. (6.3.9) in Eq. (6.3.4) one obtains the following equation.

$$u = a_0\exp(-\zeta\omega_n t)\cos(\omega_n t + \beta_0) + O(\varepsilon) \tag{6.3.10}$$

This equation is same as the expression one may obtain by finding the complementary function of the differential equation (6.3.2). Using the u_0 and $\dot{u}_0$ as the initial displacement and velocity respectively, one may write Eq. (6.3.10) as

$$u = \exp(-\zeta\omega_n t)\left[u_0\cos\omega_d t + \left((\dot{u}_0 + \zeta\omega_n u_0)/\omega_d\right)\sin\omega_d t\right] \tag{6.3.11}$$

Where the damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

For over damped ($\zeta > 1$) system one may use the following expressions for the response.

$$\begin{aligned} u = &\left(\dot{u}_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n u_0\right)/\left(2\omega_n\sqrt{\zeta^2-1}\right)\exp\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t + \\ &\left(-\dot{u}_0 + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n u_0\right)/\left(2\omega_n\sqrt{\zeta^2-1}\right)\exp\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t \end{aligned} \tag{6.3.12}$$

For critically damped ($\zeta = 1$) system one may write the response as follows.

$$u = \left(u_0 + (\dot{u}_0 + \omega_n u_0)t\right)\exp(-\omega_n t) \tag{6.3.13}$$

A Matlab code is given below to plot the under damped, critically damped and over damped response of a system as shown in figure 6.3.1.

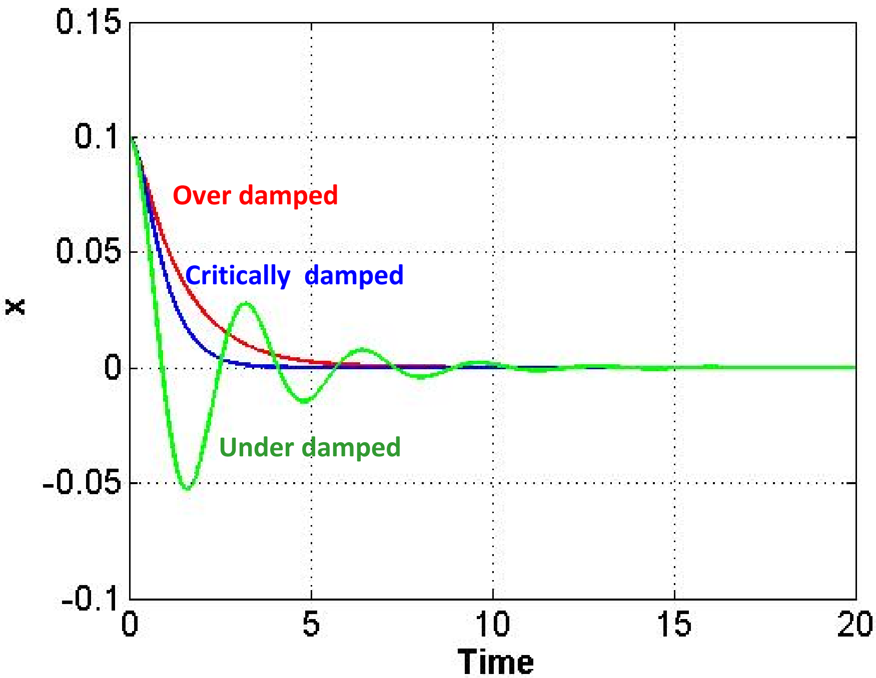


Figure 6.3.1(a): Time response of a linear single degree of freedom with viscous damping.

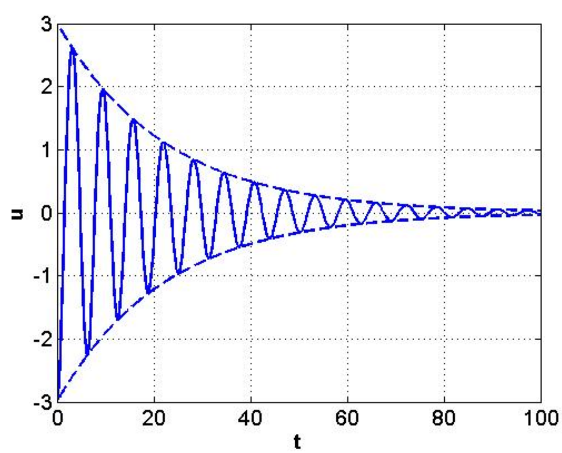


Figure 6.3.1(b): Time response of the system with linear damping. $\left(a_0 = 3, \omega_n = 1, \varepsilon = 0.5, \mu = 0.09, \beta_0 = -3.15\right)$

Using Eq. (6.3.10) the time response is shown in Fig. 6.3.1(b). It may be noted that the response decreases exponentially. The corresponding Matlab code is given below

Matlab code 6.3.1:

```
%Free Vibration response of a linear single degree of freedom system

x0=0.1;
xt0=0.001;
wn=2;
zeta=1.5;
t=0:0.001:20;

%overdamped

z1=-zeta+sqrt(zeta^2-1);
z2=-zeta-sqrt(zeta^2-1);
z3=2*wn*sqrt(zeta^2-1);
A=(xt0-z2*wn*x0)/z3;
B=(-xt0+z1*wn*x0)/z3;
x1=A*exp(z1*wn*t)+B*exp(z2*wn*t);

%critically damped

x2=(x0+(xt0+wn*x0)*t).*exp(-wn*t)

%underdamped

zt=0.2  %Damping factor
wd=wn*sqrt(1-zt^2);
x3=exp(-zt*wn*t).*(((xt0+zt*wn*x0)/wd).*sin(wd*t)+x0*cos(wd*t));

plot(t,x1,'r',t,x2,'b',t,x3,'g','linewidth',2)
grid on

set(gca,'FontSize',15) % For changing fontsize of tick no
xlabel('\bf Time','Fontsize',15)
ylabel('\bf x','Fontsize',15)
```

Matlab code 6.3.2:

```
% plotting of linear damping (Eq. 6.3.10).
clc
clear all
a0=3;
ep=.5;
mu=.09;
t=0:0.1:100;
omega=1;
beta=-3.15;
a=a0*exp(-ep*mu*t);
u=a0*exp(-ep*mu*t).*cos(omega*t+beta);
plot(t,u,t,a,'--',t,-a,'--')
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('u','fontsize',14,'fontweight','b');
grid on
```

Single degree of freedom system with quadratic damping.

Here, the equation of motion of the system can be written as

$$\ddot{u} + \omega_n^2 u = f\left(u, \dot{u}\right) = -\varepsilon \dot{u}\,|\,\dot{u}\,| \tag{6.3.14}$$

Similar to viscous damping here also using KB method the solution can be written as

$$u = a\sin(\omega_n t + \beta) \tag{6.3.15}$$

Here, a and β can be given by Eq. (6.3.5) and (6.3.6). Now using the expression for $f\left(u,\dot{u}\right)$ in Eq. (6.3.5) and (6.3.6), one may write

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega_n}\int_0^{2\pi}\sin\phi\Big(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\Big)d\phi = -\frac{\varepsilon a^2\omega_n}{2\pi}\int_0^{2\pi}\sin^2\phi\left|\sin\phi\right|d\phi$$

$$= -\frac{\varepsilon a\omega_n}{2\pi}\left[\int_0^{2\pi}\sin^3\phi\,d\phi - \int_\pi^{2\pi}\sin^3\phi\,d\phi\right] = -\frac{4}{3\pi}\varepsilon a^2\omega_n \tag{6.3.16}$$

and $$\dot{\beta} = -\frac{\varepsilon}{2\pi\omega_n a}\int_0^{2\pi}\cos\phi\Big(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\Big)d\phi = -\frac{\varepsilon\omega_n a}{2\pi}\int_0^{2\pi}\sin\phi\cos\phi\left|\sin\phi\right|d\phi = 0 \tag{6.3.17}$$

Solving Eq. (6.3.16) and Eq. (6.3.17) one may write Eq. (6.3.15) as

$$u = \frac{a_0}{1 + \dfrac{4\varepsilon\omega_n a_0}{3\pi}t}\cos\left(\omega_n t + \beta_0\right) + O\left(\varepsilon\right) \tag{6.3.18}$$

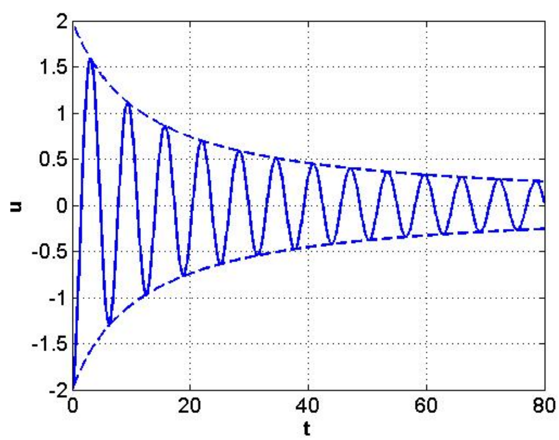


Figure 6.3.2: Time response of the system with quadratic damping. $\left(a_0 = 2, \omega_n = 1, \varepsilon = 0.1, \beta_0 = -3.15\right)$

Using Eq. (6.3.18) the time response is shown in Fig. 6.3.2. It may be noted that unlike the linear system the response does not decreases exponentially but decreases algebraically. The corresponding Matlab code is given in Matlab code 6.3.3.

Matlab code 6.3.3:

```
% plotting of quadratic damping. (Eq. 6.3.18)
clc
clear all
a0=2;
ep=.1;
t=0:0.1:80;
omega=1;
beta=-3.15;
a=a0./(1+(4*ep*omega*a0*t)/(3*pi));
u=a.*cos(omega*t+beta);
plot(t,u,t,a,'--',t,-a,'--')
% title('SYSTEM WITH QUADRATIC DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('u','fontsize',14,'fontweight','b');
grid on
```

System with Coulomb damping

In this case the equation of motion of the system can be given by

$$m\ddot{x} + kx + F_c = 0$$

$$F_c = \mu N \operatorname{sgn}(\dot{x}) = \begin{cases} \mu N \text{ for } \dot{x} > 0 \\ -\mu N \text{ for } \dot{x} < 0 \end{cases} \tag{6.3.19}$$

Using method of averaging this equation can be written as

$$\ddot{x} + \omega_0^2 x = f = -F_c / m = \begin{cases} -\mu g \text{ for } \dot{x} > 0 \\ \mu g \text{ for } \dot{x} < 0 \end{cases} \tag{6.3.20}$$

The solution of the above equation can be written as

$x = a\cos(\omega_0 t + \beta)$

where

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega_n}\int_0^{2\pi} \sin\phi \left(f\left(a\cos\phi, -\omega_n a \sin\phi\right)\right) d\phi = -\frac{\varepsilon\mu g}{2\pi\omega_n}\left[\int_0^{\pi} \sin\phi d\phi - \int_{\pi}^{2\pi} \sin\phi d\phi\right] = -\frac{2\varepsilon\mu g}{\pi\omega_n} \tag{6.3.21}$$

$$\dot{\beta} = -\frac{\varepsilon}{2\pi\omega_n a}\int_0^{2\pi} \cos\phi \left(f\left(a\cos\phi, -\omega_n a \sin\phi\right)\right) d\phi = -\frac{\varepsilon\mu g}{2\pi\omega_n}\left[\int_0^{\pi} \cos\phi d\phi - \int_{\pi}^{2\pi} \cos\phi d\phi\right] = 0 \tag{6.3.22}$$

Integrating Eq. (6.3.21) and Eq. (6.3.22) one may obtain

$$a = a_0 - \frac{2\pi\mu g}{\pi\omega_n}t,\ \beta = \beta_0 \tag{6.3.23}$$

Substituting Eq. (6.3.23) in Eq. (6.3.20) the response of the system can be written as

$$x = \left(a_0 - \frac{2\pi\mu g}{\pi\omega_n}t\right)\cos\left(\omega_n t + \beta_0\right) + O(\varepsilon) \tag{6.3.24}$$

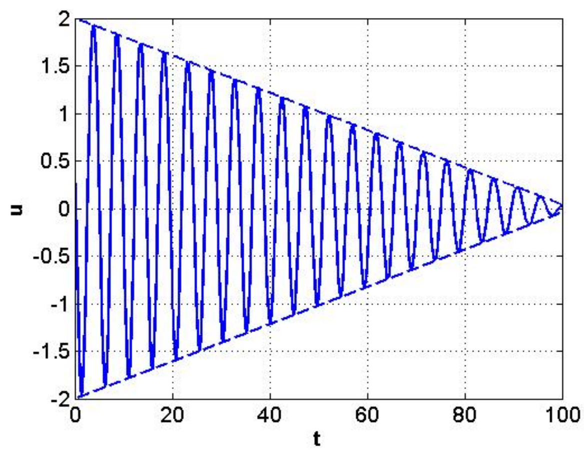


Figure 6.3.3: Time response of the system with Coulomb damping.
$\left(a_0 = 2, \omega_n = 1.3, \varepsilon = .1, \mu g = 0.4, \beta_0 = -3.15\right)$

Here it may be noted that the response of the system decreases linearly. A Matlab code is given below.

Matlab code 6.3.4:

```
% plotting of time response for system with Coulomb damping. (Eq.6.3.24)
clc
clear all
a0=2;
ep=.1;
mug=.4;
t=0:0.1:100;
omega=1.3;
phi=omega;
beta=-3.15;
a=a0-((2*ep*mug*t)/(pi*omega));
u=a.*cos(omega*t+phi);
plot(t,u,t,a,'--',t,-a,'--')
% title('SYSTEM WITH COULOMB DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('u','fontsize',14,'fontweight','b');
grid on
```

Exercise problem :

1. Find the response of a single degree of freedom system with mass 1 kg, stiffness 100 N/m and damping factor 10 N.s/m. Plot the time response and phase portrait. Also plot the phase portrait considering coulomb damping and quadratic damping. Develop a Matlab code for finding the time response and phase portrait by using second order governing differential equation of motion (Use Runge-Kutta method).

Hints-The Matlab code for the system with viscous damping is given below

Matlab code 6.3.5:

```
%Use Runge-Kutta method to obtain the response of a sdof vibrating system

m=input( 'mass of the system in kg =  ' )
k=input( 'Stiffness of the system in N/m =  ' )
c=input( 'damping  factor of the system  in N.S/m=  ' )
u0=input('initial Displacement  in  m= '
v0=input('initial velocity  in  m= '
omega_n=sqrt(k/m),
zeta=c/(2*m*omega_n);

if (zeta>1)
display(over damp system)
u=
  end
if(zeta==1)
display('critically damped system')

u=
end

if(zeta<1)
display('under damped system')
u=u0*sin(omega_n*t)+(v0/omega_n)*cos(omega_n*t)
end

plot(t,u)

[T,u]=ode45(@ex631f,[0,20],[0.1,0.01]

Xlabel
Ylabel
Title
function
Ex631f
```

2. Find the response of a single degree of freedom system with Hysteretic damping. The equation of motion in this case is given by.

$$\ddot{x}+\omega_0^2 x=\varepsilon f$$

$$-f=\begin{cases} x+x_s-x_b & x_b \ge x \ge x_c \\ -x_s & x_c \ge x \ge x_d \\ x-x_s-x_d & x_a \ge x \ge x_d \\ x_s & x_d \ge x \ge x_a \end{cases}$$

$$x_c = x_b - 2x_s \quad \text{and} \quad x_a = x_d + 2x_s$$

3. Find the response of a single degree of freedom system with material damping by considering (a) Maxwell model (spring and dashpot) in series, (b) Kelvin-Viogot Model (spring and dashpot) in parallel. Consider soft spring with cubic nonlinearity in both the cases.

Module 6 Lecture 4

FREE VIBRATION OF SYSTEMS WITH NEGATIVE DAMPING

In this lecture initially the system with negative damping will be discussed. Then the free vibration response of systems similar to van der Pol type of oscillator will be discussed with the help of numerical examples. Finally the nonlinear response of a simple pendulum with viscous damping will be illustrated.

There are many systems which can be modeled as a system with negative damping. This type of system particularly occurs in control system where the derivative gains if not properly adjusted will give rise to negative damping. Also this type of damping can be found in the high voltage transmission lines. The equation of motion for this type of system for example that of a Rayleigh oscillator can be given by

$$\ddot{u}+\omega_0^2 u=\varepsilon f=\varepsilon\left(\dot{u}-\dot{u}^3\right) \tag{6.4.1}$$

There are many systems which can be modeled as a system similar Rayleigh oscillator.

Using KB method the response amplitude a and phase β of the system can be given by

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega_n}\int_0^{2\pi}\sin\phi\left(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\right)d\phi = -\frac{\varepsilon a}{2\pi}\int_0^{2\pi}\left(\sin^2\phi\omega_n^2 a^2\sin^4\phi\right)d\phi$$

$$= \frac{1}{2} - \varepsilon a\left(1 - \frac{3}{4}\omega_n^2 a^2\right) \tag{6.4.2}$$

$$\dot{\beta} = -\frac{\varepsilon}{2\pi\omega_n a}\int_0^{2\pi}\cos\phi\left(f\left(a\cos\phi, -\omega_n a\sin\phi\right)\right)d\phi = -\frac{\varepsilon}{2\pi}\left[\int_0^{2\pi}\left(1-\omega_0^2 a^2\sin^2\phi\right)\sin\phi\cos\phi d\phi\right] = 0 \tag{6.4.3}$$

Solving Eq. (6.4.2) one can write

$$a^2 = \frac{a_0^2}{\frac{3}{4}\omega_0^2 a_0^2 + \left(1 - \frac{3}{4}\omega_0^2 a_0^2\right)\exp\left(-\varepsilon t\right)} \tag{6.4.4}$$

The time responses obtained by using Eq. (6.4.4) are shown in figure 6.4.1 for large and small initial disturbance condition. It is observed that while with large initial disturbance the response decreases to

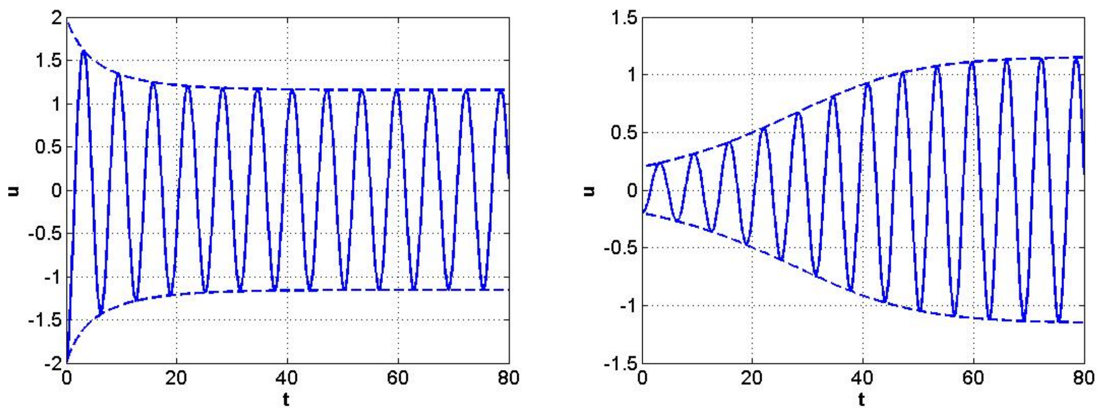


Figure 6.4.1: Time response of the system with Rayleigh damping. (a) Large initial disturbance ($a_0 = 2$) (b) Small initial disturbance ($a_0 = 0.2$); $\left(\omega_n = 1.3, \varepsilon = .1, \mu g = 0.4, \beta_0 = -3.15\right)$

attain the steady state periodic response, and in case of small initial disturbance the response grows to attain the steady state periodic response. Matlab code 6.4.1 may be used for plotting the time response of the system with Rayleigh damping.

Matlab code 6.4.1:

```
% plotting of time response of the system with Rayleigh damping.Equation no-6.4.4.
clc
clear all
a0=2;
%a0=.2;
ep=.1;
t=0:0.1:80;
omega=1;
```

```
beta=-3.15;
a1=0.75*(omega*a0)^2;
a2=a0.^2./(a1+(1-a1)*exp(-ep*t));
a=sqrt(a2);
u=a.*cos(omega*t+beta);
plot(t,u,t,a,'--',t,-a,'--')
% title('SYSTEM WITH Rayleigh DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('u','fontsize',14,'fontweight','b');
grid on
```

THE VAN DER POL OSCILLATOR

There are many systems which can be modeled as a system similar to van der Pol's oscillator which is named after the Dutch physicist Balthasar van der Pol (27 January 1889 – 6 October 1959). Mostly this equation is used in electrical circuits but it can also be used in some mechanical system where self oscillation takes place due to negative damping.

The van der Pol's equation can be written as

$$\frac{d^2u}{dt^2}+u=\varepsilon(1-u^2)\frac{du}{dt} \tag{6.4.5}$$

Using method of multiple scales the solution of this equation can be given by

$$u(t;\varepsilon)=u_0(T_0,T_1)+\varepsilon u_1(T_0,T_1)+....... \tag{6.4.6}$$

Where $T_n=\varepsilon^n t,\ n=0,1,2,\cdots$. Using Eq. (6.4.6) in Eq. (6.4.5) and separating the terms with different order of ε, one obtains the following equations.

$$D_0^2u_0+u_0=0 \tag{6.4.7}$$

$$D_0^2u_1+u_1=-2D_0D_1u_0+(1-u_0^2)D_0u_0 \tag{6.4.8}$$

$$D_0^2u_2+u_2=-2D_0D_1u_1-D_1^2u_0-2D_0D_2u_0+(1-u_0^2)D_0u_1+(1-u_0^2)D_1u_0-2u_0u_1D_0u_0 \tag{6.4.9}$$

The solution of Eq.(6.4.7) can be written as

$$u_0=A(T_1,T_2,)e^{iT_0}+\bar{A}(T_1,T_2,)e^{-iT_0} \tag{6.4.10}$$

Substituting Eq. (6.4.10) in Eq. (6.4.8)

$$D_0^2u_1+u_1=\underbrace{-i(2D_1A-A+A^2\bar{A}]}_{\text{secular term}}e^{iT_0}-iA^3e^{3iT_0}+cc \tag{6.4.11}$$

Eliminating the secular term marked in Eq. (6.4.11) one can write

$$2D_1A = A - A^2\bar{A} \tag{6.4.12}$$

Now the solution of Eq.(6.4.11) can be written as

$$u_1 = B(T_1,T_2)e^{iT_0} + \frac{1}{8}iA^3e^{3iT_0} + cc \tag{6.4.13}$$

$$A = \frac{1}{2}a(T_1,T_2)\exp\left(i\phi(T_1,T_2)\right) \tag{6.4.14}$$

Substituting Eq. (6.4.14) in Eq. (6.4.12) one can write

$$2\left(\frac{1}{2}\frac{\partial a}{\partial T_1}\exp\left(i\phi\right) + \frac{1}{2}ia\frac{\partial \phi}{\partial T_1}\exp\left(i\phi\right)\right) = \frac{1}{2}a\exp\left(i\phi\right) - \left(\frac{1}{2}a\exp\left(i\phi\right)\right)^2\frac{1}{2}a\exp\left(-i\phi\right) \tag{6.4.15}$$

Separating the real and imaginary terms one can write

$$\frac{\partial \phi}{\partial T_1} = 0, \ \ \frac{\partial a}{\partial T_1} = \frac{1}{2}\left(1 - \frac{1}{4}a^2\right)a \tag{6.4.16}$$

Hence, $\phi = \phi(T_2)$, and $a^2 = \dfrac{4}{1 + c(T_2)e^{-T_1}}$ (6.4.17)

So the first order solution of the system can be given by

$$u = a\cos t + o(\varepsilon) \tag{6.4.18}$$

Where

$$a^2 = \frac{4}{1 + \left(\dfrac{4}{a_0^2} - 1\right)\exp\left(-\varepsilon t\right)} \tag{6.4.19}$$

To obtain the second or higher order solution one may use the expression for u_0 and u_1 in Eq. (6.4.9) which yields the following equation.

$$D_0^2u_2 + u_2 = Q(T_1,T_2,)e^{iT_0} + \bar{Q}(T_1,T_2,)e^{iT_0} + \text{NST} \tag{6.4.20}$$

Where NST contains non-secular terms.

$$Q = -2iD_1B + i(1 - 2A\bar{A})B - iA^2\bar{B} - 2iD_2A - D_1^2A + (1 - 2A\bar{A})D_1A - A^2D_1\bar{A} + \frac{A^3\bar{A}^2}{8} \tag{6.4.21}$$

Now again substituting Eq. (6.4.14) in the secular term of Eq. (6.4.20) and separating the real and imaginary parts of the resulting equation one obtains the following equation.

$$\frac{\partial a}{\partial T_2} = 0, a = a(T_3) \tag{6.4.22}$$

$$2\frac{\partial b}{\partial T_1}-\frac{2}{a}\frac{da}{dT_1}b=-2a\left(\frac{d\phi}{dT_2}+\frac{1}{16}\right)+\left(\frac{7}{16}a^2-\frac{1}{4}\right)\frac{da}{dT_1} \tag{6.4.23}$$

$$d\left(\frac{b}{a}\right)=-\left(\frac{d\phi}{dT_2}+\frac{1}{16}\right)dT_1+\left(\frac{7}{32}a-\frac{1}{8a}\right)da \tag{6.4.24}$$

Integrating one obtains

$$b=-a\left(\frac{d\phi}{dT_2}+\frac{1}{16}\right)T_1+\frac{7}{64}a^3-\frac{1}{8}a\ln a+ab_0(T_2) \tag{6.4.25}$$

In order that the solution to be bounded for all T_1, the coefficient of T_1 in the above equation for b must vanish. Hence one obtains $\left(\frac{d\phi}{dT_2}+\frac{1}{16}\right)=0$.

So , $\phi=-\frac{1}{16}T_2+\phi_0$ (6.4.26)

Here ϕ_0 is a constant. Now using the expression for u_0 and u_1 the second order solution can be given by

$$u=a\cos\left[\left(1-\frac{1}{16}\varepsilon^2\right)t+\phi_0\right]-\varepsilon\left\{\begin{array}{l}\left(\frac{7}{64}a^2-\frac{1}{8}\ln a+ab_0\right)\sin\left[\left(1-\frac{1}{16}\varepsilon^2\right)t+\phi_0\right]\\+\frac{1}{32}a^3\sin 3\left[\left(1-\frac{1}{16}\varepsilon^2\right)t+\phi_0\right]\end{array}\right\}+o(\varepsilon^2) \tag{6.4.27}$$

Here a can be given by the Eq. (6.4.19). Equation (6.4.27) can also be written as

$$u=a\cos(t-\theta)-\frac{1}{32}\varepsilon a^3\sin 3(t-\theta)+o(\varepsilon^2) \tag{6.4.28}$$

where $\theta=\frac{1}{16}\varepsilon^2 t+\frac{1}{8}\varepsilon\ln a-\frac{7}{64}\varepsilon a^2+\theta_0$ (6.4.29)

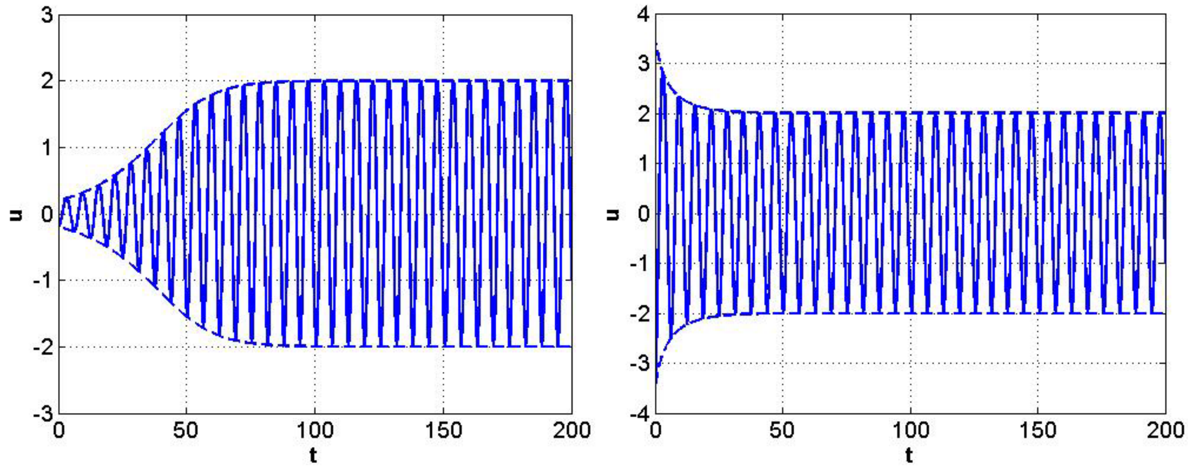


Figure 6.4.2: Time response of the system with van der Pol's equation $\left(\omega_n = 1, \varepsilon = .1, \theta_0 = -3.15\right)$ (a) Small initial disturbance ($a_0 = 0.2$) (b) Large initial disturbance ($a_0 = 3.5$).

Example: 6.4.2 Simple pendulum with quadratic damping

The equation of motion of a simple pendulum with quadratic damping can be given by

$$\ddot{\theta} + 2\varepsilon\mu\dot{\theta}\left|\dot{\theta}\right| + \omega_0^2 \sin\theta = \ddot{\theta} + 2\varepsilon\mu\dot{\theta}\left|\dot{\theta}\right| + \omega_0^2\left(\theta - \frac{1}{6}\theta^3\right) = 0 \tag{6.4.30}$$

Using method of multiple scales with different time scales $T_n = \varepsilon^n t,\ n = 0,1,2,\cdots$ and writing the motion of the pendulum θ as follows

$$\theta(t;\varepsilon) = \varepsilon\theta_1(T_0, T_1, T_2) + \varepsilon^2\theta_2(T_0, T_1, T_2) + \varepsilon^3\theta_3(T_0, T_1, T_2) + ... \tag{6.4.31}$$

and substituting Eq. (6.4.31) in (6.4.29) and separating the terms with different order of ε one obtains the following equations.

$$D_0^2\theta_1 + \omega_0^2\theta_1 = 0 \tag{6.4.32}$$

$$D_0^2\theta_2 + \omega_0^2\theta_2 = -2D_0D_1\theta_1 \tag{6.4.33}$$

$$D_0^2\theta_3 + \omega_0^2\theta_3 = -2D_0D_1\theta_2 - 2D_0D_2\theta_1 - D_1^2\theta_1 - 2\mu D_0\theta_1\left|D_0\theta_1\right| + \frac{1}{6}\theta_1^3 \tag{6.4.34}$$

The solution of Eq. (6.4.32) can be written as

$$\theta_1 = A(T_1, T_2)\exp(i\omega_0 T_0) + cc \tag{6.4.35}$$

Using Eq. (6.4.35) in (6.4.33) it can be written as

$$D_0^2\theta_2 + \omega_0^2\theta_2 = -2D_0D_1\theta_1 = -2\left(i\omega_0 D_1 A\exp\left(i\omega_0 T_0\right)\right) + cc \tag{6.4.36}$$

As the terms in the right hand sides are secular terms, it will be eliminated if $D_1 A = 0$. Hence, A is not a function of T_1. After eliminating the secular term, the solution of Eq. (6.4.36) will contain only auxiliary part and hence, θ_2 may be dropped. So one can write

$$\theta_1 = A(T_2)\exp\left(i\omega_0 T_0\right) + cc \tag{6.4.37}$$

Now substituting (6.4.37) in (6.4.34) one obtains

$$D_0^2\theta_3 + \omega_0^2\theta_3 = -2i\omega_0\left[A'\exp\left(i\omega_0 T_0\right) - \bar{A}'\exp\left(-i\omega_0 T_0\right)\right] - 2\mu\left[i\omega_0 A\exp\left(i\omega_0 T_0\right) - i\omega_0\bar{A}\exp\left(-i\omega_0 T_0\right)\right]$$
$$\left|i\omega_0 A\exp\left(i\omega_0 T_0\right) - i\omega_0\bar{A}\exp\left(-i\omega_0 T_0\right)\right| + \frac{1}{2}A^2\bar{A}\exp\left(i\omega_0 T_0\right) + \frac{1}{2}\bar{A}A^2\exp\left(-i\omega_0 T_0\right) +$$
$$\underbrace{\frac{1}{6}A^3\exp\left(3i\omega_0 T_0\right) + \frac{1}{6}\bar{A}^3\exp\left(-3i\omega_0 T_0\right)}_{\text{non secular terms}} \tag{6.4.38}$$

Now substituting $A = \frac{1}{2}a\exp(i\beta)$ in Eq. (6.4.38) one can write

$$D_0^2\theta_3 + \omega_0^2\theta_3 = 2\omega_0 a'\sin\gamma + 2\omega_0 a\beta'\cos\gamma + \frac{1}{8}a^3\cos\gamma + \frac{1}{24}a^3\cos 3\gamma + 2\mu\omega_0^2 a^2\sin\gamma\left|\sin\gamma\right| \tag{6.4.39}$$

where $\gamma = \omega_0 t + \beta$. It may be noted that the damping term is periodic and can be expanded in Fourier series as

$$\sin\gamma\left|\sin\gamma\right| = \sum_{n=1}^{\infty} f_n \sin n\gamma \tag{6.4.40}$$

Where $f_n = \frac{8}{3\pi}$. The secular terms will be eliminated from () if

$$a' + \frac{8\mu\omega_0}{3\pi}a^2 = 0, \qquad \beta' + \frac{a^2}{16\omega_0} = 0 \tag{6.4.41}$$

Solving Eq. (6.4.41) one may write

$$a = \frac{3\pi a_0}{3\pi + 8\mu\omega_0 a_0 T_2}, \qquad \beta = \frac{9\pi^2 a_0}{128\mu\omega_0^2\left(3\pi + 8\mu\omega_0 a_0 T_2\right)} + \beta_0 \tag{6.4.42}$$

Hence, the response of the system can be given by

$$\theta = \frac{3\pi\theta_0}{3\pi + 8\hat{\mu}\omega_0\theta_0 t}\cos\left[\omega_0 t + \frac{9\pi^2\theta_0}{128\omega_0^2\hat{\mu}\left(3\pi + 8\hat{\mu}\omega_0\theta_0 t\right)} - \frac{3\pi\theta_0}{128\omega_0^2\hat{\mu}}\right] + O\left(\varepsilon^3\right) \qquad (6.4.43)$$

EXERCISE PROBLEM:

1. Obtain the response of a simple pendulum with quadratic damping. The equation of motion for this system can be written as follows.

(a) $\ddot{\theta} - 2\mu\left(\dot{\theta} - \dot{\theta}\left|\dot{\theta}\right|\right) + \omega_0^2 \sin\theta = 0$, (b) $\ddot{\theta} + 2\mu\left(\dot{\theta} - \dot{\theta}\left|\dot{\theta}\right|\right) + \omega_0^2 \sin\theta = 0$

Plot the phase portrait and discuss about the equilibrium solution.

2. Obtain the response of a simple pendulum with viscous damping. The equation of motion for this system can be written as

$$\ddot{\theta} + 2\mu\dot{\theta} + \omega_0^2 \sin\theta = \ddot{\theta} + 2\mu\dot{\theta} + \omega_0^2\left(\theta - \frac{1}{6}\theta^3\right) = 0$$

Plot the phase portrait and discuss about the equilibrium solution. Compare the results with that obtained in problem 6.4.1.

Module 6 Lecture 5

Forced Vibration of single degree of freedom system with cubic nonlinearities

In this lecture, the response of a nonlinear single degree of freedom system with cubic nonlinearies will be discussed considering a weak forcing function. The simplest form of this equation can be given by the forced Duffing equation as follows.

$$\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha u^3 = \varepsilon f \cos\Omega t \tag{6.5.1}$$

As discussed in the previous lectures, in the absence of external force, the free vibration response amplitude of such system is a function of the natural frequency ω_0 of the system. Similar to linear vibration of the system here we may consider the behaviour of the system near the resonance condition, i.e., when the external frequency is equal to the natural frequency of the system. This condition is known as the primary resonance condition ($\Omega \approx \omega_0$). In case of multi-degree of freedom system one may reduce the system into a number of single degree of freedom system and follow a procedure as outlined here.

To study the behaviour of the system near the primary resonance condition, one may use the detuning parameter which represents the nearness of the external frequency to that of the natural frequency. Hence one may write

$$\Omega = \omega_0 + \varepsilon\sigma \tag{6.5.2}$$

Using method of multiple scales the solution of Eq. (6.5.1) can be written as

$$u(t;\varepsilon) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + ... \tag{6.5.3}$$

Where $T_n = \varepsilon^n t$. Substituting Eq. (6.5.3) in Eq. (6.5.1) and separating terms with different order of ε, one obtains the following equations.

$$D_0^2 u_0 + \omega_0^2 = 0 \tag{6.5.4}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - 2\mu D_0 u_0 - \alpha u_0^3 + f\cos(\omega_0 T_0 + \sigma T_1) \tag{6.5.5}$$

The solution of Eq. (6.5.4) can be written as

$$u_0 = A(T_1)\exp(i\omega_0 T_0) + \bar{A}(T_1)\exp(-i\omega_0 T_0) \tag{6.5.6}$$

Substituting Eq. (6.5.6) in Eq. (6.5.5) one obtains the following equation.

$$D_0^2 u_1 + \omega_0^2 u_1 = \underbrace{-\left[2i\omega_0\left(A' + A\mu\right) + 3\alpha A^2\bar{A}\right]\exp\left(i\omega_0 T_0\right)}_{\text{Secular term}} - \alpha A^3\exp\left(3i\omega_0 T_0\right) + \underbrace{\frac{1}{2} f\exp\left[i\left(\omega_0 T_0 + \sigma T_1\right)\right]}_{\text{Mixed Secular term}} + cc \tag{6.5.7}$$

To eliminate the secular and near secular terms from Eq. (6.5.7), one can write

$$2i\omega_0\left(A'+\mu A\right)+3\alpha A^2\bar{A}-\frac{1}{2}f\exp\left(i\sigma T_1\right)=0\,. \tag{6.5.8}$$

Now substituting $A=\frac{1}{2}a\exp\left(i\beta\right)$ in Eq.(6.5.8) and separating the real and imaginary parts following reduced equations are obtained.

$$a'=-\mu a+\frac{1}{2}\frac{f}{\omega_0}\sin\left(\sigma T_1-\beta\right) \tag{6.5.9}$$

$$a\beta'=\frac{3}{8}\frac{\alpha}{\omega_0}a^3-\frac{1}{2}\frac{f}{\omega_0}\cos\left(\sigma T_1-\beta\right) \tag{6.5.10}$$

To write these two equations in its autonomous form one may use $\gamma=\sigma T_1-\beta$ and obtained the following equations.

$$a'=-\mu a+\frac{1}{2}\frac{f}{\omega_0}\sin\gamma \tag{6.5.11}$$

$$a\gamma'=\sigma a-\frac{3}{8}\frac{\alpha}{\omega_0}a^3+\frac{1}{2}\frac{f}{\omega_0}\cos\gamma \tag{6.5.12}$$

One should solve these two equations to obtain a and γ and can write the first order solution of the system in the following form

$$\begin{aligned} u &= a\cos\left(\omega_0 t+\beta\right)+O(\varepsilon)=a\cos\left(\omega_0 t+\sigma T_1-\gamma\right)+O(\varepsilon) \\ &= a\cos\left(\omega_0 t+\varepsilon\sigma t-\gamma\right)+O(\varepsilon)=a\cos\left(\Omega t-\gamma\right)+O(\varepsilon) \end{aligned} \tag{6.5.13}$$

Now for steady state as a' and γ' equals to 0, one can write Eq. (6.5.11-12) as

$$\mu a=\frac{1}{2}\frac{f}{\omega_0}\sin\gamma \tag{6.5.14}$$

$$a\sigma-\frac{3}{8}\frac{\alpha}{\omega_0}a^3=-\frac{1}{2}\frac{f}{\omega_0}\cos\gamma \tag{6.5.15}$$

Now eliminating γ from the above equations, one obtains

$$\left[\mu^2+\left(\sigma-\frac{3}{8}\frac{\alpha}{\omega_0}a^2\right)^2\right]a^2=\frac{f^2}{4\omega_0^2} \tag{6.5.16}$$

Eq. (6.5.16) is a 6th order Polynomial in a, but quadratic Polynomial in σ. Hence, by solving this quadratic equation, one can write the expression for the frequency response curve as follows.

$$\sigma = \frac{3}{8}\frac{\alpha}{\omega_0}a^2 \pm \left(\frac{f^2}{4\omega_0^2 a^2} - \mu^2\right)^{\frac{1}{2}} \tag{6.5.17}$$

Hence, for a particular value of detuning parameter one can get different amplitude of the response and the phase. Now to check the stability of the obtained steady state response, one can perturb the Eq. (6.5.11) and (6.5.12) to obtain the following Jacobian matrix.

$$J = \begin{vmatrix} -\mu & -a_0\left(\sigma - \frac{3\alpha a_0^2}{8\omega_0}\right) \\ \frac{1}{a_0}\left(\sigma - \frac{9\alpha a_0^2}{8\omega_0}\right) & -\mu \end{vmatrix} \tag{6.5.18}$$

Now to find the stability of the steady state response one can find the eigenvalues by finding the determinant of the $J - \lambda I$ matrix. This leads to expression

$$\lambda^2 + 2\mu\lambda + \mu^2 + \left(\sigma - \frac{3\alpha a_0^2}{8\omega_0}\right)\left(\sigma - \frac{9\alpha a_0^2}{8\omega_0}\right) = 0 \tag{6.5.19}$$

The system will be unstable when the real part of at least one of the eigenvalue becomes positive. This gives rise to the following relation.

$$\Gamma = \left(\sigma - \frac{3\alpha a_0^2}{8\omega_0}\right)\left(\sigma - \frac{9\alpha a_0^2}{8\omega_0}\right) + \mu^2 < 0$$

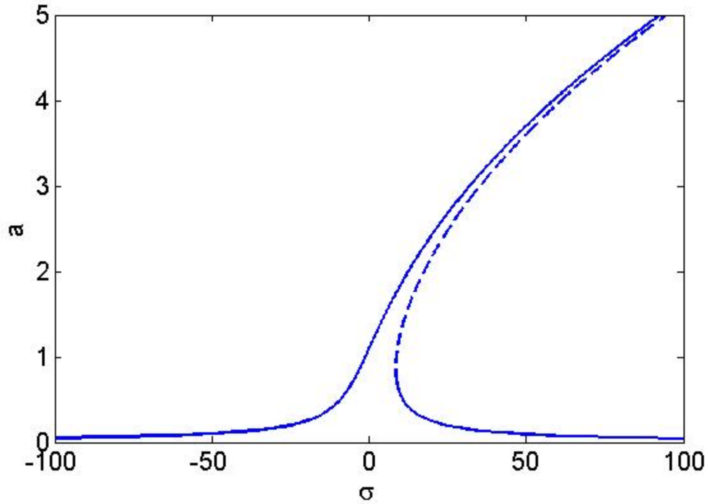


Fig:6.5.1 Frequency response curves for ω=1,f=10, α=10,μ=0.1

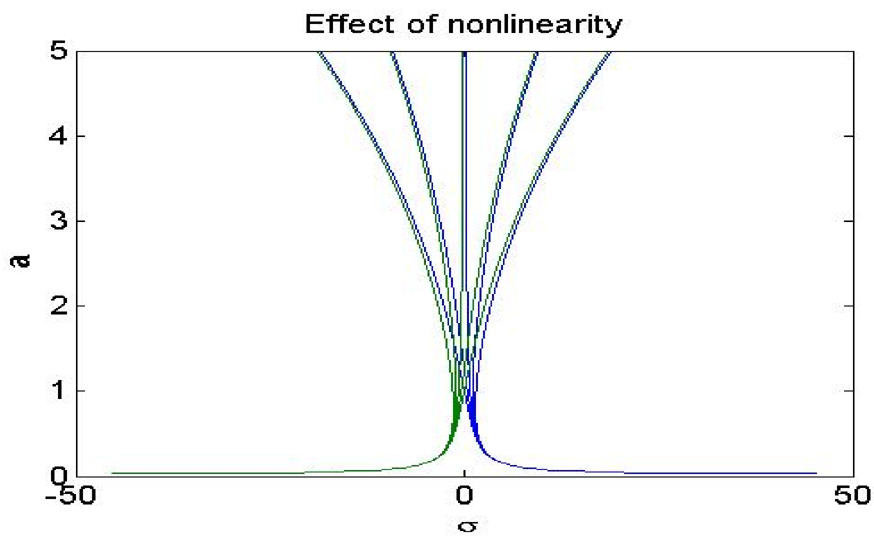


Fig:6.5.2 Frequency response curves with different values of α

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Module 6 Lect 6

System with non resonant hard excitations

In the previous lecture a single degree of freedom nonlinear system is considered when the amplit ude of the external excitation (*f*) is one order less than the linear term (i.e. ω_0^2). In the present lecture the forcing term is assumed to be of same the order as that of the linear term. So the equation of motion considered in this case is

$$\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha u^3 = f\cos\Omega t \tag{6.6.1}$$

Following similar procedure of method of multiple scales, one may write

$$u(t;\varepsilon) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \ldots \tag{6.6.2}$$

Now separating the terms with different order of ε one obtains the following equations.

$$D_0^2 u_0 + \omega_0^2 u_0 = f\cos\Omega T_0 \tag{6.6.3}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - 2\mu D_0 u_0 - \alpha u_0^3 \tag{6.6.4}$$

The solution of Eq. (6.6.3) can be written as

$$u_0 = A(T_1)\exp(i\omega_0 T_0) + \Lambda\exp(i\Omega T_0) + cc \tag{6.6.5}$$

Where $\Lambda = \dfrac{f}{2(\omega_0^2 - \Omega^2)}$

It may be noted that unlike the previous lecture, where only the complementary part of the solution was present, in this case both complimentary and particular integral parts are present in the solution of u_0. Now substituting Eq. (6.6.5) in Eq. (6.6.4) one obtains the following equation.

$$D_0^2 u_1 + \omega_0^2 u_1 = -\underbrace{\left[2i\omega_0\left(A' + \mu A\right) + 6\alpha A\Lambda^2 + 3\alpha A^2 \bar{A}\right]\exp\left(i\omega_0 T_0\right)}_{\text{Secular term}}$$

$$-\alpha\left\{\begin{array}{l} A^3 \exp\left(3i\omega_0 T_0\right) + \underbrace{\Lambda^3 \exp\left(3i\Omega T_0\right)}_{\text{Mixed Secular term 1}} + 3A^2\Lambda \exp\left[i\left(2\omega_0 + \Omega\right)T_0\right] \\ + \underbrace{3\bar{A}^2\Lambda \exp\left[i\left(\Omega - 2\omega_0\right)T_0\right]}_{\text{Mixed Secular term 2}} + 3A\Lambda^2 \exp\left[i\left(\omega_0 + 2\Omega\right)T_0\right] + \underbrace{3\bar{A}\Lambda^2 \exp\left[i\left(\omega_0 - 2\Omega\right)T_0\right]}_{\text{Mixed Secular term 3}} \end{array}\right\} \quad (6.6.6)$$

$$-\underbrace{\Lambda\left[2i\mu\Omega + 3\alpha\Lambda^2 + 6\alpha A\bar{A}\right]\exp\left(i\Omega T_0\right)}_{\text{Mixed Secular term 4}} + cc$$

It may be noted from Eq. (6.6.6) that when the exponent terms of the marked mixed secular terms are equal to ω_0 a resonance condition will occur. Hence, resonance will be observed in the system when

$\Omega = \omega_0$ (Primary resonance)

$\Omega - 2\omega_0 = \omega_0 \Rightarrow \Omega = 3\omega_0$ (Sub harmonic resonance)

$3\Omega = \omega_0 \Rightarrow \Omega = \frac{1}{3}\omega_0$ (Super harmonic resonance)

$\omega_0 - 2\Omega = \omega_0 \Rightarrow \Omega = 0$

For the non resonant case, i.e., when the external frequency is away from 0, ω_0, $\frac{1}{3}\omega_0$ or 3 ω_0, from Eq. (6.6.6) eliminating the secular terms yield the following equation.

$$2i\omega_0\left(A' + \mu A\right) + 6\alpha\Lambda^2 A + 3\alpha A^2 \bar{A} = 0 \quad (6.6.7)$$

Now using $A = \frac{1}{2}a\exp\left(i\beta\right)$ in Eq.(6.6.7) and separating the real and imaginary parts following reduced equations are obtained.

$$D_1 a = -\mu a \Rightarrow a = a_0 \exp(-\mu T_1) = a_0 \exp(-\varepsilon\mu t) \quad (6.6.8)$$

and $\omega_0 a D_1 \beta = 3\alpha\left(\Lambda^2 + \frac{1}{8}a^2\right)a \Rightarrow \beta = 3\alpha\left(\Lambda^2 + \frac{1}{8}a^2\right)T_1 + \beta_0 = 3\varepsilon\alpha\left(\Lambda^2 + \frac{1}{8}a^2\right)t + \beta_0$ (6.6.9)

$$u = a\cos\left(\omega_0 t + \beta\right) + \frac{f}{\left(\omega_0^2 - \Omega^2\right)}\cos\Omega t + O\left(\varepsilon\right)$$
$$= a_0 \exp(-\varepsilon\mu t)\cos\left(\left(\omega_0 + 3\varepsilon\alpha\left(\Lambda^2 + \frac{1}{8}a^2\right)\right)t + \beta_0\right) + \frac{f}{\left(\omega_0^2 - \Omega^2\right)}\cos\Omega t + O\left(\varepsilon\right) \qquad (6.6.10)$$

The free oscillation solution decays with time and hence the steady state response consists of forced solution only similar to the linear case.

Superharmonic Resonance $(\Omega \approx \frac{1}{3}\omega_0)$

To express the nearness of the external excitation frequency to one third of the natural frequency one may use the detuning parameter (σ) as follows.

$$3\Omega = \omega_0 + \varepsilon\sigma \qquad (6.6.11)$$

Now to include the mixed secular (or nearly secular or small divisor) term 1in Eq. (6.6.6) in this resonance condition one may write

$$3\Omega T_0 = \left(\omega_0 + \varepsilon\sigma\right)T_0 = \omega_0 T_0 + \varepsilon\sigma T_0 = \omega_0 T_0 + \sigma T_1 \qquad (6.6.12)$$

Now to eliminate the secular and near secular terms from Eq. (6.6.6) one can write

$$2i\omega_0\left(A' + \mu A\right) + 6\alpha\Lambda^2 A + 3\alpha A^2\bar{A} + \alpha\Lambda^3\exp\left(i\sigma T_1\right) = 0 \qquad (6.6.13)$$

Using $A = \frac{1}{2}a\exp\left(i\beta\right)$ in Eq.(6.6.13) and separating the real and imaginary parts following reduced equations are obtained.

$$a' = -\mu a - \frac{\alpha\Lambda^3}{\omega_0}\sin\left(\sigma T_1 - \beta\right) \qquad (6.6.14)$$

$$a\beta' = \frac{3\alpha}{\omega_0}\left(\Lambda^2 + \frac{1}{8}a^2\right)a + \frac{\alpha\Lambda^3}{\omega_0}\cos\left(\sigma T_1 - \beta\right) \qquad (6.6.15)$$

Now to express the above equations in their autonomous form one may use the following transformation.

$$\gamma = \sigma T_1 - \beta \qquad (6.6.16)$$

Hence, Eq. (6.6.14-15) can be written as

$$a' = -\mu a - \frac{\alpha\Lambda^3}{\omega_0}\sin\gamma \quad (6.6.17)$$

$$a\gamma' = \left(\sigma - \frac{3\alpha\Lambda^2}{\omega_0}\right)a - \frac{3\alpha}{8\omega_0}a^3 - \frac{\alpha\Lambda^3}{\omega_0}\cos\gamma \quad (6.6.18)$$

By solving the above two equations, one can obtain a and γ, and then can write the solution of the system as

$$u = u_0 = a\cos(3\Omega t - \gamma) + \frac{f}{(\omega_0^2 - \Omega^2)}\cos\Omega t + O(\varepsilon) \quad (6.6.19)$$

For steady state as the time derivative terms vanish, Eq. (6.6.17) and (6.6.18) can be written as

$$-\mu a = \frac{\alpha\Lambda^3}{\omega_0}\sin\gamma \quad (6.6.20)$$

$$\left(\sigma - 3\frac{\alpha\Lambda^2}{\omega_0}\right)a - \frac{3\alpha}{8\omega_0}a^3 = \frac{\alpha\Lambda^3}{\omega_0}\cos\gamma \quad (6.6.21)$$

Now eliminating γ from the above two equations, one can obtain a closed form equation which can be used for finding the frequency response of the system.

$$\left[\mu^2 + \left(\sigma - 3\frac{\alpha\Lambda^2}{\omega_0} - \frac{3\alpha}{8\omega_0}a^2\right)^2\right]a^2 = \frac{\alpha^2\Lambda^6}{\omega_0^2} \quad (6.6.22)$$

Solving Eq. (6.6.22) one may write the relation between the detuning parameter and amplitude of the response as follows.

$$\sigma = 3\frac{\alpha\Lambda^2}{\omega_0} + \frac{3\alpha}{8\omega_0}a^2 \pm \left(\frac{\alpha^2\Lambda^6}{\omega_0^2 a^2} - \mu^2\right)^{1/2} \quad (6.6.23)$$

Hence, in this resonance condition, the free oscillation term does not decay to zero inspite of the presence of damping. Moreover, the nonlinearity adjusts the frequency of the free oscillation term to exactly three times the frequency of the excitations so that the response is periodic. Since the frequency of the free oscillation term is 3 times the frequency of excitation, such resonances are called super harmonic resonances or overtones.

From Eq. (6.6.20) the peak amplitude of the free oscillation term is given by

$$a_p = \frac{\alpha\Lambda^3}{\mu\omega_0} \quad (6.6.24)$$

$$\sigma_p = \frac{3\alpha\Lambda^2}{\omega_0}\left(1 + \frac{\alpha^2\Lambda^4}{8\mu^2\omega_0^2}\right) \quad (6.6.25)$$

Subharmonic Resonance $\Omega = 3\omega_0$

When the external frequency is nearly 3 times the natural frequency of the system, using detuning parameter one can write

$$\Omega = 3\omega_0 + \varepsilon\sigma, \quad (6.6.26)$$

Or, $\left(\Omega - 2\omega_0\right)T_0 = \omega_0 T_0 + \varepsilon\sigma T_0$ (6.6.27)

Using a similar procedure of method of multiple scales, to eliminate the secular terms from Eq. (6.6.6) one can write

$$2i\omega_0\left(A' + \mu A\right) + 6\alpha\Lambda^2 A + 3\alpha A^2\bar{A} + \alpha\Lambda\bar{A}^2\exp\left(i\sigma T_1\right) = 0. \quad (6.6.28)$$

Using $A = \frac{1}{2}a\exp\left(i\beta\right)$ in Eq.(6.6.13) and separating the real and imaginary parts following reduced equations are obtained.

$$a' = -\mu a - \frac{3\alpha\Lambda}{4\omega_0}a^2\sin\left(\sigma T_1 - 3\beta\right) \quad (6.6.29)$$

$$a\beta' = \frac{3\alpha}{\omega_0}\left(\Lambda^2 + \frac{1}{8}a^2\right)a + \frac{3\alpha\Lambda}{4\omega_0}a^2\cos\left(\sigma T_1 - 3\beta\right) \quad (6.6.30)$$

Now to express the above equations in their autonomous form one may use the following transformation.

$$\gamma = \sigma T_1 - 3\beta \quad (6.6.31)$$

$$a' = -\mu a - \frac{3\alpha\Lambda}{4\omega_0}a^2\sin\gamma \quad (6.6.32)$$

$$a\gamma' = \left(\sigma - \frac{9\alpha\Lambda^2}{\omega_0}\right)a - \frac{9\alpha}{8\omega_0}a^3 - \frac{9\alpha\Lambda}{8\omega_0}a^2\cos\gamma \tag{6.6.33}$$

$$u = a\cos\left[\frac{1}{3}(\Omega t - \gamma)\right] + f\left(\omega_0^2 - \Omega^2\right)^{-1}\cos\Omega t + O(\in) \tag{6.6.34}$$

For steady state one can write

$$-\mu a = \frac{3\alpha\Lambda}{4\omega_0}a^2\sin\gamma \tag{6.6.35}$$

$$\left(\sigma - \frac{9\alpha\Lambda^2}{\omega_0}\right)a - \frac{9\alpha}{8\omega_0}a^3 = \frac{9\alpha\Lambda}{4\omega_0}a^2\cos\gamma \tag{6.6.36}$$

Now eliminating γ from the above two equations, one can obtain a closed form equation which can be used for finding the frequency response of the system.

$$\left[9\mu^2 + \left(\sigma - \frac{9\alpha\Lambda^2}{\omega_0} - \frac{9\alpha}{8\omega_0}a^2\right)^2\right]a^2 = \frac{81\alpha^2\Lambda^2}{16\omega_0^2}a^4 \tag{6.6.37}$$

This shows either the system will have a trivial state response (i.e., $a=0$) and a non trivial response which can be obtained by solving the following equation.

$$\left[9\mu^2 + \left(\sigma - \frac{9\alpha\Lambda^2}{\omega_0} - \frac{9\alpha}{8\omega_0}a^2\right)^2\right] = \frac{81\alpha^2\Lambda^2}{16\omega_0^2}a^2 \tag{6.6.38}$$

This equation is quadratic in a^2 and hence the solution can be written as

$$a^2 = p \pm \left(p^2 - q\right)^{1/2} \tag{6.6.39}$$

Where $p = \frac{8\omega_0\sigma}{9\alpha} - 6\Lambda^2$ (6.6.40)

and $q = \frac{64\omega_0^2}{81\alpha^2}\left[9\mu^2 + \left(\sigma - \frac{9\alpha\Lambda^2}{\omega_0}\right)^2\right]$ (6.6.41)

As q is always positive, so the nontrivial free oscillation amplitudes occur when $p > 0$ and $p^2 \geq q$. For these conditions one should have

$$\Lambda^2 < \frac{4\omega_0\sigma}{27\alpha} \text{ and } \frac{\alpha\Lambda^2}{\omega_0}\left(\sigma - \frac{63\alpha\Lambda^2}{8\omega_0}\right) - 2\mu^2 \geq 0 \qquad (6.6.42)$$

So σ and α should have same sign. From Eq. (6.6.42), for a given Λ nontrivial solutions can exist only if

$$\alpha\sigma \geq \frac{2\mu^2\omega_0}{\Lambda^2} + \frac{63\alpha^2\Lambda^2}{8\omega_0} \qquad (6.6.43)$$

Similarly for a given σ, nontrivial solution can exist only if

$$\frac{\sigma}{\mu} - \left(\frac{\sigma^2}{\mu^2} - 63\right)^{1/2} \leq \frac{63\alpha\Lambda^2}{4\omega_0\mu} \leq \frac{\sigma}{\mu} + \left(\frac{\sigma^2}{\mu^2} - 63\right)^{1/2} \qquad (6.6.44)$$

In the $\Lambda\sigma$ plane the boundary of the region where nontrivial solutions can exist can be given by

$$\frac{63\alpha\Lambda^2}{4\omega_0\mu} = \frac{\sigma}{\mu} \pm \left(\frac{\sigma^2}{\mu^2} - 63\right)^{1/2} \qquad (6.6.45)$$

Exercise Problems

1. Study the response of the single degree of freedom system with both quadratic and cubic nonlinearities. Consider primary, subharmonic and superharmonic resonance conditions. Use either method of multiple scales or the method of normal forms.

References:

1. A. H. Nayfeh and D. T. Mook: Nonlinear Oscillations, Wiley, 1979

2. A. H. Nayfeh, Method of Normal Forms, Wiley,1993

Exercise problems

1. 1. Using numerical techniques plot the time response of the system given by Eq. (6.6.1) with primary, subharmonic and superharmonic resonance. Compare the results with those obtained from the perturbation method.

2. Study different resonance conditions for the Duffing equation with two frequency excitation terms as given by the following equation.

$$\ddot{u}+\omega_0^2 u+2\varepsilon\mu\dot{u}+\varepsilon\alpha u^3 = f_1\cos\left(\Omega_1 t\ +\theta_1\right)+f_2\ \cos\left(\Omega_2 t\ +\theta_2\right)$$

Ans: One may observe following resonance conditions

$$\omega_0 \approx 3\Omega_n, \omega_0 \approx \frac{1}{3}\Omega_n,, \omega_0 \approx 1+\left|\pm 2\Omega_m \pm \Omega_n\right| \omega_0 \approx \frac{1}{2}\left(\Omega_m \pm \Omega_n\right)$$
$$\omega_0 \approx 3\Omega_1\ \ or \qquad 3\Omega_2, \omega_0 \approx \frac{1}{3}\Omega_1\ or\ \ \omega_0 \approx \frac{1}{3}\Omega_2$$
$$\omega_0 \approx \Omega_2 \pm 2\Omega_1 \qquad or \qquad 2\Omega_1 - \Omega_2, \omega_0 \approx 2\Omega_2 \pm \Omega_1, \omega_0 \approx \frac{1}{2}\left(\Omega_2 \pm \Omega_1\right)$$
$$\Omega_2 \approx 9\Omega_1 \approx 3\omega_0, \Omega_2 \approx \Omega_1 \approx 3\omega_0$$
$$\Omega_2 \approx \Omega_1 \approx \frac{1}{3}\omega_0, \Omega_2 \approx 5\Omega_1 \approx \frac{5}{3}\omega$$
$$\Omega_2 \approx 7\Omega_1 \approx \frac{7}{3}\omega_0, \Omega_2 \approx 2\Omega_1 \approx \frac{2}{3}\omega_0$$
$$\Omega_2 \approx \frac{7}{3}\Omega_1 \approx 7\omega_0, \Omega_2 \approx \frac{5}{3}\Omega_1 \approx 5\omega_0$$

Q No. 3 Plot the frequency response curves for the system with sub harmonic and super harmonic resonance condition using equation 6.6.25 and 6.6.45.

Module 6 Lecture 7

Forced vibration Single-Degree of freedom system

In this lecture briefly following analysis will be carried out

- Forced vibration of Single-Degree of freedom system with cubic and quadratic nonlinearies
- System with self sustained oscillation

System with cubic and quadratic nonlinearies

In this case considering damping, cubic nonlinearity and the forcing parameters to be one order less than quadratic nonlinearity which is one order less than the linear term, the equation of motion can be written as

$$\ddot{u}+\omega_0^2 u+2\varepsilon^2\mu\dot{u}+\varepsilon\alpha_2 u^2+\varepsilon^2\alpha_3 u^3 = \varepsilon^2 f\cos\Omega t \qquad (6.7.1)$$

Following similar procedure as in the previous two lectures one may study the primary, super harmonic and sub harmonic resonance conditions.

In case of primary resonance, taking $\Omega = \omega_0 + \varepsilon^2 \sigma$ and

$$u(t,\ \varepsilon) = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) \tag{6.7.2}$$

and applying the usual procedure of method of multiple scales, one can obtain the following reduced equations (Nayfeh and Mook, 1979).

$$a' = -\mu a - \frac{f}{2\omega_0}\sin\gamma \tag{6.7.3}$$

$$a\gamma' = a\sigma - \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^3}a^3 + \frac{f}{2\omega_0}\cos\gamma \tag{6.7.4}$$

Comparing these two equations with those obtained from Duffing equation with only cubic nonlinearities for primary resonance (Eq. 6.5.11 and 12), one may observe that both sets of equations are identical if

$$\alpha = \alpha_3 - \frac{10\alpha_2^2}{9\omega_0^2} \tag{6.7.5}$$

When $\alpha_3 = 0 \Rightarrow \alpha = -\frac{10\alpha_2^2}{9\omega_0^2}$ which is negative, the system will show softening effect. Also if $\alpha_3 < \frac{10\alpha_2^2}{9\omega_0^2}, \alpha < 0$ one obtains softening effect in which the frequency response curves bends towards the lower frequencies irrespective of the sign of α_2.

When $\alpha_3 = \frac{10\alpha_2^2}{9\omega_0^2},\ \alpha = 0$, the system behaviour will be similar to that of a linear system. Here the effect of cubic nonlinearities will be cancelled by that of the quadratic nonlinearity. Similar to the expression in the cubic nonlinear system here the following equation can be used to find the frequency response.

$$\left[\mu^2 + \left(\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}a^2\right)^2\right]a^2 = \frac{f^2}{4\omega_0^2} \tag{6.7.8}$$

The expression for the response can be given by

$$u = a\cos(\Omega t - \gamma) - \frac{1}{2}\varepsilon\frac{\alpha_2}{\omega_0^2}a^2 + \frac{1}{6}\varepsilon\frac{\alpha_2}{\omega_0^2}a^2\cos(2\Omega t + 2\gamma) + O(\varepsilon^2) \tag{6.7.9}$$

Comparing this expression with that of the system with only cubic nonlinearity, it can be observed that the oscillating motion is not centered at $u=0$ and there is a drift or shift of the steady state part by an amount of $-\frac{1}{2}\varepsilon\frac{\alpha_2}{\omega_0^2}a^2$.

Superharmonic resonance

This resonance condition may occur when one consider, the forcing term is same order that of the linear part. Also considering the damping and quadratic nonlinearity of the order of ε and cubic nonlinearity of the order of ε^2 one may write the equation of motion as (Nayfeh and Mook, 1979)

$$\ddot{u}+\omega_0^2 u+2\varepsilon\mu\dot{u}+\varepsilon\alpha_2 u^2+\varepsilon^2\alpha_3 u^3 = f\cos\Omega t \qquad (6.7.10)$$

Applying method of multiple scales while eliminating the secular term, one may observe that a resonance condition will occur when $2\Omega=\omega_0+\varepsilon\sigma$. In this case the steady state solution can be written as

$$u=\frac{f}{\omega_0^2-\Omega^2}\cos(\Omega t)-\frac{\alpha_2 f^2}{4\omega_0\left(\omega_0^2-\Omega^2\right)^2\left(\mu^2-\sigma^2\right)^{1/2}}\sin(2\Omega t-\gamma)+O\left(\varepsilon^2\right) \qquad (6.7.11)$$

$$\gamma=\tan^{-1}\left(\frac{\sigma}{\mu}\right)$$

Subharmonic resonance

For the system given by equation (6.7.10), subharmonic resonance will occur when external frequency is nearly equal to $\Omega=2\omega_0+\varepsilon\sigma$. In this case one may obtain the expression for the frequency of resulting oscillation as

$$\lambda=-\mu\pm\left(\frac{\alpha_2^2\Lambda^2}{\omega_0^2}-\frac{\sigma^2}{4}\right)^{1/2} \qquad (6.7.12)$$

Hence if $\sigma^2>4\frac{\alpha_2^2\Lambda^2}{\omega_0^2}$, the motion is oscillatory and decays with time

If $4\frac{\alpha_2^2\Lambda^2}{\omega_0^2}>\sigma^2>4\left(4\frac{\alpha_2^2\Lambda^2}{\omega_0^2}-\mu^2\right)$, the response decays without oscillating

If $4\left(4\frac{\alpha_2^2\Lambda^2}{\omega_0^2}-\mu^2\right)>\sigma^2$, the system becomes unstable as the response grows

Systems with self-sustained oscillations

Let us consider the van der Pol's oscillator with soft harmonic excitation which is given by the following equation.

$$\ddot{u}+\omega_0^2 u-\varepsilon\left(\dot{u}-\frac{1}{3}\dot{u}^3\right)=\varepsilon f\cos\Omega t \tag{6.7.13}$$

Let us consider the primary resonance case in which the external frequency is assumed to be near the linear system frequency ω_0. So using detuning parameter one may write

$$\Omega=\omega_0 t+\varepsilon\sigma \tag{6.7.14}$$

Now to solve Eq. (6.7.14), taking

$$u(t,\ \varepsilon)\ =\ u_0(T_0,T_1)+\varepsilon u_1(T_0,T_1)+o(\varepsilon^2) \tag{6.7.15}$$

one will obtain the following reduced equations

$$a'=\frac{1}{2}\left(1-\frac{1}{4}\omega_0^2 a^2\right)a+\frac{f}{2\omega_0}\sin\gamma \tag{6.7.16}$$

$$a\gamma'=a\sigma+\frac{f}{2\omega_0}\cos\gamma \tag{6.7.17}$$

The steady state solution can be given by

$$u=a\cos(\Omega t-\gamma)+O(\varepsilon) \tag{6.7.18}$$

For steady state taking $a'=\gamma'$ =0 and eliminating γ form the equations (6.7.16-17) one can obtain

$$\rho(1-\rho^2)+4\sigma^2\rho=\frac{f^2}{4} \tag{6.7.19}$$

Where $\rho=\frac{1}{4}\omega_0^2 a^2$ (6.7.20)

Figure 6.7.1 shows the frequency response obtained using equation 6.7.20 for different values of forcing parameter.

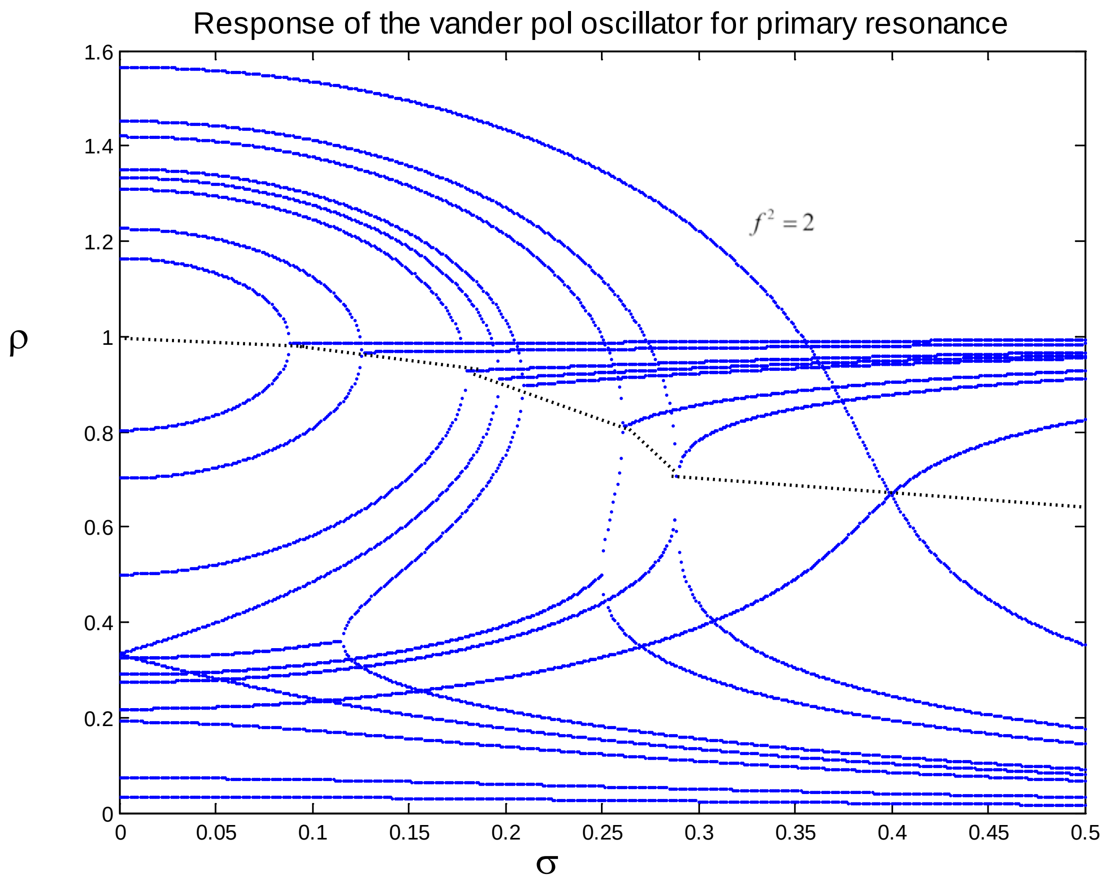


Fig. 6.7.1: Frequency response curves for primary resonance for van der Pol's oscillator

Exercise Problems:

1. Using numerical techniques plot the time response of the system given by Eq. (6.7.1) with primary, subharmonic and superharmonic resonance. Compare the results with those obtained from the perturbation method.

2. Find the frequency response curves for van der Pol's oscillator considering strong forcing term. Study the subharmonic and superharmonic resonance conditions.

3. Find the expressions for frequency response curves for a single degree of freedom system for different resonance conditions when subjected to 2 and 3 frequency excitations.

Module 6 Lecture 8

Free and Forced vibration of Multi-Degree of freedom system

In this lecture briefly we will discuss about the free vibration of multi-degree of freedom nonlinear systems. Initially the system with quadratic nonlinearity and then the system with cubic nonlinearity will be considered.

Free vibration of the system with quadratic nonlinearities

Let us consider a two degree of freedom system where the equation of motion can be given by

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\hat{\mu}_1 \dot{u}_1 - \alpha_1 u_1 u_2 = 0 \tag{6.8.1}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\hat{\mu}_2 \dot{u}_2 - \alpha_2 u_1^2 = 0 \tag{6.8.2}$$

To use method of multiple scales one may assume the solution of these equations using different time scales T_0, T_1 $\left(T_n = \varepsilon^n t\right)$ as follows.

$$u_1 = \varepsilon u_{11}(T_0, T_1) + \varepsilon^2 u_{12}(T_{0,} T_1) + \ldots \tag{6.8.3}$$

$$u_2 = \varepsilon u_{21}(T_0, T_1) + \varepsilon^2 u_{22}(T_{0,} T_1) + \ldots \tag{6.8.4}$$

Substituting Eq. (6.8.3) and (6.8.4) in Eq. (6.8.1) and (6.8.2) and separating the terms with different order of ε one obtains

Order of ε

$$D_0^2 u_{11} + \omega_1^2 u_{11} = 0$$
(6.8.5)

$$D_0^2 u_{21} + \omega_2^2 u_{21} = 0$$
(6.8.6)

Order of ε^2

$$D_0^2 u_{12} + \omega_1^2 u_{12} + -2D_0(D_1 u_{11} + \mu_1 u_{11}) + \alpha_1 u_{11} u_{21} \tag{6.8.7}$$

$$D_0^2 u_{22} + \omega_2^2 u_{22} + -2D_0(D_1 u_{21} + \mu_2 u_{21}) + \alpha_1 u_{11}^2 \tag{6.8.8}$$

The solution of Eq. (6.8.5) and (6.8.6) can be given by

$$u_{11} = A_1(T_1)\exp(i\omega_1 T_0) + cc \qquad (6.8.9)$$

$$u_{21} = A_2(T_1)\exp(i\omega_2 T_0) + cc \qquad (6.8.10)$$

Substituting Eq. (6.8.9) and (6.8.10) in Eq. (6.8.7) and (6.8.8) yields

$$D_0^2 u_{12} + \omega_1^2 u_{12} = \underbrace{-2\omega_1(D_1 A_1 + \mu_1 A_1)\exp(i\omega_1 T_0)}_{\text{Secular term}} + \alpha_1 \left\{ A_1 A_2 \exp\left[i(\omega_1 + \omega_2)T_0\right]\right\} + cc \qquad (6.8.11)$$

$$D_0^2 u_{22} + \omega_2^2 u_{22} = \underbrace{-2\omega_2(D_1 A_2 + \mu_2 A_2)\exp(i\omega_2 T_0)}_{\text{Secular Term}} + \alpha_2 \left[A_1^2 \exp(2i\omega_1 T_0) + A_1 \bar{A}_1 \right] + cc \qquad (6.8.12)$$

To eliminate the secular term from Eq. (6.8.11) and Eq. (6.8.12) one can write

$$D_1 A_1 + \mu_1 A_1 = 0 \quad \text{and} \quad D_1 A_2 + \mu_2 A_2 = 0 \qquad (6.8.13)$$

Solving Eq. (6.8.13) one can write

$$A_1 = a_1 \exp(-\mu_1 T_1) \quad \text{and} \quad A_2 = a_2 \exp(-\mu_2 T_1)$$
(6.8.14)

Substituting Eq. (6.8.14) in (6.8.3) and (6.8.4), the first order solution of the system can be given by

$$u_1 = \varepsilon \exp(-\varepsilon\mu_1 t)\left[a_1 \exp(i\omega_1 t) + cc\right] + O(\varepsilon^2) \qquad (6.8.15)$$

$$u_2 = \varepsilon \exp(-\varepsilon\mu_2 t)\left[a_2 \exp(i\omega_2 t) + cc\right] + O(\varepsilon^2) \qquad (6.8.16)$$

Hence, for steady state as time tends to infinity, both the response $u_1 = u_2 = 0$.

Resonant case (System with internal resonance)

Considering internal resonance of 1:2, i.e., when the second frequency is nearly equal to twice the first frequency one can write

$$\omega_2 = 2\omega_1 + \varepsilon\sigma \qquad (6.8.17)$$

So, $2\omega_1 T_0 = \omega_2 T_0 - \varepsilon\sigma T_0 = \omega_2 T_0 - \sigma T_1$ and $(\omega_2 - \omega_1)T_0 = \omega_1 T_0 + \varepsilon\sigma T_0 = \omega_1 T_0 + \sigma T_1$ (6.8.18)

Following similar procedure of method of multiple scales to eliminate the secular terms one obtains the following equations.

$$-2i\omega_1\left(A_1^{'}+\mu_1 A_1\right)+\alpha_1 A_2\bar{A}_1\exp\left(i\sigma T_1\right)=0 \text{ and } -2i\omega_2\left(A_2^{'}+\mu_2 A_2\right)+\alpha_2 A_1^2\exp\left(-i\sigma T_1\right)=0 \quad (6.8.19)$$

Now using $A_1=\frac{1}{2}a_1\exp\left(i\theta_1\right)$ and $A_2=\frac{1}{2}a_2\exp\left(i\theta_2\right)$ and introducing $\gamma=\theta_2-2\theta_1+\sigma T_1$ and separating the real and imaginary parts the following reduced equations are obtained.

$$a_1^{'}=-\mu_1 a_1+\frac{\alpha_1}{4\omega_1}a_1 a_2\sin\gamma \text{ and } a_2^{'}=-\mu_2 a_2-\frac{\alpha_2}{4\omega_2}a_1^2\sin\gamma \quad (6.8.20)$$

$$a_1\theta_1^{'}=-\frac{\alpha_1}{4\omega_1}a_1 a_2\cos\gamma \text{ and } a_2\theta_2^{'}=-\frac{\alpha_2}{4\omega_2}a_1^2\cos\gamma \quad (6.8.21)$$

Or, $$a_2\gamma'=\sigma a_2+\left(\frac{\alpha_1 a_2^2}{2\omega_1}-\frac{\alpha_2 a_1^2}{4\omega_2}\right)\cos\gamma \quad (6.8.22)$$

For steady state using $a_1^{'}=a_2^{'}=\gamma^{'}=0$, one may write

$$-\mu_1 a_1+\frac{\alpha_1}{4\omega_1}a_1 a_2\sin\gamma=0 \quad (6.8.23)$$

$$-\mu_2 a_2-\frac{\alpha_2}{4\omega_2}a_1^2\sin\gamma=0 \quad (6.8.24)$$

$$\left(\frac{\alpha_1}{2\omega_1}a_2^2-\frac{\alpha_2}{4\omega_2}a_1^2\right)\cos\gamma+\sigma a_2=0 \quad (6.8.25)$$

Eliminating γ from Eq. (6.8.23) and (6.8.24) one obtains the following equations.

$$a_1^2+\frac{\mu_2\omega_2\alpha_1}{\mu_1\omega_1\alpha_2}a_2^2=0 \quad (6.8.26)$$

If α_1 and α_2 are of different sign, a_1 and a_2 can differ from zero, that is the system may have nontrivial response. Hence, in the presence of internal resonance, even though there is no external forcing energy will transfer from 1st mode to second mode and self sustained oscillation will continue.

Forced vibration Multi-Degree of freedom system with quadratic nonlinearities

Let us consider a two degree of freedom system with quadratic nonlinearies given by the following equations.

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\varepsilon\mu_1\dot{u}_1 + \alpha_1 u_1 u_2 = \varepsilon f_1 \cos(\Omega t + \tau_1) \tag{6.8.27}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\varepsilon\mu_2\dot{u}_2 + \alpha_2 u_1^2 = \varepsilon^2 f_2 \cos(\Omega t + \tau_2) \tag{6.8.28}$$

Now to solve these equations let us use method of multiple scales by considering

$$u_1 = \varepsilon u_{11}(T_0, T_1) + \varepsilon^2 u_{12}(T_{0,} T_1) + \dots \text{ and } u_2 = \varepsilon u_{21}(T_0, T_1) + \varepsilon^2 u_{22}(T_{0,} T_1) + \dots \tag{6.8.29}$$

Substituting Eq. (6.8.29) in Eqs. (6.8.27-6.8.28) and equating coefficients of like power of ε one obtains

$$D_0^2 u_{11} + \omega_1^2 u_{11} = f_1 \cos(\Omega T_0 + \tau_1) \tag{6.8.30}$$

$$D_0^2 u_{21} + \omega_2^2 u_{21} = 0 \tag{6.8.31}$$

$$D_0^2 u_{12} + \omega_1^2 u_{12} = -2D_0(D_1 u_{11} + \mu_1 u_{11}) - \alpha_1 u_{11} u_{21} \tag{6.8.32}$$

$$D_0^2 u_{22} + \omega_2^2 u_{22} = -2D_0(D_1 u_{21} + \mu_2 u_{21}) - \alpha_2 u_{11}^2 + f_2 \cos(\Omega T_0 + \tau_2) \tag{6.8.34}$$

Now, solution of Eq. (6.8.30) and Eq. (6.8.31) can be given by

$$u_{11} = A_1(T_1)\exp(i\omega_1 T_0) + \Lambda \exp\left[i(\Omega T_0 + \tau_1)\right] + cc \tag{6.8.35}$$

$$u_{21} = A_2(T_1)\exp(i\omega_2 T_0) + cc$$

(6.8.36) Here, $\Lambda = f_1 / 2(\omega_1^2 - \Omega^2)$ Now substituting these two equations in Eq. (6.8.33) and (6.8.34) one can write

$$D_0^2 u_{12} + \omega_1^2 u_{12} = \underbrace{-2\omega_1(A_1' + \mu_1 A_1)\exp(i\omega_1 T_0)}_{\text{secular term}} - \alpha_1 A_2 A_1 \exp\left[i(\omega_2 + \omega_1)T_0\right]$$

$$-\alpha_1\left(\underbrace{A_2\bar{A}_1 \exp\left[i(\omega_2 - \omega_1)T_0\right]}_{\text{near secular in case of internal resonance } \omega_2 \approx 2\omega_1} + \Lambda A_2 \exp\left[i(\Omega + \omega_2)T_0 + i\tau_1\right]\right)$$

$$-\underbrace{\alpha_1 \Lambda \bar{A}_2 \exp\left[i(\Omega - \omega_2)T_0 + i\tau_1\right]}_{\text{near secular term if } \Omega \approx \omega_1 + \omega_2} - \underbrace{2i\mu_1\Omega\Lambda \exp\left[i(\Omega T_0 + \tau_1)\right]}_{\text{near secular term if } \Omega \approx \omega_1} + cc \tag{6.8.37}$$

And $$D_0^2 u_{22} + \omega_2^2 u_{22} = -\underbrace{2\omega_2(A_2' + \mu_2 A_2)\exp(i\omega_2 T_0)}_{\text{Secular term}} - \alpha_2 A_1^2 \exp(2i\omega_1 T_0)$$

$$-\alpha_2\left(A_1\bar{A}_1 + \Lambda^2 + 2A_1\Lambda \exp\left[i(\omega_1 + \Omega)T_0 + i\tau_1\right]\right)$$

$$-\alpha_2\left(\underbrace{2\bar{A}_1\Lambda\exp\left[i\left(\Omega-\omega_1\right)T_0+i\tau_1\right]}_{\text{may be near secular term considering internal resonance}}+\Lambda^2\exp\left[2i\left(\Omega T_0+\tau_1\right)\right]\right)$$

$$+\underbrace{\frac{1}{2}f_2\exp\left[i\left(\Omega T_0+\tau_2\right)\right]}_{\text{near secular term if }\Omega\approx\omega_2}+cc \tag{6.8.38}$$

From Eq. (6.8.37) and (6.8.38) one may observe that one may get many resonance conditions such as (a) $\Omega\approx\omega_1$, (b) $\Omega\approx\omega_2$ (c) $\Omega\approx\omega_1+\omega_2$. Also, some resonance condition occurs when one consider internal resonance i.e. $\omega_2\approx 2\omega_1$.

Let us consider when the external frequency is nearly equal to the second mode frequency i.e.,

$\Omega=\omega_2+\varepsilon\sigma_1$ (6.8.39).

Hence, without considering internal resonance, to eliminate secular terms one can write

$$A_1^{'}+\mu_1A_1=0 \tag{6.8.40}$$

and $2i\omega_2\left(A_2^{'}+\mu_2A_2\right)=\frac{1}{2}f_2\exp\left[i\left(\sigma_1T_1+\tau_2\right)\right]$ (6.8.41)

Now solution of Eq. (6.8.40) can be written as

$$A_1=\frac{1}{2}a_1\exp(-\mu_1T_1+i\theta_1) \tag{6.8.42}$$

$$A_2=\frac{1}{2}a_2\exp(-\mu_2T_1+i\theta_2)-\frac{if_2}{4\omega_2\left(\mu_2+i\sigma_1\right)}\exp\left[i\left(\sigma_1T_1+\tau_2\right)\right] \tag{6.8.43}$$

For steady state as t tends to 0, one obtains

$$A_1=0,\ \text{and}\ A_2=-\frac{if_2}{4\omega_2\left(\mu_2+i\sigma_1\right)}\exp\left[i\left(\sigma_1T_1+\tau_2\right)\right] \tag{6.8.44}$$

Hence one obtains,

$$u_1=\frac{F_1}{\left(\omega_1^2-\Omega^2\right)}\cos\left(\Omega t+\tau_1\right)+O\left(\varepsilon^2\right) \tag{6.8.45}$$

$$u_2 = \frac{F_2}{2\varepsilon\omega_2\sqrt{\left(\mu_2^2+\sigma_1^2\right)}}\sin\left(\Omega t+\tau_2-\gamma_0\right)+O\left(\varepsilon^2\right) \tag{6.8.46}$$

where

$$\gamma_0 = \tan^{-1}\left(\frac{\sigma_1}{\mu_2}\right) \tag{6.8.47}$$

Now if one considers the system with internal resonance i.e., $\omega_2 \approx 2\omega_1$ or $\omega_2 = 2\omega_1 + \varepsilon\sigma_2$, then to eliminate secular term one can write

$$-2i\omega_1\left(A_1^{'}+\mu_1 A_1\right)-\alpha_1 A_2 A_1^{'}\exp\left(i\sigma_2 T_1\right)=0 \tag{6.8.48}$$

$$-2i\omega_2\left(A_2^{'}+\mu_2 A_2\right)-\alpha_2 A_1^2\exp\left(i\sigma_2 T_1\right)+\frac{1}{2}f_2\exp\left[i\left(-\sigma_1 T_1+\tau_2\right)\right]=0 \tag{6.8.49}$$

Substituting $A_n = \frac{1}{2}a_n\exp\left(i\beta_n\right)$ where n =1, 2, in Eqs. (6.8.48) and (6.8.49) one obtains

$$a_1^{'} = -\mu_1 a_1 - \frac{\alpha_1}{4\omega_1}a_1 a_2\sin\gamma_2 \tag{6.8.50}$$

$$a_2^{'} = -\mu_2 a_2 + \frac{\alpha_2}{4\omega_2}a_1^2\sin\gamma_2 + \frac{1}{2}\omega_2^{-1}f_2\sin\gamma_1 \tag{6.8.51}$$

$$a_1\beta_1^{'} = \frac{\alpha_1}{4\omega_1}a_2 a_1\cos\gamma_2 \tag{6.8.52}$$

$$a_2\beta_2^{'} = \frac{\alpha_2}{4\omega_2}a_1^2\cos\gamma_2 - \frac{1}{2\omega_2}f_2\cos\gamma_1 \tag{6.8.53}$$

Where, $\gamma_1 = \sigma_1 T_1 - \beta_2 + \tau_2$ and $\gamma_2 = \beta_2 - 2\beta_1 - \sigma_2 T_1$ (6.8.54)

For steady state response, $a_1^{'}, a_2^{'}, \beta_1^{'}\ \beta_2^{'}$ are zero and there are two possibilities. In the first case one will obtain the linear solution similar to Eq. 6.8.44. In the second case

$$a_1 = 2\left[-P_1 \pm\left(\frac{1}{4}f_2^2 - P_2^2\right)^{1/2}\right]^{1/2} \tag{6.8.55}$$

$$a_2 = 2\omega_1 \left[4\mu_1^2 + \left(\sigma_1 - \sigma_2\right)^2 \right]^{1/2} \quad (6.8.56)$$

Here, $P_1 = 2\omega_1\omega_2 \left[\sigma_1 \left(\sigma_2 - \sigma_1\right) + 2\mu_1\mu_2 \right]$

$$P_2 = 2\omega_1\omega_2 \left[2\sigma_1\mu_1 - \mu_2 \left(\sigma_2 - \sigma_1\right) \right] \quad (6.8.57)$$

Substituting these values in u_1 and u_2 one can obtain the response of the system. It may be noted that the second mode amplitude a_2 is not a direct function of the external excitation. Using these equations one may plot the frequency response curves and observe many resonance phenomena similar to those observed in case of single degree of freedom system.

Exercise Problems:

1. Plot the frequency response curves using Eq. (6.8.55) and (6.8.56) taking $\alpha_1 = \alpha_2$ =-1. Observe different nonlinear phenomena. Study the stability of the system.

2. Study the resonant and non resonant free vibration of two degree of freedom system with cubic nonlinearities. The equation of motion of this system can be given by

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\hat{\mu}_1 \dot{u}_1 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_2^2 + \alpha_4 u_2^3 = 0$$

$$\ddot{u} + \omega_1^2 u_1 + 2\hat{\mu}_1 \dot{u}_1 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_2^2 + \alpha_4 u_2^3 = 0$$

Module 6 Lect 9

Parametrically excited system

In this lecture, the parametrically excited system will be considered and using Floquet theory the conditions for instability regions will be determined. It may be noted that a simple parametrically excited system can be represented by the following equation.

$$\ddot{u} + p_1(t)\dot{u} + p_2(t)u = 0 \tag{6.9.1}$$

Where, the term $p_1(t)$ and $p_2(t)$ are periodic function of time. It may be noted that as these time varying terms are coefficients of the response and its derivative, this equation is called the equation of a parametrically excited system. One can have many variation of this equation by including different nonlinear terms and forcing terms. One can consider a single or multi degree of freedom system also. Eq. (6.9.1) can also be written in the following form by substituting

$$u = x\exp\left(-\frac{1}{2}\int p_1(t)dt\right)$$

in Eq. (6.9.1). The resulting equation can be written as

$$\ddot{x} + p(t)x = 0 \tag{6.9.2}$$

where $p(t) = p_2 - \frac{1}{4}p_1^2 - \frac{1}{2}\dot{p}_1$ (6.9.3)

Equation (6.9.2) is called the Hill's equation who studied this system in 1886 (Nayfeh and Mook 1979).

Now by substituting $p(t) = \delta + 2\varepsilon\cos 2t$ in Eq. (6.9.2) one can write

$$\ddot{x} + (\delta + 2\varepsilon\cos 2t)x = 0 \tag{6.9.4}$$

This equation is known as Mathieu's equation. It may be noted that the basics of parametrically excited systems are based on Mathieu-Hill types of equation. As pointed out in module 4, one may use Floquet theory to study the stability of these systems.

Example 6.9.1: Study the stability of the Hill's equation given in Eq. (6.9.2) by taking the following initial condition.

$$u_1(0)=1, \quad \dot{u}_1(0)=0, \quad u_2(0)=0, \quad \dot{u}_2(0)=1 \tag{6.9.5}$$

Solution

Writing the fundamental sets of solution as

$$\begin{aligned} u_1(t+T) &= a_{11}u_1(t)+a_{12}u_2(t) \\ u_2(t+T) &= a_{21}u_1(t)+a_{22}u_2(t) \end{aligned} \tag{6.9.6}$$

$$\begin{aligned} \dot{u}_1(t+T) &= a_{11}\dot{u}_1(t)+a_{12}\dot{u}_2(t) \\ \dot{u}_2(t+T) &= a_{21}\dot{u}_1(t)+a_{22}\dot{u}_2(t) \end{aligned} \tag{6.9.7}$$

one can obtain

$$a_{11}=u_1(T), \quad a_{21}=u_2(T), \quad a_{12}=\dot{u}_1(T), \quad a_{22}=\dot{u}_2(T) \tag{6.9.8}$$

Or $A=\begin{bmatrix} u_1(T) & \dot{u}_1(T) \\ u_2(T) & \dot{u}_2(T) \end{bmatrix}$ (6.9.9)

Finding the determinant of $A-\lambda I$ matrix one may write

$$\lambda^2-2\alpha\lambda+\Delta=0 \tag{6.9.10}$$

where

$$\alpha=\tfrac{1}{2}[u_1(T)+\dot{u}_2(T)], \Delta=u_1(T)\dot{u}_2(T)-\dot{u}_1(T)u_2(T) \tag{6.9.11}$$

The parameter Δ is known as the Wronskian determinant of $u_1(T)$ and $u_2(T)$.

In case of Hill's equation the Wronskian determinant can also be obtained as follows.

As, $u_1(t)$ and $u_2(t)$ are the fundamental set of solution, hence they must satisfy Eq. (6.9.2). Hence, one can write

$$\begin{aligned} \ddot{u}_1+p(t)u_1 &= 0 \\ \ddot{u}_2+p(t)u_2 &= 0 \end{aligned} \tag{6.9.12}$$

Or, $u_1\ddot{u}_2-\ddot{u}_1u_2=0$ (6.9.13)

Which can be integrated to obtain, $\Delta(t)=u_1(t)\dot{u}_2(t)-\dot{u}_1(t)u_2(t)$ =constant (6.9.14)

At t =0, $\Delta(t)=1$. So the roots of the Eq. (6.9.10) can be given by

$$\lambda_{1,2}=\alpha\pm\sqrt{\alpha^2-1} \tag{6.9.15}$$

Or, $\lambda_1\lambda_2=1$ (6.9.16)

From Eq.(6.9.15) it may be noted that $|\alpha|$ >1 one of the root will be greater than unity while the other root is less than one. Hence one of the normal solutions is unbounded and the other is bounded. When $|\alpha|$ <1 both the roots will be complex conjugate and their absolute value will be less than one. Hence they will be in the unit circle. So the solutions will be bounded. It may be observed that the transistion from stable to unstable will takes place when $|\alpha|$ =1. This corresponds to a periodic solution of period T when $\lambda_1=\lambda_2=1$ and a periodic solution of period 2T when $\lambda_1=\lambda_2=-1$.

Example 6.9.2: Study the stability of the Mathieu's equation using same initial condition given in example 6.9.1.

Solution: In case of Mathieu equation

$$\ddot{x}+(\delta+2\varepsilon\cos 2t)x=0 \tag{6.9.17}$$

α is a function of δ and ε. The values of δ and ε for which $|\alpha|$ >1 are called unstable values while those for which $|\alpha|$ =1 are called transition values. The locus of transition values separates the $\varepsilon\ \delta$ plane into regions of stability and instability as shown in the fallowing figure.

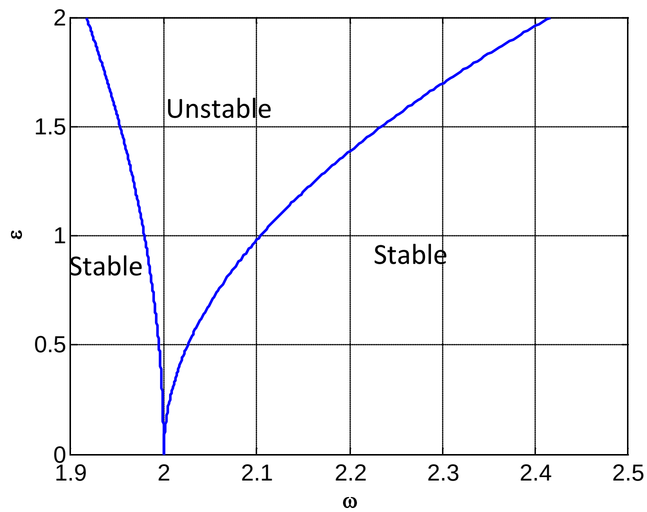


Fig 6.9.1 Transition curve for Matheiu equation

Hill's Infinite Determinat

One may also use hill's infinite determinant method to find the transition curve which is explained below.

$$\ddot{u} + \left(\delta + 2\varepsilon \cos 2t\right)u = 0 \tag{6.9.18}$$

Using Floquet theory one may assume the solution of the equation (6.9.18) as

$$u = \exp\left(\gamma t\right)\phi\left(t\right) \tag{6.9.19}$$

Where $\phi\left(t\right) = \phi\left(t+T\right)$. One may expand $\phi\left(t\right)$ in a Fourier series to obtain the following equation.

$$u = \sum_{n=-\infty}^{\infty} \phi_n \exp\left[\left(\gamma + 2in\right)t\right] \tag{6.9.20}$$

Where ϕ_n is constant. Substituting Eq. (6.9.20) in (6.9.18) one obtains

$$\sum_{n=-\infty}^{\infty} \left\{\left[\left(\gamma + 2in\right)^2 + \delta\right]\phi_n \exp\left[\left(\gamma + 2in\right)t\right]\right\} + \in \sum_{n=-\infty}^{\infty} \phi_n \left\{\exp\left[\gamma t + 2i\left(n+1\right)t\right] + \exp\left[\gamma t + 2i\left(n-1\right)t\right]\right\} = 0 \tag{6.9.21}$$

Equating each of the coefficients of the exponential functions to the zero one can obtain the following infinite set of linear, algebraic, homogenous equations for ϕ_n

$$\left[\left(\gamma + 2in\right)^2 + \delta\right]\phi_n + \varepsilon\left(\phi_{n-1} + \phi_{n+1}\right) = 0 \tag{6.9.22}$$

For a non trivial solution the determinant of the coefficient matrix must vanish. Since the determinant is infinite one may divide the mth row by $\delta - 4m^2$ for convergence considerations to obtain the following Hill's determinant.

$$\Delta(\gamma)=\begin{vmatrix} \cdots & . & . & . & . & . & . & . & . & . & \cdots \\ \cdots & 0 & \frac{\varepsilon}{\delta-4^2} & \frac{\delta+(\gamma-4i)^2}{\delta-4^2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{\varepsilon}{\delta-2^2} & \frac{\delta+(\gamma-2i)^2}{\delta-2^2} & \frac{\varepsilon}{\delta-2^2} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{\varepsilon}{\delta} & \frac{\delta+\gamma^2}{\delta} & \frac{\varepsilon}{\delta} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \frac{\varepsilon}{\delta-2^2} & \frac{\delta+(\gamma-2i)^2}{\delta-2^2} & \frac{\varepsilon}{\delta-2^2} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \frac{\varepsilon}{\delta-4^2} & \frac{\delta+(\gamma-4i)^2}{\delta-4^2} & \frac{\varepsilon}{\delta-4^2} & 0 & \cdots \end{vmatrix} \tag{6.9.23}$$

The determinant can be rewritten as (Whittaker and Watson 1962, Nayfeh and Mook 1979)

$$\Delta(\gamma)=\Delta(0)-\frac{\sin^2\left(\frac{1}{2}i\pi\gamma\right)}{\sin^2\left(\frac{1}{2}\pi\sqrt{\delta}\right)} \tag{6.9.24}$$

Since the characteristic exponents are solution of $\Delta(\gamma)=0$, they are given by

$$\gamma=\pm\frac{2i}{\pi}\sin^{-1}\left[\Delta(0)\sin^2\left(\frac{1}{2}\pi\sqrt{6}\right)\right]^{\frac{1}{2}} \tag{6.9.25}$$

Once γ is known ϕ_n can be related to ϕ_0 using equation (6.9.22).

One may also consider the central three rows and columns to approximate the characteristic equation as follows

$$\Delta(\gamma)=\begin{vmatrix} \delta+(\gamma-2i)^2 & \varepsilon & 0 \\ \varepsilon & \delta+\gamma^2 & \varepsilon \\ 0 & \varepsilon & \delta+(\gamma+2i)^2 \end{vmatrix}=0$$

Or $\left[\delta+(\gamma+2i)^2\right](\delta+\gamma^2)\left[\delta+(\gamma-2i)^2\right]-\varepsilon^2\left[\delta+(\gamma+2i)^2\right]-\varepsilon^2\left[\delta+(\gamma-2i)^2\right]=0$ (6.9.26)

The transition curve separating stability from instability correspond to $\gamma=0$ (i.e., periodic solution with period π) or $\gamma=\pm i$ (i.e., periodic motion with period 2π).

When $\gamma=0$ Eq. (6.9.26) gives the transition curves

$$\delta=-\frac{1}{2}\varepsilon^2 \text{ and } \delta=4+\frac{1}{2}\varepsilon^2 \tag{6.9.27}$$

When $\gamma=\pm i$ Eq. (6.9.26) gives the transition curves

$$\delta=1\pm\varepsilon \quad \text{and} \quad \delta=9+\frac{1}{8}\varepsilon^2 \tag{6.9.28}$$

Exercise problem

Problem 6.9.1: Use Floquet theory to study the stability of the periodic motion corresponding to primary resonance of Duffing equation.

Problem 6.9.2: Use Method of Multiple Scales to determine the equations for transition curves for Matheiu equation. Plot these transition curves near $\delta=\omega^2=1$, 4. Taking few points in the stable and unstable regions and by using numerical method to solve the Mathieu's equation, plot the time responses to check whether the marked instability regions are correct.

Module 6 Lect 10

Multi-degree-of freedom parametrically excited system

Case study: Instability Region of a Sandwich beam

In this lecture a case study has been taken by considering a three-layered, soft-cored, symmetric sandwich beam subjected to a periodic axial load. For completeness purpose the derivation of the governing equation of motion is given here and then the parametric instability regions for simple and combination resonances are investigated for simply supported and clamped-free end conditions by modified Hsu's method.

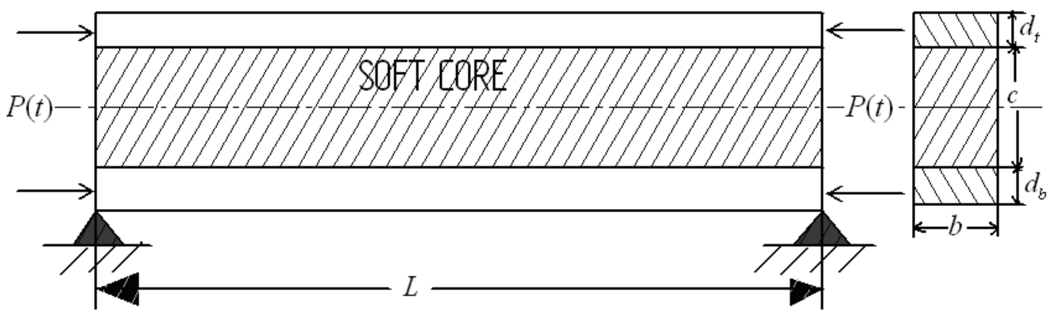
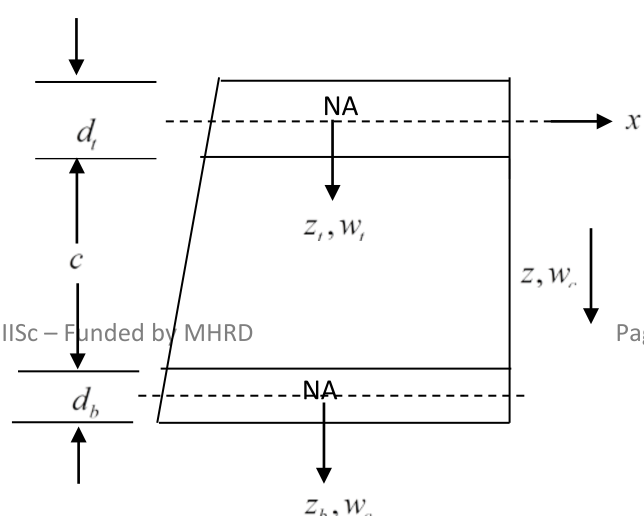


Figure 6.10.1 Symmetric three-layered soft cored sandwich beam with periodic axial load

Figure 6.10.1 shows a simply supported, symmetric, three-layered sandwich beam of length L and width b with a flexible soft core. The top, core and bottom layer thickness are d_t, c and d_b, respectively. The upper and lower layers (face layer) of the beam are of the same elastic material and the core is of soft viscoelastic material. The sandwich beam is subjected to an axial periodic load $P(t) = P_0 + P_1 \cos \omega t$, ω being the frequency of the applied load, t being the time and P_0 and P_1 are the amplitudes of static and dynamic load, respectively.

Figure 6.10.2 shows the geometry of the sandwich beam, the load and internal forces and moments in different layers and the deflection in x and z directions before and after deformations. Here, Q_{xx} is the shear force, N_{xx} is the axial force and M_{xx} is the bending moment. Superscripts t and b represent the top and bottom layer, respectively. The assumptions made for deriving the governing equations are similar to that by Frostig [1-4] and are (i) the face sheets of the sandwich beam are modeled as Euler-Bernoulli beams (ii) the transversely flexible core layer is considered as a two dimensional elastic medium with small deformations where its height may change under loading, and its cross section does not remain planar. The longitudinal (in-plane) stresses in the core are neglected and (iii) the interface layers between the face sheets and the core are assumed to be infinitely rigid and provide perfect continuity of the deformations at the interfaces.



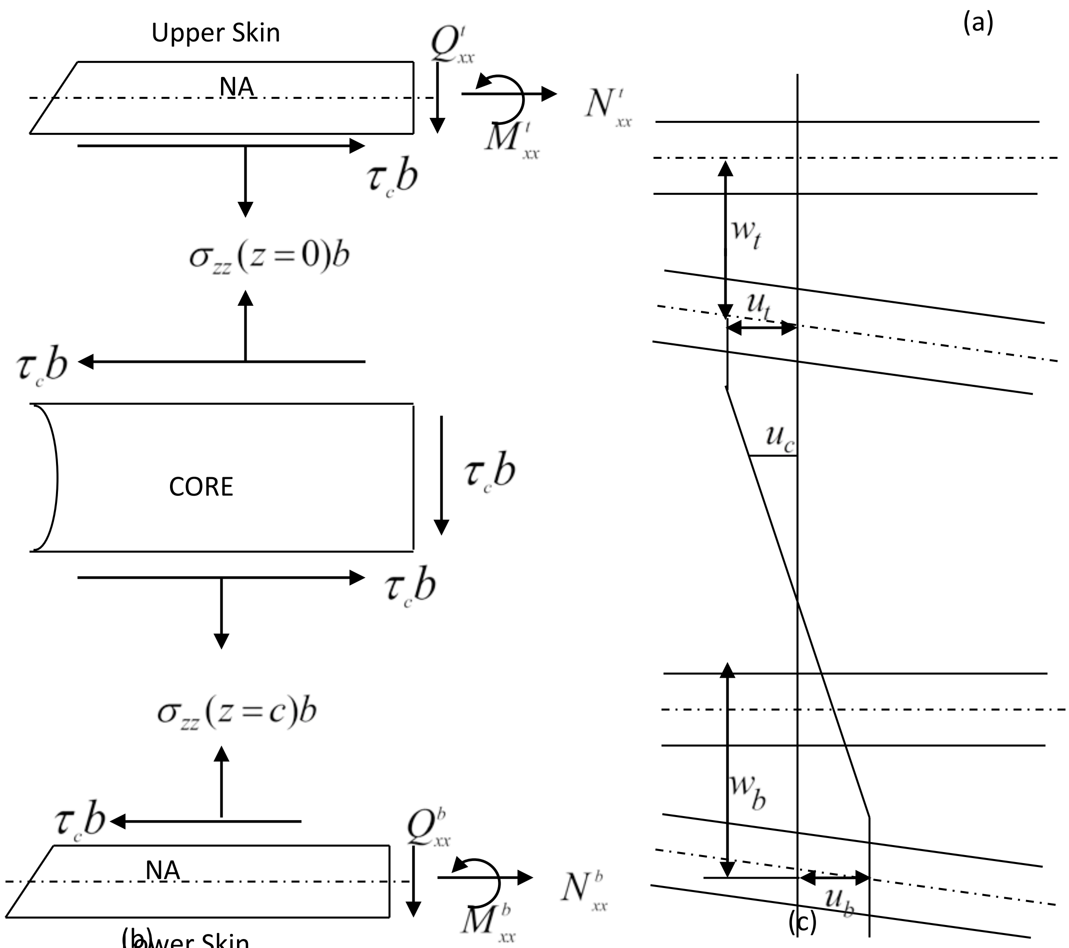


Figure 6.10.2 (a) Geometry, (b) load, internal forces and moments (c) displacement pattern through depth of section. N.A is the neutral axis.

The internal potential energy (U) in terms of direct stresses (σ) and shear stress τ and strains (ε, γ) is given by,

$$U = \int_{v_{top}} \sigma_{xx}\varepsilon_{xx}dv + \int_{v_{bot}} \sigma_{xx}\varepsilon_{xx}dv + \int_{v_{core}} \tau_c\gamma_c dv + \int_{v_{core}} \sigma_{zz}\varepsilon_{zz}dv, \tag{6.10.1}$$

where, v_{top}, v_{bot}, and v_{core} are the volume of the top, bottom and core layer, respectively. One may note that, as the core is taken to be flexible, deformation takes place in the transverse direction (z direction) and the last term of equation (1) takes care of that effect. The kinetic energy T can be given by

$$T = (1/2)\left\{\int_0^L m_t\left(\dot{u}_t^2 + \dot{w}_t^2\right)dx + \int_0^L m_b\left(\dot{u}_b^2 + \dot{w}_b^2\right)dx + \int_{v_{core}} \rho_c \dot{u}_c^2\, dv + \int_{v_{core}} \rho_c \dot{w}_c^2 dv\right\}. \qquad (6.10.2)$$

Here $m_{t,}$ and m_b are the mass per unit length of the top and bottom layer, respectively and ρ_c is the density of the core material; u_t and u_b are the displacement at the neutral axis of the top and bottom layer along *x* (longitudinal) direction, respectively; w_t and w_b are the displacement at top and bottom layer along *z* (vertical) direction, respectively (Fig. 6.10.2(c)). Also, u_c and w_c are the displacement of the core along *x* and *z* directions, respectively and can be given by [1]

$$u_c = u_t - (d_t/2)w_{t,x} + \frac{\left\{u_b + (d_b/2)w_{b,x} - \left(u_t - (d_t/2)w_{t,x}\right)\right\}z}{c},\; w_c = w_t + \frac{(w_b - w_t)\,z}{c} \qquad (6.10.3)$$

Here $()_{,x}$ represents the differentiation with respect to *x* and subscripts *t* , *b*, *c* represent top, bottom and core layer, respectively.

The non-conservative work done due to the applied load can be given by

$$W_{nc} = (1/2)\left(\int_0^L P w_{t,x}^2 dx + \int_0^L P w_{b,x}^2 dx\right). \qquad (6.10.4)$$

The following non-dimensional parameters are used in this analysis.

$$\bar{P}_0 = P_0 L^2/(2E_q I_q),\, \bar{P}_1 = P_1 L^2/(2E_q I_q),\, \xi_c = G_c^* A_c L^2/E,\, \phi_t = E_t A_t L^2/E,\, \phi_b = E_b A_b L^2/E,\quad \phi_c = E_c A_c L^2/E,$$
$$g = G_c/\left(E_t(c/d_t)(L/d_t)^2 + E_b(c/d_b)(L/d_b)^2\right), \bar{t} = t/t_0,\ \bar{x} = x/L, \qquad \bar{u}_q = u_q/L, \qquad \bar{w}_q = w_q/L,$$
$$\bar{m}_q = m_q/m,\, \bar{m}_c = m_c/m. \qquad (6.10.5)$$

Here, $\bar{P}_0$ and $\bar{P}_1$ are, respectively, the non-dimensional static and dynamic load amplitudes; E_q, I_q and A_q are the Young's modulus, moment of inertia and the area of cross-section of the *q*th layer (*q* equal to *t* for top layer and *b* for bottom layer); $E = E_t I_t + E_b I_b$; E_c, A_c and m_c are the Young's modulus, area of cross-section and mass per unit length of the core, respectively. The non-dimensional time, $t_0 = (mL^4/E)^{(1/2)}$, where m is the total mass per unit length. The complex shear modulus of the viscoelastic core is given by $G_c^* = G_c(1 + j\eta_c)$, where

G_c is the phase shear modulus, $j=\sqrt{-1}$ and η_c is the core loss factor. The non-dimensional term g is known as the shear parameter of the system.

Using equations (6.10.1-6.10.5), the governing non-dimensional equations of motion and the boundary conditions are derived by applying the extended Hamilton's principle. These resulting governing equations of motion are as follows.

$$
\begin{aligned}
&(\bar{m}_t+\bar{m}_c/3)\ddot{\bar{w}}_t-(\bar{m}_c/12)(d_t/c)^2(c/L)^2\ddot{\bar{w}}_{t,\overline{xx}}+(\bar{m}_c/576)(d_t/c)\{1+(d_t/c)\}(c/L)^4(\xi_c/\phi_c)\ddot{\bar{w}}_{t,\overline{xxxx}}\\
&+(\bar{m}_c/24)(d_t/c)(d_b/c)(c/L)^2\ddot{\bar{w}}_{b,\overline{xx}}+(\bar{m}_c/576)(d_b/c)\{1+(d_t/c)\}(c/L)^4(\xi_c/\phi_c)\ddot{\bar{w}}_{b,\overline{xxxx}}\\
&+(\bar{m}_c/6)\ddot{\bar{w}}_b+(\bar{m}_c/6)(d_t/c)(c/L)\ddot{\bar{u}}_{t,\bar{x}}-(1/48)(\bar{m}_t+\bar{m}_c/6)\{1+(d_t/c)\}(c/L)^3(\xi_c/\phi_c)\ddot{\bar{u}}_{t,\overline{xxx}}\\
&+(\bar{m}_c/12)(d_t/c)(c/L)\ddot{\bar{u}}_{b,\bar{x}}+(1/48)(\bar{m}_b+\bar{m}_c/6)\{1+(d_t/c)\}(c/L)^3(\xi_c/\phi_c)\ddot{\bar{u}}_{b,\overline{xxx}}\\
&-\phi_c(L/c)^2\bar{w}_t-(\xi_c/4)\{1+(d_t/c)\}^2\bar{w}_{t,\overline{xx}}-\phi_c(L/c)^2\bar{w}_b-(\xi_c/4)\{1+(d_t/c)\}\{1+(d_b/c)\}\bar{w}_{b,\overline{xx}}\\
&+(\xi_c/2)(L/c)\{1+(d_t/c)\}\bar{u}_{t,\bar{x}}+(\xi_c/\phi_c)\{1+(d_t/c)\}(\phi_t/48)(c/L)^3\bar{u}_{t,\overline{xxxxx}}\\
&-(\xi_c/2)(L/c)\{1+(d_t/c)\}\bar{u}_{b,\bar{x}}-(\xi_c/\phi_c)\{1+(d_t/c)\}(\phi_b/48)(c/L)^3\bar{u}_{b,\overline{xxxxx}}\\
&+(\phi_t/12)(d_t/c)^2(c/L)^2\bar{w}_{t,\overline{xxxx}}+(PL^2/E)\bar{w}_{t,\overline{xx}}=0
\end{aligned}
\tag{6.10.6}
$$

$$
\begin{aligned}
&(\bar{m}_c/6)\ddot{\bar{w}}_t+(\bar{m}_c/24)(d_t/c)(d_b/c)(c/L)^2\ddot{\bar{w}}_{t,\overline{xx}}+(\bar{m}_c/576)(d_t/c)\{1+(d_b/c)\}(c/L)^4(\xi_c/\phi_c)\ddot{\bar{w}}_{t,\overline{xxxx}}\\
&+(\bar{m}_b+\bar{m}_c/3)\ddot{\bar{w}}_b-(\bar{m}_c/12)(d_t/c)^2(c/L)^2\ddot{\bar{w}}_{b,\overline{xx}}+(\bar{m}_c/576)(d_b/c)\{1+(d_b/c)\}(c/L)^4(\xi_c/\phi_c)\ddot{\bar{w}}_{b,\overline{xxxx}}\\
&-(\bar{m}_c/12)(d_b/c)(c/L)\ddot{\bar{u}}_{t,\bar{x}}-(1/48)(\bar{m}_t+\bar{m}_c/6)\{1+(d_b/c)\}(c/L)^3(\xi_c/\phi_c)\ddot{\bar{u}}_{t,\overline{xxx}}-(\bar{m}_c/6)(d_b/c)(c/L)\ddot{\bar{u}}_{b,\bar{x}}\\
&+(1/48)(\bar{m}_b+\bar{m}_c/6)\{1+(d_b/c)\}(c/L)^3(\xi_c/\phi_c)\ddot{\bar{u}}_{b,\overline{xxx}}+\phi_c(L/c)^2\bar{w}_b-(\xi_c/4)\{1+(d_t/c)\}\{1+(d_b/c)\}\bar{w}_{t,\overline{xx}}\\
&-\phi_c(L/c)^2\bar{w}_t-(\xi_c/4)\{1+(d_b/c)\}^2\bar{w}_{b,\overline{xx}}+(\xi_c/2)(L/c)\{1+(d_b/c)\}\bar{u}_{t,\bar{x}}\\
&+(\xi_c/\phi_c)\{1+(d_b/c)\}(\phi_t/48)(c/L)^3\bar{u}_{t,\overline{xxxxx}}-(\xi_c/2)(L/c)\{1+(d_b/c)\}\bar{u}_{b,\bar{x}}\\
&-(\xi_c/\phi_c)\{1+(d_b/c)\}(\phi_b/48)(c/L)^3\bar{u}_{b,\overline{xxxxx}}+(\phi_b/12)(d_b/c)^2(c/L)^2\bar{w}_{b,\overline{xxxx}}+(PL^2/E)\bar{w}_{b,\overline{xx}}=0
\end{aligned}
\tag{6.10.7}
$$

$$
\begin{aligned}
&(\bar{m}_c/6)(d_t/c)(c/L)\ddot{\bar{w}}_{t,\bar{x}}-(\bar{m}_c/288)(d_t/c)(c/L)^3(\xi_c/\phi_c)\ddot{\bar{w}}_{t,\overline{xxx}}\\
&-(\bar{m}_c/12)(d_b/c)(c/L)\ddot{\bar{w}}_{b,\bar{x}}-(\bar{m}_c/288)(d_b/c)(c/L)^3(\xi_c/\phi_c)\ddot{\bar{w}}_{b,\overline{xxx}}\\
&+(1/24)(\bar{m}_t+\bar{m}_c/6)(c/L)^2(\xi_c/\phi_c)\ddot{\bar{u}}_{t,\overline{xx}}-(\bar{m}_t+\bar{m}_c/3)\ddot{\bar{u}}_t\\
&-(1/24)(\bar{m}_b+\bar{m}_c/6)(c/L)^2(\xi_c/\phi_c)\ddot{\bar{u}}_{b,\overline{xx}}-(\bar{m}_c/6)\ddot{\bar{u}}_b+(\xi_c/2)\{1+(d_t/c)\}(L/c)\bar{w}_{t,\bar{x}}\\
&+(\xi_c/2)(L/c)\{1+(d_b/c)\}\bar{w}_{b,\bar{x}}+\phi_t\bar{u}_{t,\overline{xx}}-(L/c)^2\xi_c\bar{u}_t-(\xi_c/\phi_c)(\phi_t/24)(c/L)^2\bar{u}_{t,\overline{xxxx}}\\
&+(L/c)^2\xi_c\bar{u}_b+(\xi_c/\phi_c)(\phi_b/24)(c/L)^2\bar{u}_{b,\overline{xxxx}}=0
\end{aligned}
\tag{6.10.8}
$$

$$
\begin{aligned}
&(\bar{m}_c/12)(d_t/c)(c/L)\ddot{\bar{w}}_{t,\bar{x}} + (\bar{m}_c/288)(d_t/c)(c/L)^3(\xi_c/\phi_c)\ddot{\bar{w}}_{t,\bar{x}\bar{x}\bar{x}} \\
&-(\bar{m}_c/6)(d_b/c)(c/L)\ddot{\bar{w}}_{b,\bar{x}} + (\bar{m}_c/288)(d_b/c)(c/L)^3(\xi_c/\phi_c)\ddot{\bar{w}}_{b,\bar{x}\bar{x}\bar{x}} \\
&-(\bar{m}_c/6)\ddot{\bar{u}}_t - (1/24)(\bar{m}_t + \bar{m}_c/6)(c/L)^2(\xi_c/\phi_c)\ddot{\bar{u}}_{t,\bar{x}\bar{x}} - (\bar{m}_b + \bar{m}_c/3)\ddot{\bar{u}}_b \\
&+(1/24)(\bar{m}_b + \bar{m}_c/6)(c/L)^2(\xi_c/\phi_c)\ddot{\bar{u}}_{b,\bar{x}\bar{x}} - (\xi_c/2)\{1+(d_t/c)\}(L/c)\bar{w}_{t,\bar{x}} \\
&-(\xi_c/2)(L/c)\{1+(d_b/c)\}\bar{w}_{b,\bar{x}} + (L/c)^2\xi_c\bar{u}_t + (\xi_c/\phi_c)(\phi_t/24)(c/L)^2\bar{u}_{t,\bar{x}\bar{x}\bar{x}\bar{x}} \\
&+\phi_b\bar{u}_{b,\bar{x}\bar{x}} - (L/c)^2\xi_c\bar{u}_b - (\xi_c/\phi_c)(\phi_b/24)(c/L)^2\bar{u}_{b,\bar{x}\bar{x}\bar{x}\bar{x}} = 0
\end{aligned}
\tag{6.10.9}
$$

As the above equations of motion (6.10.6-6.10.9) are in space and time co-ordinates, generalized Galerkin's principle is used to reduce these equations to their temporal form. For multi-mode discretization one may take

$$
\bar{w}_t = \sum_{p=1}^{N} f_p(\bar{t})w_p(\bar{x}), \quad \bar{w}_b = \sum_{q=N+1}^{2N} f_q(\bar{t})w_q(\bar{x}), \quad \bar{u}_t = \sum_{r=2N+1}^{3N} f_r(\bar{t})u_r(\bar{x}) \text{ and}
$$

$$
\bar{u}_b = \sum_{s=3N+1}^{4N} f_s(\bar{t})u_s(\bar{x}). \tag{6.10.10}
$$

Here, N is a positive integer representing the number of modes taken in the analysis, and $f_p(\bar{t})$, $f_q(\bar{t})$, $f_r(\bar{t})$ and $f_s(\bar{t})$ are the generalized co-ordinates and $w_p(\bar{x})$, $w_q(\bar{x})$, $u_r(\bar{x})$ and $u_s(\bar{x})$ are the shape functions chosen to satisfy as many as the boundary conditions. The resulting equation of motion becomes

$$
[M]\{\ddot{f}\} + [K]\{f\} - \bar{P}_1\cos\bar{\omega}\bar{t}\,[H]\{f\} = \{\phi\}. \tag{6.10.11}
$$

Here, $(\dot{\cdot}) = d(\;)/d\bar{t}$, $\{f\} = \left\{\{f_p\}^T\{f_q\}^T\{f_r\}^T\{f_s\}^T\right\}^T$, and $[K] = [K_1] - \bar{P}_0[H]$,

where

$$
[M] = \begin{bmatrix} [M_{11}] & [M_{12}] & [M_{13}] & [M_{14}] \\ [M_{21}] & [M_{22}] & [M_{23}] & [M_{24}] \\ [M_{31}] & [M_{32}] & [M_{33}] & [M_{34}] \\ [M_{41}] & [M_{42}] & [M_{43}] & [M_{44}] \end{bmatrix}, \quad [K_1] = \begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] & [K_{14}] \\ [K_{21}] & [K_{22}] & [K_{23}] & [K_{24}] \\ [K_{31}] & [K_{32}] & [K_{33}] & [K_{34}] \\ [K_{41}] & [K_{42}] & [K_{43}] & [K_{44}] \end{bmatrix}
$$

$$[H] = \begin{bmatrix} [H_{11}] & [\phi] & [\phi] & [\phi] \\ [\phi] & [H_{22}] & [\phi] & [\phi] \\ [\phi] & [\phi] & [\phi] & [\phi] \\ [\phi] & [\phi] & [\phi] & [\phi] \end{bmatrix},$$ $\{\phi\}$ and $[\phi]$ are null matrices.

The elements of the various sub matrices are given below.

$$(M_{11})_{ij} = (\overline{m}_t + m_c/3)\left(\int_0^1 w_i w_j d\overline{x}\right) + \left\{(\overline{m}_c/12)(d_t/L)^2\right\}\left(\int_0^1 w_i' w_j' d\overline{x}\right)$$
$$+\left\{(\overline{m}_c/576)(d_t/c)(1+d_t/c)(c/L)^4(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' w_j'' d\overline{x}\right)$$

$$(M_{12})_{ij} = (\overline{m}_c/6)\left(\int_0^1 w_i w_j d\overline{x}\right) - \left\{(\overline{m}_c/24)(d_t\, d_b/l^2)\right\}\left(\int_0^1 w_i' w_j' d\overline{x}\right)$$
$$+\left\{(\overline{m}_c/576)(d_b/c)(1+d_t/c)(c/L)^4(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' w_j'' d\overline{x}\right)$$

$$(M_{13})_{ij} = \left\{(\overline{m}_t + \overline{m}_c/6)(1/48)(1+d_t/c)(c/L)^3(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' u_j' d\overline{x}\right) + \left\{(m_c/6)(d_t/L)\right\}\left(\int_0^1 w_i' u_j d\overline{x}\right)$$

$$(M_{14})_{ij} = -\left\{(1/48)(\overline{m}_b + \overline{m}_c/6)(1+d_t/c)(L)^3(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' w_j' d\overline{x}\right) + \left\{(\overline{m}_c/12)(L)\right\}\left(\int_0^1 w_i' u_j d\overline{x}\right)$$

$$(M_{21})_{ij} = (\overline{m}_c/6)\left(\int_0^1 w_i w_j d\overline{x}\right) - \left\{(\overline{m}_c/24)(d_t\, d_b/L^2)\right\}\left(\int_0^1 w_i' w_j' d\overline{x}\right)$$
$$+\left\{(\overline{m}_c/576)(d_t/c)(1+d_b/c)(c/L)^4(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' w_j'' d\overline{x}\right)$$

$$(M_{22})_{ij} = (\overline{m}_b + \overline{m}_c/3)\left(\int_0^1 w_i w_j d\overline{x}\right) + \left\{(\overline{m}_c/12)(c/L)^2\right\}\left(\int_0^1 w_i' w_j' d\overline{x}\right)$$
$$+\left\{(\overline{m}_c/576)(d_b/c)(1+d_b/c)(c/L)^4(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' w_j'' d\overline{x}\right)$$

$$(M_{23})_{ij} = \left\{(1/48)(\overline{m}_t + \overline{m}_c/6)(1+d_b/c)(c/L)^3(\xi_c/\phi_c)\right\}\left(\int_0^1 w_i'' u_j' d\overline{x}\right) - \left\{(\overline{m}_c/12)(c/L)\right\}\left(\int_0^1 w_i' u_j d\overline{x}\right)$$

$$\left(M_{24}\right)_{ij}=-\left\{(1/48)\left(\bar{m}_b+\bar{m}_c/6\right)\left(1+d_b/c\right)\left(c/L\right)^3\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 w_i''u_j'd\bar{x}\right)-\left(\bar{m}_c/6\right)\left(d_b/L\right)\left(\int_0^1 w_i'u_j d\bar{x}\right)$$

$$\left(M_{31}\right)_{ij}=\left\{\left(\bar{m}_c/288\right)\left(d_t/c\right)\left(c/L\right)^3\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i''w_j'd\bar{x}\right)+\left\{\left(\bar{m}_c/6\right)\left(d_t/L\right)\right\}\left(\int_0^1 u_i'w_j d\bar{x}\right)$$

$$\left(M_{32}\right)_{ij}=\left\{\left(\bar{m}_c/12\right)\left(d_b/L\right)\right\}\left(\int_0^1 u_i'w_j d\bar{x}\right)-\left\{\left(\bar{m}_c/288\right)\left(d_b/c\right)\left(c/L\right)^3\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i''w_j'd\bar{x}\right)$$

$$\left(M_{33}\right)_{ij}=-\left\{(1/24)\left(\bar{m}_t+\bar{m}_c/6\right)\left(c/L\right)^2\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i'u_j'd\bar{x}\right)-\left(\bar{m}_t+\bar{m}_c/3\right)\left(\int_0^1 u_i u_j d\bar{x}\right)$$

$$\left(M_{34}\right)_{ij}=\left\{\left(-\bar{m}_c/6\right)\right\}\left(\int_0^1 u_i u_j d\bar{x}\right)+\left\{(1/24)\left(\bar{m}_b+\bar{m}_c/6\right)\left(c/L\right)^2\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i'u_j'd\bar{x}\right)$$

$$\left(M_{41}\right)_{ij}=-\left\{\left(\bar{m}_c/12\right)\left(d_t/L\right)\right\}\left(\int_0^1 u_i'w_j d\bar{x}\right)+\left\{\left(\bar{m}_c/288\right)\left(d_t/c\right)\left(c/L\right)^3\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i''w_j'd\bar{x}\right)$$

$$\left(M_{42}\right)_{ij}=\left\{\left(\bar{m}_c/6\right)\left(d_b/L\right)\right\}\left(\int_0^1 u_i'w_j d\bar{x}\right)+\left\{\left(\bar{m}_c/288\right)\left(d_b/c\right)\left(c/L\right)^3\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i'w_j d\bar{x}\right)$$

$$\left(M_{43}\right)_{ij}=\left\{\left(-\bar{m}_c/6\right)\right\}\left(\int_0^1 u_i u_j d\bar{x}\right)+\left\{(1/24)\left(\bar{m}_t+\bar{m}_c/6\right)\left(c/L\right)^2\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i'u_j'd\bar{x}\right)$$

$$\left(M_{44}\right)_{ij}=-\left\{\left(\bar{m}_b+\bar{m}_c/3\right)\right\}\left(\int_0^1 u_i u_j d\bar{x}\right)-\left\{(1/24)\left(\bar{m}_b+\bar{m}_c/6\right)\left(c/L\right)^2\left(\xi_c/\phi_c\right)\right\}\left(\int_0^1 u_i'u_j'd\bar{x}\right)$$

$$\left(K_{11}\right)_{ij}=\left\{(1/4)\left(1+d_t/c\right)^2\xi_c\right\}\left(\int_0^1 w_i'w_j'd\bar{x}\right)+\left\{(1/12)\left(d_t/L\right)^2\phi_c\right\}\left(\int_0^1 w_i''w_j''d\bar{x}\right)+\left\{\phi_c\left(L/c\right)^2\right\}\left(\int_0^1 w_i w_j d\bar{x}\right)$$

$$\left(K_{12}\right)_{ij}=-\left\{\phi_c\left(L/c\right)^2\right\}\left(\int_0^1 w_i w_j d\bar{x}\right)+\left\{(1/4)\left(1+d_t/c\right)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 w_i'w_j'd\bar{x}\right)$$
$$-\left\{(1/48)\left(c/L\right)^3\left(1+d_t/c\right)\left(\xi_c/\phi_c\right)\phi_b\right\}\left(\int_0^1 w_i'''u_j''d\bar{x}\right)$$

$$\left(K_{13}\right)_{ij}=-\left\{(1/2)\left(L/c\right)\left(1+d_t/c\right)\xi_c\right\}\left(\int_0^1 w_i'u_j d\bar{x}\right)-\left\{(1/48)\left(c/L\right)^3\left(1+d_t/c\right)\left(\xi_c/\phi_c\right)\phi_t\right\}\left(\int_0^1 w_i'''u_j''d\bar{x}\right)$$

$$\left(K_{14}\right)_{ij}=\left\{(1/2)\left(L/c\right)\left(1+d_t/c\right)\xi_c\right\}\left(\int_0^1 w_i'u_j d\bar{x}\right)+\left\{(1/48)\left(c/L\right)^3\left(1+d_t/c\right)\left(\xi_c/\phi_c\right)\phi_b\right\}\left(\int_0^1 w_i'''u_j''d\bar{x}\right)$$

$$\left(K_{21}\right)_{ij}=-\left\{\phi_c\left(L/c\right)^2\right\}\left(\int_0^1 w_i w_j d\bar{x}\right)+\left\{(1/4)\left(1+d_t/c\right)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 w_i'w_j'd\bar{x}\right)$$

$$\left(K_{22}\right)_{ij}=\left\{(1/4)\left(1+d_b/c\right)^2\xi_c\right\}\left(\int_0^1 w_i' w_j' d\overline{x}\right)+\left\{(1/12)\left(d_b/L\right)^2\phi_c\right\}\left(\int_0^1 w_i'' w_j'' d\overline{x}\right)+\left\{\phi_c\left(L/c\right)^2\right\}\left(\int_0^1 w_i w_j d\overline{x}\right)$$

$$\left(K_{23}\right)_{ij}=\left\{(-1/2)(L/c)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 w_i' u_j d\overline{x}\right)-\left\{(1/48)(c/L)^3\left(1+d_b/c\right)\left(\xi_c/\phi_c\right)\phi_t\right\}\left(\int_0^1 w_i''' u_j'' d\overline{x}\right)$$

$$\left(K_{24}\right)_{ij}=\left\{(1/2)(L/c)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 w_i' u_j d\overline{x}\right)+\left\{(1/48)(c/L)^3\left(1+d_b/c\right)\left(\xi_c/\phi_c\right)\phi_b\right\}\left(\int_0^1 w_i''' u_j'' d\overline{x}\right)$$

$$\left(K_{31}\right)_{ij}=\left\{(-1/2)(L/c)\left(1+d_t/c\right)\xi_c\right\}\left(\int_0^1 u_i' w_j d\overline{x}\right)$$

$$\left(K_{32}\right)_{ij}=\left\{(-1/2)(L/c)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 u_i' w_j d\overline{x}\right)$$

$$\left(K_{33}\right)_{ij}=\left(-\phi_t\right)\left(\int_0^1 u_i' u_j' d\overline{x}\right)-\left\{(L/c)^2\xi_c\right\}\left(\int_0^1 u_i u_j d\overline{x}\right)-\left\{(1/24)(c/L)^2\left(\xi_c/\phi_c\right)\phi_t\right\}\left(\int_0^1 u_i'' u_j'' d\overline{x}\right)$$

$$\left(K_{34}\right)_{ij}=\left\{(L/c)^2\xi_c\right\}\left(\int_0^1 u_i u_j d\overline{x}\right)+\left\{(1/24)(c/L)^2\left(\xi_c/\phi_c\right)\phi_b\right\}\left(\int_0^1 u_i'' u_j'' d\overline{x}\right)$$

$$\left(K_{41}\right)_{ij}=\left\{(1/2)(L/c)\left(1+d_t/c\right)\xi_c\right\}\left(\int_0^1 u_i' w_j d\overline{x}\right)$$

$$\left(K_{42}\right)_{ij}=\left\{(1/2)(L/c)\left(1+d_b/c\right)\xi_c\right\}\left(\int_0^1 u_i' w_j d\overline{x}\right)$$

$$\left(K_{43}\right)_{ij}=\left\{(L/c)^2\xi_c\right\}\left(\int_0^1 u_i u_j d\overline{x}\right)+\left\{(1/24)(c/L)^2\left(\xi_c/\phi_c\right)\phi_t\right\}\left(\int_0^1 u_i'' u_j'' d\overline{x}\right)$$

$$\left(K_{44}\right)_{ij}=\left(-\phi_b\right)\left(\int_0^1 u_i' u_j' d\overline{x}\right)-\left\{(L/c)^2\xi_c\right\}\left(\int_0^1 u_i u_j d\overline{x}\right)-\left\{(1/24)(c/L)^2\left(\xi_c/\phi_c\right)\phi_b\right\}\left(\int_0^1 u_i'' u_j'' d\overline{x}\right)$$

$$\left(F\right)_{11}=\int_0^1 w_i' w_j' d\overline{x}$$

$$\left(F\right)_{22}=\int_0^1 w_i' w_j' d\overline{x}$$

In the above equations $()' = \partial()/\partial \bar{x}$.

Equation (6.10.11) is a set of coupled Mathieu-Hill equations with complex coefficients. For numerical calculations following shape functions (Ray and Kar [7]) are considered.

For simply supported beam

$$w_p(\bar{x}) = \sin(p\pi\bar{x}),\ w_q(\bar{x}) = \sin(\bar{q}\pi\bar{x}),\ u_r(\bar{x}) = \cos(\bar{r}\pi\bar{x}), \text{ and } u_s(\bar{x}) = \cos(\bar{s}\pi\bar{x}). \qquad (6.10.12)$$

These shape functions satisfy all the boundary conditions. Here $p = 1, 2\cdots, N$,

$$\bar{q} = (q - N),\ \bar{r} = (r - 2N) \text{ and } \bar{s} = (s - 3N). \qquad (6.\ 10.13a)$$

For clamped-free beam, the shape functions are as follows (Ray and Kar [7])

$$\begin{aligned} w_i(\bar{x}) &= (i+3)(i+2)\{(i+2)(i+1) - \mu_2\}\bar{x}^{(i+1)} + \left[2(i+3)(i+1)\{\mu_2 - i(i+2)\}\mu_1 i(i+1)/\{(i+2)(i+1) - \mu_2\}\right] \\ &+ \left[(i+2)(i+1)\{-\mu_2 + i(i+1)\} - \mu_1 i(i+1)^2 / \{(i+3)(i+2)(i+1) - (i+3)\mu_2\}\right]\bar{x}^{(i+3)}, \\ u_k(\bar{x}) &= (\bar{k}+1)\bar{x}^{\bar{k}} - \bar{k}\bar{x}^{(\bar{k}+1)} \end{aligned} \qquad (6.10.13b)$$

Here i and $\bar{k}$ are same as the previous boundary conditions.

If $[L]$ is a normalized modal matrix of $[M]^{-1}[K]$, then the linear transformation

$$\{f\} = [L]\{U\}, \qquad (6.10.14)$$

transforms equation (6.10.11) to,

$$\ddot{U}_q + (\omega_q^*)^2 U_q + 2\varepsilon\cos\bar{\omega}\bar{t}\sum_{p=1}^{4N} b_{qp}^* U_p = 0, \qquad q = 1\ldots.4\text{N}; \qquad (6.10.15)$$

where $(\omega_q^*)^2$ are the distinct eigen values of $[M]^{-1}[K]$ and b_{qp}^* are the elements of

$[B] = -[L]^{-1}[M]^{-1}[H][L]$. Also, $\varepsilon = \bar{P}_1/2 < 1$ for the present analysis. The complex frequency and forcing parameters in terms of real and imaginary parts are given by

$$\omega_q^* = \omega_{q,R} + j\omega_{q,I} \text{ and } b_{qp}^* = b_{qp,R} + jb_{qp,I}\,. \tag{6.10.16}$$

The boundaries of the regions of instability for simple and combination resonances are obtained by the modified Hsu's [9] method. When the system is excited at a frequency nearly equal to twice the natural frequencies principal parametric resonance and when it is excited near a frequency, which is equal to the sum or differences of any two modal frequencies combination resonances of sum or difference types take place. Following relations are used to obtain the boundaries of the regions of instability for simple and combination resonances [7].

(1) Simple resonance case

$$\left|(\bar{\omega}/2) - \omega_{\alpha,R}\right| < \frac{1}{4}\chi_\alpha, \qquad \alpha = 1, 2\ldots..4N \tag{6.10.17}$$

where $$\chi_\alpha = \left[\frac{4\varepsilon^2(b^2{}_{\alpha\alpha,R} + b^2{}_{\alpha\alpha,I}}{\omega^2{}_{\alpha,R}} - 16\omega^2{}_{\alpha,I}\right]. \tag{6.10.18}$$

(2) Combination resonance of sum type

$$\left|\bar{\omega} - (\omega_{\alpha,R} + \omega_{\beta,R})\right| < \chi_{\alpha\beta} \tag{6.10.19}$$

when damping is present,

$$\chi_{\alpha\beta} = \frac{(\omega_{\alpha,I} + \omega_{\beta,I})}{4(\omega_{\alpha,I}\omega_{\beta,I})^{1/2}}\left[\frac{4\varepsilon^2(b_{\alpha\beta,R}b_{\beta\alpha,R} + b_{\alpha\beta,I}b_{\beta\alpha,I})}{\omega_{\alpha,R}\omega_{\beta,R}} - 16\omega_{\alpha,I}\omega_{\beta,I}\right]^{1/2}, \tag{6.10.20}$$

and for the undamped case,

$$\chi_{\alpha\beta} = \varepsilon\left[\frac{b_{\alpha\beta,R}\ b_{\beta\alpha,R}}{\omega_{\alpha,R}\ \omega_{\beta,R}}\right]^{1/2}. \qquad \alpha \neq \beta,\quad \alpha,\ \beta = 1, 2\ldots..4N \tag{6.10.21}$$

(3) Combination resonance of difference type

$$\left|\bar{\omega} - (\omega_{\beta,R} - \omega_{\alpha,R})\right| < \Lambda_{\alpha\beta} \qquad \alpha > \beta\ ,\quad \alpha,\ \beta = 1, 2\ldots..4N \tag{6.10.22}$$

when damping is present,

$$\Lambda_{\alpha\beta} = \frac{(\omega_{\alpha,I} + \omega_{\beta,I})}{4(\omega_{\alpha,I}\omega_{\beta,I})^{1/2}} \left[\frac{4\varepsilon^2 (b_{\alpha\beta,I} b_{\beta\alpha,I} - b_{\alpha\beta,R} b_{\beta\alpha,R})}{\omega_{\alpha,R}\omega_{\beta,R}} - 16\omega_{\alpha,I}\omega_{\beta,I} \right]^{1/2}, \tag{6.10.23}$$

and for the undamped case,

$$\Lambda_{\alpha\beta} = \varepsilon \left[-\frac{b_{\alpha\beta,R}\ b_{\beta\alpha,R}}{\omega_{\alpha,R}\ \omega_{\beta,R}} \right]^{1/2}. \tag{6.10.24}$$

Numerical Results and Discussions

Here the parametric instability regions of a three-layered symmetric sandwich beam with simply supported, and clamped-free boundary conditions have been determined numerically using MATLAB. For visco-elastic materials, core loss factor (η_c) is a measure energy dissipation capacity and the shear parameter $g = G_c / \left(2E_t (c/d_t)(L/d_t)^2 \right)$ is a measure of stiffness of the material and is important in determining how much energy gets into the visco-elastic material. So these two parameters are varied in determining the instability regions for the parametrically excited beams. Also the effects of core and skin thickness on the instability regions are studied for all these boundary conditions. In the parametric instability regions shown in the following figures, the regions enclosed by the curves are unstable and the regions outside the curves are stable. Here the ordinate $\overline{P}_1$ is the amplitude of non-dimensional dynamic load and the abscissa ϖ is the non-dimensional forcing frequency. Following physical parameters are taken for the numerical analysis. The span of the beam, L=300 cm, width, b=50 mm, the top and bottom face thickness $d_t = d_b = 2$ mm and the core thickness, c = 30 mm. The non-dimensional static load amplitude $\overline{P}_0 = 0.1$ for all the figures except it is specifically mentioned. The top and bottom faces are of steel and the core is of soft plastic foam (Divinycell H60). The mechanical properties of steel and Divinycell H60 are given in Table (1).

Table 1:Material properties of sandwich beam [10]

Material	Young's modulus E, Gpa	Shear modulus G, Gpa	Poisson's ratio ν	Density ρ, kg/m^3
Steel	210	81	0.3	7900

Divinycell H60	0.056	0.022	0.27	60

Simply supported beam

Using the shape functions given in equation (6.10. 12) the instability regions for the simply supported beam are determined and shown in Figures (6.10.3-6.11.6) for the first three modes. Figure 6.10.3 shows the parametric instability regions obtained using both the higher-order theory and classical theory [10] for simple resonances. One may observe that for all the three modes, the region of instability starts at a lower frequency for higher order theory in comparison to the classical theory, which is due to the fact that, the core is considered to be more flexible in higher order theory than in case of classical theory. Also, it is clearly observed from these figures that the instability region is wider in case of higher order theory as compared to the classical theory. With change in $\overline{P}_0$ (say $\overline{P}_0$ =0.1), while instability region with higher order theory remains almost unchanged, it is observed that for lower value of $\overline{P}_1$, the instability regions with classical theory shifts towards left.

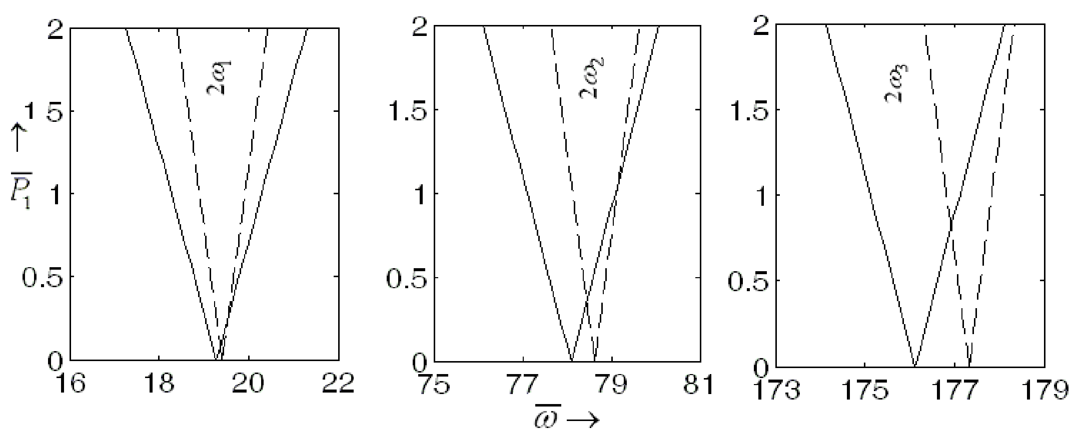


Figure 6.10.3: Comparison of instability regions using higher-order and classical theories, $\overline{P}_0 = 0.5$, $\eta_c = 0.1$; g=0.05; ———, higher-order theory; ------, classical theory.

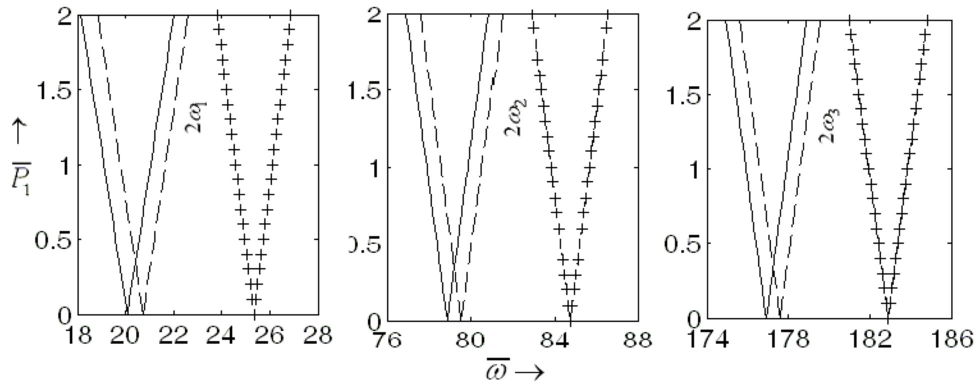


Figure 6.10.4: Effect of shear parameter on instability regions for $\eta_c = 0.0$. ——— , g = 0.05; -----, g = 0.1; +++ , g = 0.5.

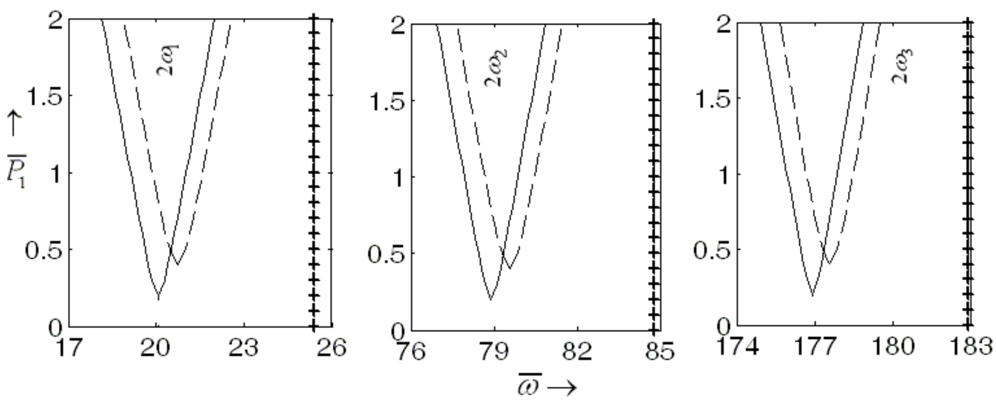


Figure 6.10.5: Effect of shear parameter on instability regions for $\eta_c = 0.3$.———, g = 0.05; -----, g = 0.1; +++, g = 0.5.

Figures (6.10.4-5) show the influence of core loss factor (η_c) and the shear parameter (g) upon the instability region obtained by using higher order theory. It is clearly observed that increase in core loss factor improves the stability by shifting the instability zones upwards and reducing the area of instability, which is similar to those, obtained by classical theory. It is also observed that with increase in shear parameter stability of the system improves. From the above figures it is clearly understood that to get a more stable system one may go for higher value of core loss factor (η_c) and shear parameter (g).

Clamped-Free beam.

Using the shape functions (equation (6.10.15)) for the clamped-free beam, the instability regions for the first three modes are determined and shown in Figures (6.10.6-8).

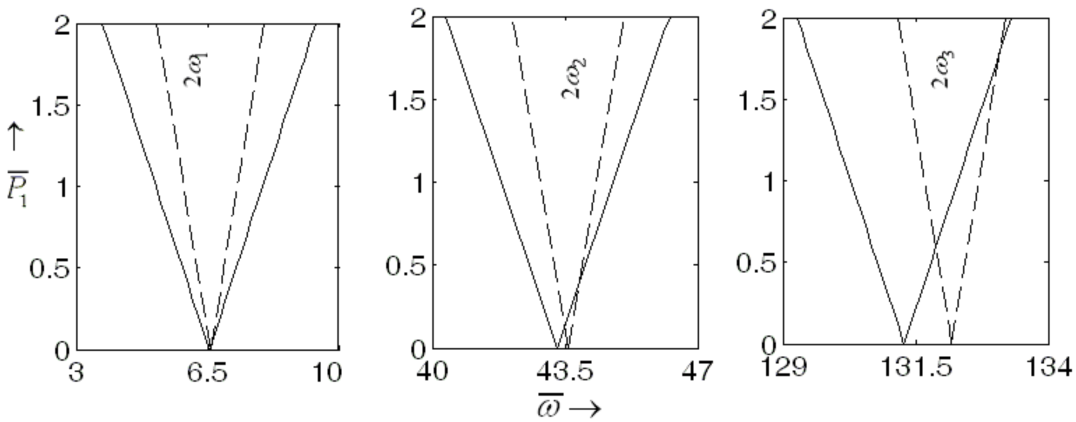


Figure 6.10.6: Comparison of instability regions using higher order and classical theories for $\overline{P}_0 = 0.5$, $\eta_c = 0.1$, g = 0.05. ——— , higher-order theory; ------, classical theory.

Using higher order theory and classical theory the parametric instability regions for simple resonances are shown in Figure 6.10.6. Here also, higher order theory gives a conservative design for lower modes.

Figures (6.10.7-8) show the influence of core loss factor (η_c) and the shear parameter (g) upon the instability region and it is observed that with increase in core loss factor and shear parameter stability of the system improves.

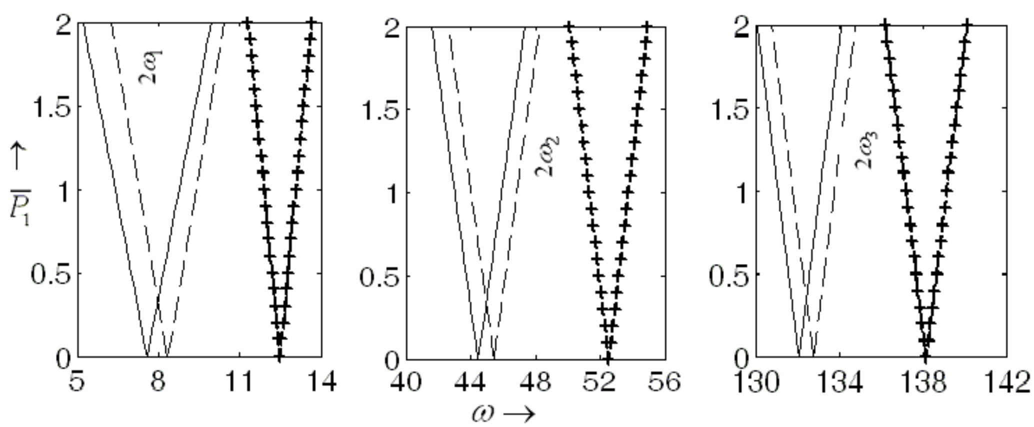


Figure 6-10.7: Effect of shear parameter on instability regions for $\eta_c = 0$. ———; g = 0.05;

-----, g = 0.1; +++, g = 0.5.

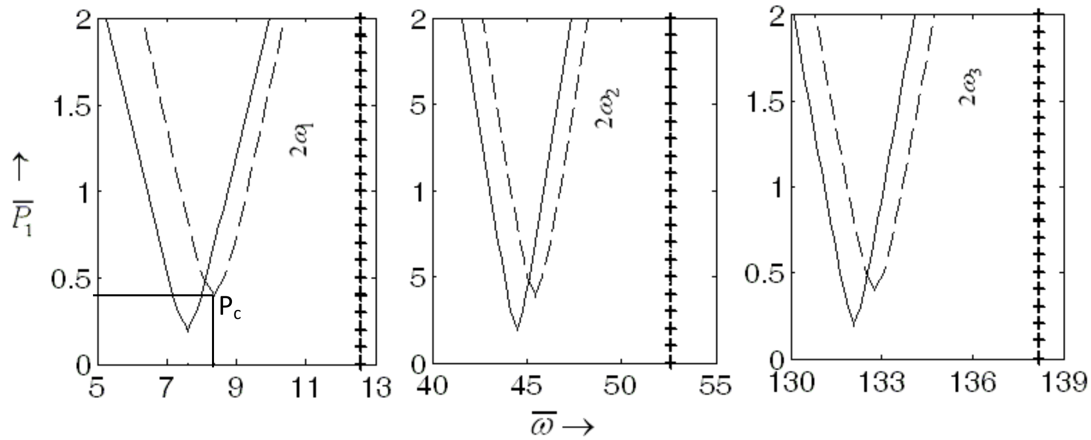


Figure 6.10.8: Effect of shear parameter on instability regions for $\eta_c = 0.3$. ——, g = 0.05; -----, g = 0.1; +++, g = 0.5.

For all the boundary conditions the system is always found to be stable at combination resonances of sum and difference type. In these cases, for simple resonances it is observed that with increase in shear parameter the instability plot moves upward implying that there exists critical forcing amplitude below which the system is always stable. For example, when a cantilevered sandwich beam with η_c =0.3 and g= 0.1 is excited near twice the first natural frequency ($\bar{\varpi} \simeq 8.2$), the system will not vibrate if the forcing amplitude is less than 0.485 (point P$_c$ on figure 6.10.8). But for the same η_c and g = 0.05, with same amplitude of forcing, the system will vibrate at a slightly less frequency (say $\bar{\varpi} \simeq 7.8$). Again with increase in shear parameter, the instability region shifts towards right and hence, for same forcing amplitude, the system becomes unstable at a higher frequency. As the shear parameter $g = G_c / \left(2E_t (c/d_t)(L/d_t)^2\right)$, is a function of dimension and material properties of both skin and core material, using the above stability charts, a designer will be able to construct sandwich beams having very less or vibration free structures.

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Module 6 Lecture 11

Parametrically excited continuous system

Case study: Nonlinear Vibration of a Magneto-Elastic Cantilever Beam With Tip Mass

In this work the effect of the application of alternating magnetic field on the large transverse vibration of a cantilever beam with tip mass is investigated. The governing equation of motion is derived using the D' Alembert's principle which is reduced to its non-dimensional temporal form by using the generalize Galerkin's method. The temporal equation of motion of the system contains the nonlinearities of geometric and inertial type along with parametric excitation and non-linear damping term. Method of multiple scales is used to determine the instability region and frequency response curves of the system. The influences of the damping, tip mass, amplitude of magnetic field strength, permeability and conductivity of the beam material on the frequency response curves are investigated. These perturbation results are found to be in good agreement with those obtained by numerically solving the temporal equation of motion and experimental results.

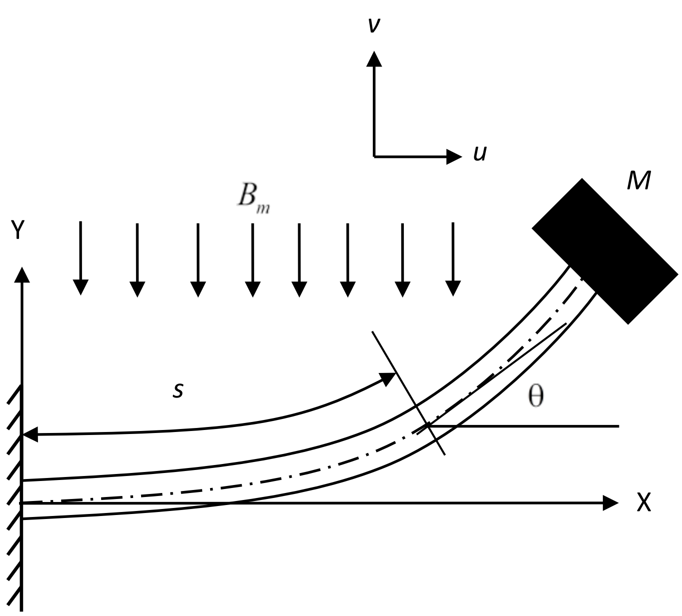

Fig. 6.11.1: Schematic diagram of a flexible single-link cantilever beam with tip mass.

Figure 6.11.1 shows a flexible cantilever beam with a tip mass *M*. The beam is subjected to a harmonic transverse magnetic field $B_0 = B_m \cos \Omega t$ where B_m and Ω are respectively, the amplitude and frequency of the magnetic field strength. In this work, the flexible cantilever beam with tip mass is modeled as an Euler-Bernoulli beam with a tip mass. For the purpose of completeness a brief derivation of the equation of motion using d' Alembert's principle is given below. The bending moment $M(s)$ of the beam at a distance s from the fixed end (Fig. 1) can be expressed as [7,11]

$$M(s) \approx EI\left(v_{ss} + \frac{1}{2}v_s^2 v_{ss}\right). \tag{6.11.1}$$

Here, v is the transverse displacement of the beam. $(\ )_s$ is the first derivative with respect to s. One may write the inextensibility condition of the beam in terms of longitudinal displacement $u(\xi,t)$and transverse displacement $v(\xi,t)$ as [7]

$$v_s^2 + (1+u_s)^2 = 1. \quad \text{or,} \quad u(\xi,t) = \xi - \int_0^{\xi}\left(1-v_\eta^2\right)^{\frac{1}{2}} d\eta. \tag{6.11.2}$$

Here ξ, η are the integration variables. Considering the inertia forces $\rho A\ddot{u}$, $\rho A\ddot{v}, M\ \ddot{u}$ and $M\ \ddot{v}$, and using the d' Alembert's principle, one may write Eq. (6.11.1) as follows

$$M(s) - M_\xi(s) - M_L(s) = 0. \tag{6.11.3}$$

Here $M_\xi(s)$ is the summation of the moment due to inertia force of beam and the moment due to external magnetic force a distance ξ from the roller support and the couple due to magnetic field. $M_L(s)$ is the moment due to inertia force for the pay load at the tip of the manipulator. The expressions for these moments are given below.

$$\begin{aligned} M_\xi(s) = & -\int_s^L \rho A\ddot{u}\int_s^{\xi}\sin\theta\, d\eta\, d\xi - \int_s^L\left(\rho A\ddot{v} + C_d\dot{v}\right)\int_s^{\xi}\cos\theta\, d\eta\, d\xi \\ & -\int_s^L p\, d\xi\int_s^{\xi}\sin\theta\, d\eta\ - c\int_s^{\xi}\cos\theta\, d\eta\ , \end{aligned} \tag{6.11.4}$$

Here, p and c are the body force and body couple of the beam due to the magnetic field B_0 which are expressed as [3, 5, 6, 9]

$$p = -\sigma h d B_0^2 \left(1 - \frac{1}{2} v_s^2\right) \int_0^{\xi} \left(v_s \dot{v}_s - \frac{1}{2} v_s v_s^2 \dot{v}_s \right) d\xi, \text{ and } c = \frac{\chi_m}{\mu_0 \mu_r} h d B_0^2 v_s. \quad (6.11.5)$$

$$\text{Also, } M_L(s) = -M \ddot{u} \int_s^L \sin\theta \, d\xi - M \ddot{v} \int_s^L \cos\theta \, d\xi. \quad (6.11.6)$$

By differentiating Eq. (6.11.3) twice with respect to s and applying the Leibnitz's rules one may obtain the following governing differential equation of motion.

$$\begin{aligned} & EI\left(v_{ssss} + \frac{1}{2} v_s^2 v_{ssss} + 3 v_s v_{ss} v_{sss} + v_{ss}^3 \right) + \rho A v_s \left(\int_0^s \left(\dot{v}_\xi^2 + v_\xi \ddot{v}_\xi \right) d\xi \right) + v_s v_{ss} \\ & \left(\int_s^L \left(\rho A \ddot{v} + C_d \dot{v} \right) d\eta \right) + M \ddot{v} v_s v_{ss} - v_{ss} \left(\int_s^L \rho A \int_0^{\xi} \left(\dot{v}_\xi^2 + v_\xi \ddot{v}_\xi \right) d\xi d\eta + M \int_0^{\xi} \left(\dot{v}_\xi^2 + v_\xi \ddot{v}_\xi \right) d\xi \right) \\ & \left(1 - \frac{1}{2} v_s^2 \right) \left(\rho A \ddot{v} + C_d \dot{v} \right) - \left(v_{ss} \int_s^L (p \, d\xi) - p v_s \right) - \left(\frac{dc}{ds} \left(1 - \frac{1}{2} v_s^2 \right) + v_s v_{ss} \left(1 + \frac{1}{2} v_s^2 \right) c \right) = 0. \end{aligned} \quad (6.11.7)$$

To obtain the temporal equation of motion, one may discretize the governing equation of motion (6.11.7) by using following assumed mode expression.

$$v(s,t) = r \psi(s) q(t). \quad (6.11.8)$$

Here, r is the scaling factor; $q(t)$ is the time modulation and $\psi(s)$ is the eigen-function of the cantilever beam with tip mass, which is given by [7]

$$\psi(s) = -\left(\frac{\sin \beta L + \sinh \beta L}{\cos \beta L + \cosh \beta L} \right) \left(\cos \beta s - \cosh \beta s \right) + \left(\sin \beta s - \sinh \beta s \right). \quad (6.11.9)$$

One may determine βL from the following equation.

$$1+\cos\beta L\cosh\beta L+\bar{m}\,\beta L\left(\cos\beta L\sinh\beta L-\sin\beta L\cosh\beta L\right)=0\,. \tag{6.11.10}$$

Following non-dimensional parameters are used in this analysis.

$$\bar{x}=\frac{s}{L},\ \tau=\omega_e\,t,\ \bar{\omega}=\frac{\Omega}{\omega_e},\ \ \bar{r}=\frac{r}{L},\ \bar{m}=\frac{M}{\rho AL},\ \ \chi=\frac{EI}{\rho AL^4}. \tag{6.11.11}$$

Substituting Eq. (6.11.9) into Eq. (6.11.6) and using the generalized Galerkin's method, one may obtain the resulting non-dimensional temporal equation of motion, which can be expressed as

$$\begin{aligned}\ddot{q}+2\varepsilon\zeta\dot{q}+q+\varepsilon\left(\alpha_1 q^3+\alpha_2 q^2\ddot{q}+\alpha_3\dot{q}^2 q\right)-\varepsilon f_1\cos\left(2\bar{\omega}\tau\right)q \\ -\varepsilon k_1\left(1+\cos\left(2\bar{\omega}\tau\right)\right)\dot{q}q^2=0.\end{aligned} \tag{6.11.12}$$

The expressions for the coefficients (i.e. $\zeta,\ \alpha_1,\ \alpha_2,\alpha_3,\ f_1,k_1$) in this equation are given below.

The natural frequency of the lateral vibration of an elastic beam

$$\omega_e^2=\frac{E_1I}{mL^4}\frac{h_1}{h_{14}}-\frac{B_m^2}{2}\frac{\chi_m\,h\,d}{\mu_0\mu_r mL^2}\left(\frac{h_{15}}{h_{14}}\right),$$

$$=\frac{E_1I}{mL^4}\frac{h_1}{h_{14}}\left(1-\frac{B_m^2}{2}\frac{\chi_m\,h\,d\,L^2}{\mu_0\mu_r E_1I}\left(\frac{h_{15}}{h_1}\right)\right)=\omega_L^2\left(1-\bar{B}_m\right). \tag{A1}$$

Here, $\omega_L^2=\dfrac{E_1I}{mL^4}\dfrac{h_1}{h_{14}}$, and $\bar{B}_m=\dfrac{B_m^2\chi_m\,h\,d\,L^2}{2\mu_0\mu_r E_1I}\left(\dfrac{h_{15}}{h_1}\right)$.

Damping ratio due to the viscous damping to the system,

$$\zeta = \frac{C_d}{2\varepsilon m \omega_e} = \bar{\mu}\left(\frac{\omega_L}{\omega_e}\right) = \bar{\mu}\,\delta, \qquad \text{(A2)}$$

Coefficient of the nonlinear geometric term q^3 =

$$\alpha_1 = \frac{EI\,\bar{r}^2}{m L^4 \varepsilon \omega_e^2}\left(\frac{h_2}{h_{14}} + \frac{h_3}{2h_{14}} + 3\frac{h_4}{h_{14}}\right), \qquad \text{(A3)}$$

Coefficient of the nonlinear inertia term $q^2\ddot{q}$ =

$$\alpha_2 = \frac{\bar{r}^2}{\varepsilon}\left(\frac{h_5}{h_{14}} + \frac{h_6}{h_{14}} + \bar{m}\frac{h_7}{h_{14}} - \frac{h_8}{h_{14}} - \bar{m}\frac{h_9}{h_{14}} - \frac{h_{10}}{h_{14}}\right), \qquad \text{(A4)}$$

Coefficient of the nonlinear inertia term $\dot{q}^2 q$ =

$$\alpha_3 = \frac{\bar{r}^2}{\varepsilon}\left(\frac{h_{11}}{h_{14}} - \frac{h_{12}}{h_{14}} - \bar{m}\frac{h_{13}}{h_{14}}\right), \qquad \text{(A5)}$$

Coefficient of the parametric excitation term $\cos(2\bar{\omega}\tau)\,q$ =

$$f_1 = \frac{B_r^2}{\omega_e^2 B_c^2} = \left(\frac{f_0}{\delta^2}\right), \text{ where } B_r^2 = \frac{B_m^2}{2} \text{ and } B_c^2 = \frac{\mu_0 \mu_r E_1 I L^2}{\chi_m h d}\left(\frac{h_1}{h_{14}}\right). \qquad \text{(A6)}$$

Coefficient of the nonlinear damping terms $\left(1+\cos(2\bar{\omega}\tau)\right) q^2\dot{q}$ =

$$k_1 = -\frac{B_m^2 \sigma h d}{2m\omega_e}\bar{r}^2\left(-\frac{h_{16}}{h_{14}} + \frac{h_{17}}{h_{14}}\right). \qquad \text{(A7)}$$

Here,

$$h_1=\int_0^1 \frac{d^4\psi(\bar{x})}{d\bar{x}}\psi(\bar{x})\,d\bar{x},\ \ h_2=\int_0^1\left(\frac{d^2\psi(\bar{x})}{d\bar{x}}\right)^3\psi(\bar{x})\,d\bar{x},\ h_3=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^4\psi(\bar{x})}{d\bar{x}}\psi(\bar{x})\,d\bar{x}$$

$$h_4=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\frac{d^3\psi(\bar{x})}{d\bar{x}^3}\psi(\bar{x})d\bar{x}\,,\ h_5=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\int_0^x\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2 d\bar{\xi}\,\psi(\bar{x})\,d\bar{x}\,,$$

$$h_6=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\int_x^1\psi(\bar{\xi})\,d\bar{\xi}\,\psi(\bar{x})\,d\bar{x}\,,\ \ h_7=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}(\psi(\bar{x}))^2d\bar{x}\,,$$

$$h_8=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\int_x^1\int_0^\eta\left(\frac{d\psi(\bar{x})}{d\bar{\xi}}\right)^2 d\bar{\xi}d\bar{\eta}\,\psi(\bar{x})\,d\bar{x}\,,\ \ h_9=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\int_0^x\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2 d\bar{\xi}\psi(\bar{x})\,d\bar{x}\,,$$

$$h_{10}=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\right)^2\left(\psi(\bar{x})\right)^2 d\bar{x}\,,\ \ h_{11}=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\int_0^x\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2 d\bar{\xi}\psi(\bar{x})\,d\bar{x}\,,$$

$$h_{12}=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\int_x^1\int_0^\eta\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2 d\bar{\xi}d\bar{\eta}\,\psi(\bar{x})\,d\bar{x}\,,\ \ h_{13}=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\int_0^x\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2 d\bar{\xi}\psi(\bar{x})\,d\bar{x}\,,$$

$$h_{14}=\int_0^1\left(\psi(\bar{x})\right)^2 d\bar{x}\,,\ \ h_{15}=\int_0^1\left[\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\psi(\bar{x})\right]d\bar{x}\,,\ h_{16}=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\left(\int_0^{\bar{x}}\frac{d\psi(\bar{\xi})}{d\bar{\xi}}d\bar{\xi}\right)\psi(\bar{x})d\bar{x}\,,$$

and $h_{17}=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\left(\int_{\bar{x}}^1\int_0^{\bar{\xi}}\frac{d\psi(\bar{\xi})}{d\bar{\xi}}d\bar{\eta}d\bar{\xi}\right)\psi(\bar{x})d\bar{x}\,.$

Here one may observe that the non-dimensional temporal Eq. (6.11.12) has parametric term $f_1\cos(2\bar{\varpi}\tau)q$ and nonlinear damping term $k_1\left(1+\cos(2\bar{\varpi}\tau)\right)\dot{q}q^2$, along with cubic geometric ($\alpha_1 q^3$) and inertial ($\alpha_2 q^2\ddot{q}+\alpha_3\dot{q}^2 q$) nonlinear terms. Hence, it may be noted that the temporal equation of motion Eq. (6.11.12) contains many nonlinear terms and it is very difficult to find the exact solution. Hence one may go for the approximate solution by using the perturbation method. Here method of multiple scales is used.

In method of multiples scales, the displacement q can be represented in terms of different time scales (T_0, T_1) and a book keeping parameter ε as follows.

$$q(\tau;\varepsilon) = q_0\left(T_0, T_1\right) + \varepsilon\, q_1\left(T_0, T_1\right) + O\left(\varepsilon^2\right). \qquad (6.11.13)$$

Here, $T_0 = \tau$, and $T_1 = \varepsilon\tau$. The transformation of first and second time derivatives are given by

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + O(\varepsilon^2), \qquad (6.11.14)$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2). \qquad (6.11.15)$$

where, $D_0 = \dfrac{\partial}{\partial T_0}$, and $D_1 = \dfrac{\partial}{\partial T_1}$. Substituting Eqs. (6.11.13- 6.11.15) into Eq. (6.11.12) and equating the coefficient of like powers of ε, yields the following equations.

Order ε^0 : $D_0^{\,2} q_0 + q_0 = 0$, (6.11.16)

Order ε^1 : $D_0^{\,2} q_1 + q_1 = -2\, D_0\, D_1\, q_0 - 2\,\zeta\, D_0 q_0 - \alpha_1\, q_0^3 - \alpha_2\left(D_0^2\, q_0\right) q_0 - \alpha_3\left(D_0 q_0\right)^2 q_0^2$

$$+ f_1 \cos\left(2\bar{\omega} T_0\right) q_0 + k_1\left(1 + \cos\left(2\bar{\omega} T_0\right)\right)\left(D_0 q_0\right) q_0^2. \qquad (6.11.17)$$

General solutions of Eq. (6.11.16) can be written as

$$q_0 = A(T_1, T_2)\exp(iT_0) + \bar{A}(T, T_2)\exp(-iT_0)\,. \qquad (6.11.18)$$

Substituting Eq. (6.11.18) into Eq. (6.11.17) leads to

$$D_0^{\,2} q_1 + q_1 = -2i\,A'\exp(iT_0) - 2i\zeta A\exp(iT_0) - \left(3\alpha_1 - 3\alpha_2 + \alpha_3 - ik_1\right)A^2\bar{A}\exp(iT_0)$$

$$+\left(-\alpha_1 + \alpha_2 + \alpha_3\right)A^3\exp\left(3iT_0\right) + ik_1A^3\exp\left(3iT_0\right) + \frac{f_1}{2}\left[A\exp i(2\bar{\omega}-1) + \bar{A}\exp i(2\bar{\omega}-1)\right]T_0$$

$$+\frac{ik_1}{2}A^3\exp i(2\bar{\omega}+3)T_0 + \frac{ik_1}{2}A^2\bar{A}\exp i(2\bar{\omega}+1)T_0 - \frac{ik_1}{2}\bar{A}^2A\exp i(2\bar{\omega}-1)T_0$$

$$+\frac{ik_1}{2}A^3\exp i(3-2\bar{\omega})T_0 + cc\;. \qquad (6.11.19)$$

Here, *cc* is the complex conjugate of the preceding terms One may observe that any solution of Eq. (6.11.19) will contain secular or small divisor terms when non-dimensional frequency of magnetic field strength ($\bar{\varpi}$) is nearly equal to 1 which may be called as simple resonance case. In this case, one may use detuning parameter σ to express the nearness of $\bar{\varpi}$ to 1, as

$$\bar{\varpi} = 1 + \varepsilon\,\sigma, \quad \text{and} \;\; \sigma = O(1). \qquad (6.11.20)$$

Substituting Eq. (6.11.20) into Eq. (6.11.19), one may obtain the following secular or small divisor terms.

$$-2i\,A'\exp(iT_0) - 2i\zeta A\exp(iT_0) - \left(3\alpha_1 - 3\alpha_2 + \alpha_3 - ik_1\right)A^2\bar{A}\exp(iT_0)$$

$$+\frac{f_1}{2}\bar{A}\exp\left(2\sigma T_1\right) + i\frac{k_1}{2}A^3\exp\left(-2\sigma T_1\right) - i\frac{k_1}{2}\bar{A}^2A\exp\left(2\sigma T_1\right) = 0. \qquad (21)$$

Putting A equal to $\frac{1}{2}a(T_1)e^{(i\beta T_1)}$ and $\gamma = 2\sigma T_1 - 2\beta$ into Eq. (6.11.21) and separating the real and imaginary terms, one may a set of reduced equations as given below.

$$\dot{a} = -\zeta a + \frac{k_1}{8}a^3 + \frac{f_1}{4}a\sin\gamma, \tag{6.11.22}$$

$$a\dot{\gamma} = 2a\left(\frac{\overline{\omega}-1}{\varepsilon}\right) - \frac{3}{4}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^3 + \frac{1}{4}a^3 k_1 \sin\gamma + \frac{f_1}{2}a\cos\gamma. \tag{6.11.23}$$

One may observe from the Eqs. (6.11.21)-(6.11.22) that the system possesses both trivial and nontrivial responses. Hence one may determine both responses by solving Eqs. (6.11.22, 6.11.23) simultaneously. To find the stability of the steady state responses, one may perturb the above Eqs. (6.11.22, 6.11.23), by substituting $a = a_o + a_1$ and $\gamma = \gamma_0 + \gamma_1$ where a_0, γ_0 are the singular points, and then investigate the eigenvalues of the Jacobian matrix (J) which is given by

$$J = \begin{bmatrix} -\zeta + \frac{3k_1}{8}a_0^2 + \frac{f_1}{4}\sin\gamma_0 & \frac{f_1}{4}a_0\cos\gamma_0 \\ -\frac{3}{2}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a_0 + \frac{1}{2}a_0 k_1 \sin\gamma_0 & \frac{1}{4}a_0^2 k_1 \cos\gamma_0 - \frac{f_1}{2}\sin\gamma_0 \end{bmatrix}. \tag{6.11.24}$$

It may be noted that the system will be stable if and only if all the real parts of the eigen-values are negative.

For trivial state instability region, one may use the following expression for the transition curve for simple resonance case which has been obtained by finding the eigen-values of Jacobian matrix (J) given in Eq.(6.11.24).

$$\bar{\Omega} = \frac{\Omega}{\omega_L} = \delta \pm \frac{\varepsilon}{2\delta}\left(\frac{f_0^2}{4} - 4\bar{\mu}^2\right) + O\left(\varepsilon^2\right). \tag{6.11.25}$$

Here, the expression for δ, $\bar{\mu}$, and f_0 are given in appendix [7].

It may be noted that this simple expression has been obtained by using first order method of multiple scales is different from the expression given in the work of Pratiher and Dwivedy [7], which was obtained by using the second order method of multiple scales.

Now the first order non-trivial steady state response of the cantilever beam with a tip mass can be given by

$$q = a\cos\left(\frac{1}{2}(\bar{\omega}\tau - \gamma)\right). \tag{6.11.26}$$

Here for numerical simulations, a steel beam similar to that considered in the work of Wu [5, 6] with length L = 0.5 m, width d = 0.005 m, depth h = 0.001 m, Young's Modulus E =1.94×10^{11} N/m^2, mass of the beam per unit length m = 0.03965 kg, and the permeability of the vacuum, μ_0 = 1.26×10^{-6} Hm^{-1} have been considered. Using these parameters, the reduced Eqs. (6.11.22, 6.10.23), have been solved numerically to obtain the instability regions and the frequency response curves. In the instability plot, the regions bounded by the curves are unstable and regions outside the curves are stable. In the frequency response curves dotted and solid lines represent, respectively the unstable and stable response of the system. The effect of the amplitude of magnetic field strength (B_m), damping (C_d), tip mass (M), material conductivity (σ), and relative permeability of the material (μ_r) on the frequency response have been investigated.

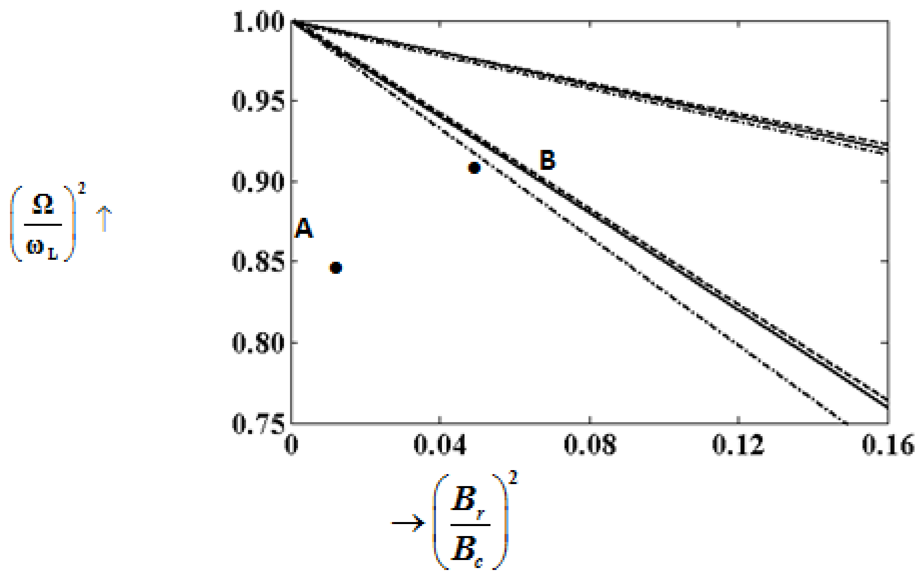

Fig. 6.11.2: The region of instability of a cantilever beam with tip mass subjected to magnetic field,

-.-.-.-.-. Moon and Pao's theoretical result, - - - - - Moon and Pao's experimental result, and ---- the present result.

For simple resonance case, the beam is subjected to a transverse magnetic field with a frequency nearly equal to the natural frequency of the system. Here, the instability regions are plotted in $(\Omega/\omega_L)^2$ Vs $(B_r/B_c)^2$ plane similar to the work of Moon and Pao [2], Wu [5, 6], and Pratiher and Dwivedy [7]. The experimental and theoretical results of Moon and Pao [2] are also being plotted in Fig.6.11.2 for comparison with the present result. It is found from Fig.6.11.2 that the result obtained in the present work is in good agreement with the experimental results Moon and Pao [2]. The accuracy of the instability region obtained by using the first order method of multiple scales can be verified by numerically solving the temporal equation of motion (6.11.12) and plotting the time response (Fig. 6.11.3) for two different points A and B as marked in Fig.6.11.2. Figure 6.11.3(i) clearly shows that the response is stable and Fig.6.11. 3(ii) shows that the response is unstable which are in good agreement with the result shown in Fig.6.11. 2. Hence, one may use the first order closed form solution (Eq. 6.11.25) for finding the instability region instead of going for a second order solution as reported in the work of Pratiher and Dwivedy [7]. But when more accurate result is required, one may use the expression given in the work of Pratiher and Dwivedy [7].

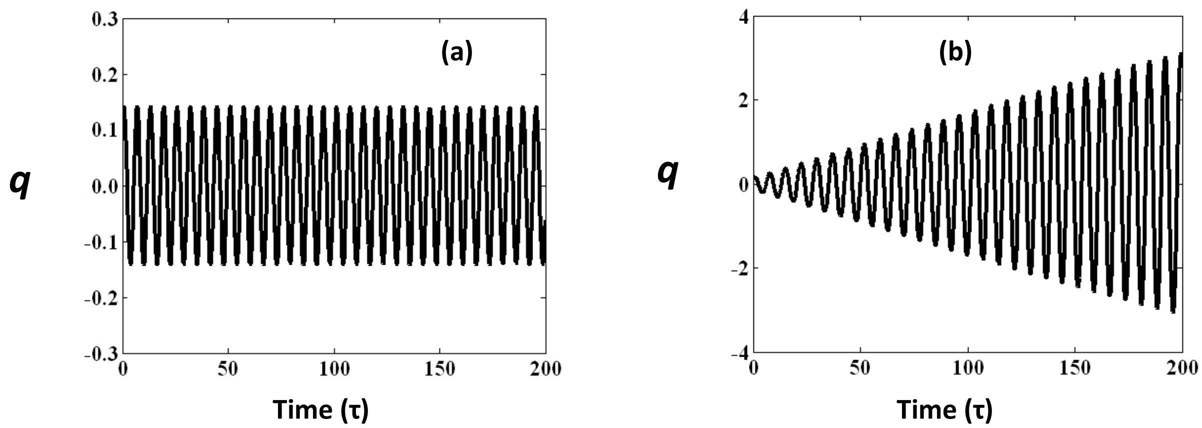

Fig.6.113. (a) Time response for the point A and (b) time response for point B marked in Fig.6.11.2.

Figure 6.11.4 shows the frequency response curve for four different values of amplitude of magnetic field strength B_m. From Fig.6.11.4, it may be noted that with increase in B_m, though the maximum response amplitude remain unchanged, the trivial state becomes unstable which is similar to that shown in Fig.6.11.2. The trivial state becomes unstable by the sub-critical pitchfork bifurcation at R_1, which ends with a super-critical pitchfork bifurcation at R_2. Here, one may observe that the system has a tendency to jump up from the unstable trivial state at R_1 to the stable non-trivial state at $R_1^{/}$.

Figures 6.11.5(a) and (b) show the transient and steady state response for point C marked in Fig. 6.11.4(c). The solid line and dotted line respectively represent the response of the system with and without magnetic field. In the presence of magnetic field, it clearly shows that the steady state response has zero response amplitude. Also it may be noted that the free vibration response of the beam shown as dotted line in Fig. 6.11.5, is reduced by applying the magnetic field.

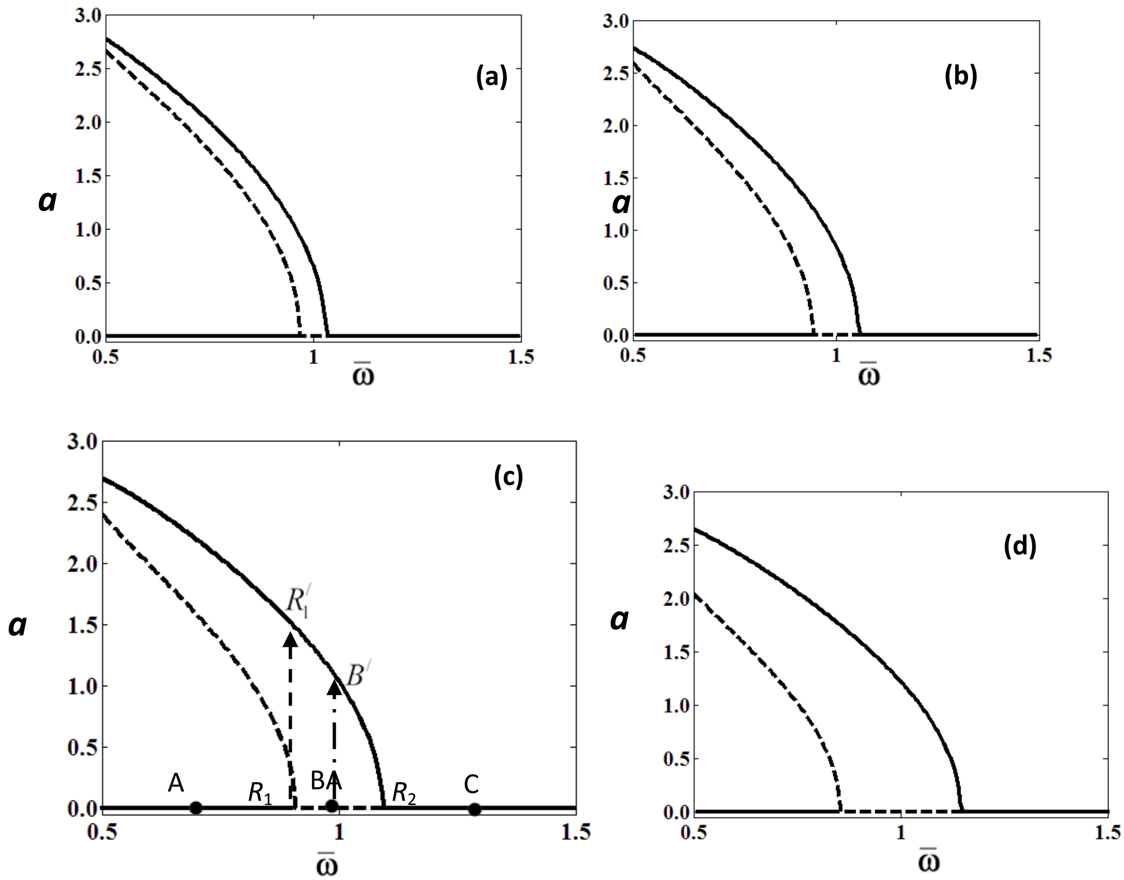

Fig.6.11.4. Effect of the magnetic field strength (B_m) on the frequency response curves for $M = 0.02$ kg, $C_d = 0.01$ N-s/m^2, $\mu_r = 3000$, $\sigma = 10^7$ Vm^{-1} (a) $B_m = 0.20$ Am^{-1} (b) $B_m = 0.25$ Am^{-1} (c) $B_m = 0.30$ Am^{-1} (d) $B_m = 0.35$ Am^{-1}.

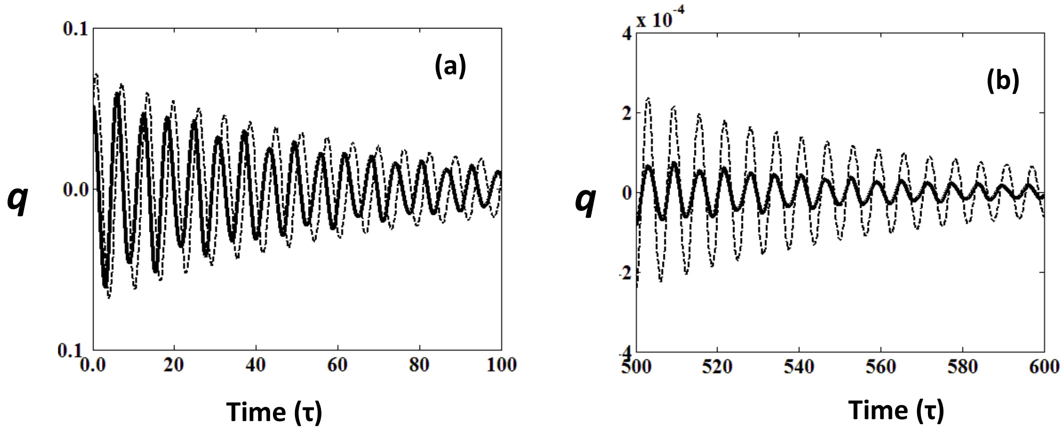

Fig.6. 11. 5. (a) Transient response and (b) steady state time response for the point C with and without magnetic field.

The effect of damping C_d on the response curves is shown in Fig.6.11.6 and it has been observed that with increase in C_d, while the non-trivial response amplitude remains unchanged the trivial state unstable region decreases, the sub-critical pitchfork bifurcation point occurs at a higher value of $\bar{\omega}$ and the corresponding jump length decreases.

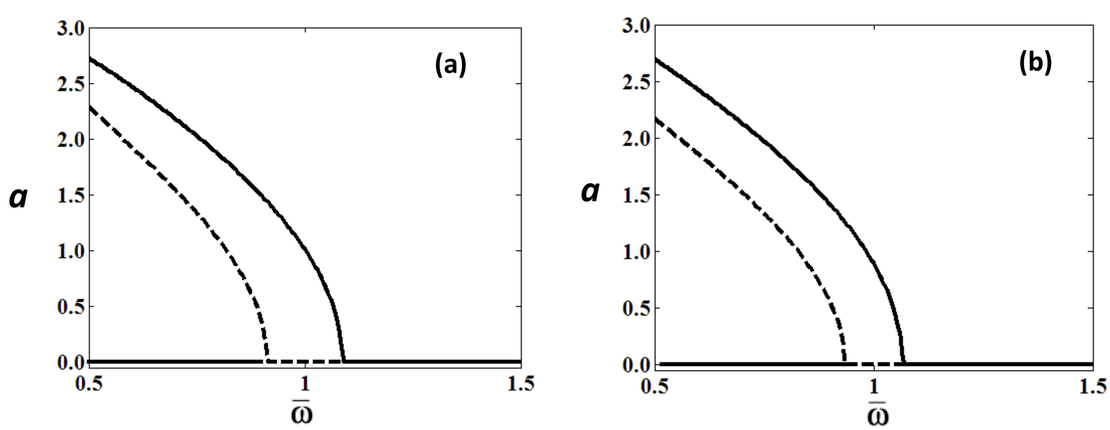

Fig.6.11.6: Influence of damping on frequency response curve for $M = 0.02$ kg, $\mu_r = 3000$, $\sigma = 10^7$ Vm^{-1}, $B_m = 0.30$ Am^{-1} (a) $C_d = 0.02$ N-s/m^2 ,(b) $C_d = 0.03$ N-s/m^2.

Similarly one can study the influence of effect of relative permeability (μ_r), material conductivity (σ) and mass ratio on the frequency response curves of the system.

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Module 6 Lect 12

Parametrically excited System with internal Resonance

In this lecture a case study is taken for a parametrically excited system with internal resonance. The system considered is a uniform cantilever beam of length L carriying a mass m at an arbitrary position d from the fixed end and subjected to base motion $z(t) = Z_0 \cos \Omega t$ as shown in Fig. 6.12.1.

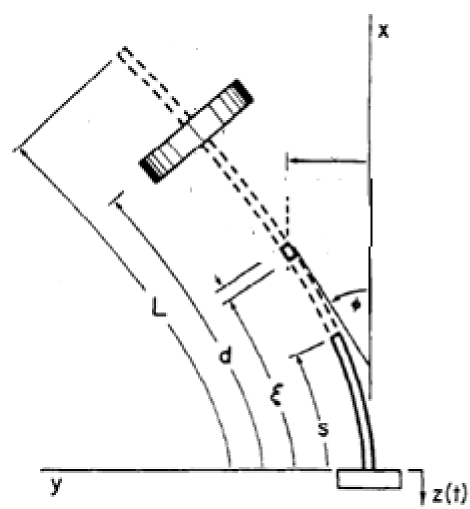

Fig. 6. 12. 1. Base Excited Cantilever beam with attached mass at arbitrary position

The equation of motion of the beam is given by Kar and Dwivedy[1999] as

$$EI\left\{\nu_{ssss}+\frac{1}{2}\nu_s^2\nu_{ssss}+3\nu_s\nu_{ss}\nu_{sss}+\nu_{ss}^3\right\}+\left(1-\frac{1}{2}\nu_s^2\right)\left\{\left[\rho+m\delta(s-d)\right]\nu_{tt}+c\nu_t\right\}$$
$$+\upsilon_s\upsilon_{sss}\int_s^L\left\{\left[\rho+m\delta(\xi-d)\right]\upsilon_{tt}+c\upsilon_t\right\}d\xi-\left[j_0\delta(s-d)(\upsilon_s)_{tt}\right]_s-(N\upsilon_s)=0 \quad (6.12.1)$$

Subject to the boundary conditions

$$\upsilon(0,t)=0,\ \upsilon_s(0,t)=0,\ \upsilon_{ss}(L,t)=0,\ \upsilon_{sss}(L,t)=0 \quad (6.12.2)$$

Where

$$N=\frac{1}{2}\rho\int_L^s\left\{\int_s^\xi\left(\upsilon_s^2\right)_{tt}d\eta\right\}d\xi+\frac{1}{2}m\int_s^L\delta(\xi-d)\times\left\{\int_0^\xi\left(\upsilon_s^2\right)_{tt}d\eta\right\}d\xi+m(z_{tt}-g)$$
$$\times\int_L^s\delta(\xi-d)d\xi+\rho L\left(1-\frac{s}{L}\right)(z_{tt}-g)-J_0\delta(s-d)\left\{\frac{1}{2}\upsilon_{stt}\upsilon_s^2+\upsilon_s\upsilon_{st}^2\right\} \quad (6.12.3)$$

Here, $(\ )_t=\frac{\partial(\ )}{\partial t},\ (\ )_s=\frac{\partial(\ )}{\partial s}$

Here, E, I and ρ are, respectively, the Young's modulus, the second moment of area of the cross-section of the beam and mass per unit length of the beam; j_0 is the moment of inertia of the concentrated mass m about its centroidal axis perpendicular to the X-Y plane; υ is the lateral displacement of the beam; g, c and z are, respectively, the acceleration due to gravity, the coefficient of viscous damping and the displacement of the base; and δ is the Dirac delta function. Assuming a solution of Eq. (6.12.1) in the form

$$\upsilon(s,t)=\sum_{n=1}^{\infty}r\varphi_n(s)u_n(t) \quad (6.12.4)$$

Where r is a scaling factor, $\varphi_n(s)$ is the shape function of the n th mode, and u_n is the time modulation of the n^{th} mode. Applying generalized Galerkin's method and using the following non-dimensional parameters,

$$x=\frac{s}{L},\quad \beta=\frac{d}{L},\quad \tau=\theta_1 t,\quad \omega_n=\frac{\theta_n}{\theta_1},\ \lambda=\frac{r}{L},\quad \mu=\frac{m}{\rho L},\quad T=\frac{Z_0}{Z_r},\ J=\frac{J_0}{\rho L r^2},\quad \phi=\frac{\Omega}{\theta_1} \quad (6.12.5)$$

Eq. (6.12.1) reduces to

$$\ddot{u}_n+2\varepsilon\xi_n\dot{u}_n+\omega_n^2u_n-\varepsilon\sum_{m=1}^{\infty}f_{nm}u_m\cos\phi\tau+\varepsilon\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left\{\alpha_{klm}^n u_k u_l u_m+\beta_{klm}^n u_k\dot{u}_l\dot{u}_m+\gamma_{klm}^n u_k\dot{u}_l\dot{u}_m\right\}=0, \quad (6.12.6)$$

Here, $n = 1, 2, \ldots$, $\left(\dot{\;}\right) = d(\;)/d\tau$. For details of the coefficients one may refer (Kar and Dwivedy 1999). The small dimensionless parameter ε is the book-keeping parameter to indicate the smallness of damping, non-linearities and excitation. So, we have n number of coupled equations with cubic geometric and inertial non-linearities, where n represents the number of modes participating in the resulting oscillation. Due to the absence of any internal and external excitation for $n \geq 3$, the amplitude of these higher modes die out in the presence of damping and hence two mode discretization in the Galerkin's method is sufficient in this particular system.

The approximate solution of Eq. (6.12.6) can be obtained using the method of multiple scales. Let

$$u_n(\tau;\, \varepsilon) = u_{n0}(T_0,\, T_1) + \varepsilon u_{n1}(T_0, T_1) + \ldots \tag{6.12.7a}$$

$$T_0 = \tau, \quad T_1 = \varepsilon\tau, \quad n = 1, 2, \ldots, \infty \tag{6.12.7b}$$

Substituting Eqs. (6.12.7a) and (6.12.7b) into Eq. (6.12.6) and equating the coefficients of ε^0 and ε to zero, yields

$$D_0^2 u_{n0} + \omega_n^2 u_{n0} = 0, \tag{6.12.8}$$

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -[2\xi_n D_0 u_{n0} + 2D_0 D_1 u_{n0} - \sum_{n,m=1}^{\infty} f_{nm} u_{m0} \cos\phi\tau$$

$$+ \sum_{klm} \alpha_{klm}^n u_{k0} u_{l0} u_{m0} + \beta_{klm}^n u_{k0} D_0 u_{l0} D_0 u_{m0} + \gamma_{klm}^n u_{k0} u_{l0} D_0^2 u_{m0} = 0 \tag{6.12.9}$$

Where $D_0 = \partial/\partial T_0$ and $D_1 = \partial/\partial T_1$. The solution of Eq. (9) is given by

$$u_{n0} = A_n(T_1)\exp(i\omega_n T_0) + cc \tag{6.12.10}$$

Where cc indicates the complex conjugate of the preceding terms and A_n is determined in the following section. Considering the Principal parametric resonance $(\phi \approx 2\omega_1)$, to express the nearness of ϕ to $2\omega_1$ the detuning parameter σ_1 is introduced. Also, to account for the internal resonance, the detuning σ_2 is used. Hence, we have

$$\varphi = 2\omega_1 + \varepsilon\sigma_1, \quad \omega_2 = 3\omega_1 + \varepsilon\sigma_2 \tag{6.12.11}$$

Substituting Eqs. (6.12.10) and (6.12.11) into Eq. (6.12.9) and eliminating the secular terms, we get for $n = 1$

$$2i\omega_1\left(\xi_1 A_1 + A_1'\right) - \frac{1}{2}\left[f_{11}\bar{A}_1 \exp(i\varepsilon\sigma_1 T_0) + f_{12} A_2 \exp\{i\varepsilon(\sigma_2 - \sigma_1)T_0\}\right]$$

$$+\sum_{j=1}^{\infty}\alpha_{e1j}A_j\overline{A}_jA_1+Q_{12}A_2\overline{A}_1^2\exp(i\varepsilon\sigma_2T_0)=0 \tag{6.12.12}$$

For n=2,

$$2i\omega_2(\xi_2A_2+A_2^{'})-\frac{1}{2}f_{21}A_1\exp\{i\varepsilon(\sigma_1-\sigma_2)T_0\}$$

$$+\sum_{j=1}^{\infty}\alpha_{e2j}A_j\overline{A}_jA_1+Q_{21}A_2\overline{A}_1^3\exp(-i\varepsilon\sigma_2T_0)=0 \tag{6.12.13}$$

For $n\geq 3$,

$$2i\omega_n(\xi_nA_n+A_n^{'})+\sum_{j=1}^{\infty}\alpha_{enj}A_j\overline{A}_jA_n=0 \tag{6.12.14}$$

Where a prime denotes the derivative with respect to T_1. Since the higher modes $(n\geq 3)$ are neither directly excited by external excitation nor indirectly excited by internal resonance, from Eq. (6.12.14) it can be shown that the response amplitude of these modes die out due to the presence of damping. Letting $A_n=\frac{1}{2}a_n(T_1)\exp\{i\beta(T_1)\}$ (where a_n and β_n are real) in Eqs. (6.12.12) and (6.12.13) and then separating into real and imaginary parts, one obtains

$$2\omega_1(\xi_1a_1+a_1^{'})-\frac{1}{2}\{f_{11}a_1\sin 2\gamma_1+f_{12}a_2\sin(\gamma_1-\gamma_2)\}+0.25Q_{12}a_2a_1^2\sin(3\gamma_1-\gamma_2)=0 \tag{6.12.15a}$$

$$\begin{aligned}&2\omega_1a_1\left(\gamma_1^{'}-\frac{1}{2}\sigma_1\right)-\frac{1}{2}\{f_{11}a_1\cos 2\gamma_1+f_{12}a_2\cos(\gamma_1-\gamma_2)\}+\\&\frac{1}{4}\sum_{j=1}^{2}\alpha_{e1j}a_j^2a_1+\frac{1}{4}Q_{12}a_2a_1^2\cos(3\gamma_1-\gamma_2)=0\end{aligned} \tag{6.12.15b}$$

$$2\omega_2(\xi_2a_2+a_2^{'})-\frac{1}{2}\{f_{21}a_1\sin(\gamma_2-\gamma_1)\}+\frac{1}{4}Q_{21}a_1^3\sin(\gamma_2-3\gamma_1)=0 \tag{6.12.15c}$$

$$2\omega_2a_2(\gamma_2^{'}+\sigma_2-1.5\sigma_1)-\frac{1}{2}f_{21}a_1\cos(\gamma_2-\gamma_1)+\frac{1}{4}\sum_{j=1}^{2}\alpha_{e2j}a_j^2a_2+\frac{1}{4}Q_{21}a_1^3\cos(\gamma_2-3\gamma_1)=0 \tag{6.12.15d}$$

Where

$$\gamma_1=-\beta_1+\frac{1}{2}\sigma_1T_1,\ \text{and}\ \ \gamma_2=-\beta_2+(1.5\sigma_1-\sigma_2)\,T_1$$

The above equations are known as the reduced equations. For steady state, $a_1^{'} = \gamma_1^{'} = a_2^{'} = \gamma_2^{'} = 0$. So, now we have a set of non-linear algebraic equations which is solved numerically to obtain the fixed point response of the system. The first-order solution of the system can be given by

$$u_1 = a_1 \cos\{(\omega_1 + \varepsilon\sigma_1 / 2)\tau - \gamma_1\} \tag{6.12.16a}$$

$$u_2 = a_2 \cos[\{\omega_2 + \varepsilon(1.5\sigma_1 - \sigma_2)\}\tau - \gamma_2] \tag{6.12.16b}$$

Stability equations of steady-state response

By directly perturbing the reduced equations, one can study the stability of the non-trivial steady sate solution. But, as the reduced Eqs. (6.12.15a-d) have the coupled terms $a_1\gamma_1^{'}$ and $a_2\gamma_2^{'}$, the perturbed equations will not contain the perturbations $\Delta\gamma_1^{'}$ or $\Delta\gamma_2^{'}$ for trivial solutions and hence the stability of the trivial state cannot be studied by directly perturbing these equations. To circumvent this difficulty, normalization method is adopted by introducing the transformation

$$p_i = a_i \cos\gamma_i, \quad q_i = a_i \sin\gamma_i, \quad i = 1, 2 \tag{6.12.17}$$

Into equations (6.12.15) to obtain the following normalized reduced equations or the Cartesian form of modulation equations:

$$2\omega_1\left(p_1^{'} + \xi_1 p_1\right) + \left(\omega_1\sigma_1 - \frac{1}{2} f_{11}\right)q_1 + \frac{1}{2} f_{12} q_2$$

$$+\frac{1}{4} Q_{12}\left\{q_2\left(q_1^2 - p_1^2\right) + 2p_1 p_2 q_1\right\} - \frac{1}{4}\sum_{j=1}^{2} \alpha_{e1j} q_1\left(p_j^2 + q_j^2\right) = 0 \tag{6.12.18a}$$

$$2\omega_1\left(q_1^{'} + \xi_1 q_1\right) + \left(\omega_1\sigma_1 - \frac{1}{2} f_{11}\right)p_1 + \frac{1}{2} f_{12} p_2$$

$$+\frac{1}{4} Q_{12}\left\{p_2\left(p_1^2 - q_1^2\right) + 2p_1 q_1 q_2\right\} - \frac{1}{4}\sum_{j=1}^{2} \alpha_{e1j} p_1\left(p_j^2 + q_j^2\right) = 0 \tag{6.12.18b}$$

$$2\omega_2\left(p_2^{'} + \xi_2 p_2\right) + \frac{1}{2} f_{21} q_1 + \omega_2(3\sigma_1 - 2\sigma_2)q_2$$

$$-\frac{1}{4} Q_{21} q_1\left(3p_1^2 - q_1^2\right) - \frac{1}{4}\sum_{j=1}^{2} \alpha_{e2j} q_2\left(p_j^2 - q_j^2\right) = 0 \tag{6.12.18c}$$

$$2\omega_2\left(q_2^{'} + \xi_2 q_2\right) - \frac{1}{2} f_{21} p_1 - \omega_2(3\sigma_1 - 2\sigma_2)p_2$$

$$+\frac{1}{4}Q_{21}p_1\left(p_1^2-3q_1^2\right)+\frac{1}{4}\sum_{j=1}^{2}\alpha_{e2j}p_2\left(p_j^2+q_j^2\right)=0 \qquad (6.12.18d)$$

Now perturbing the above equations, one obtains

$$\left\{\Delta p_1^{'},\Delta q_1^{'},\Delta p_2^{'},\Delta q_2^{'}\right\}^T=\left[J_c\right]\left\{\Delta p_1,\Delta q_1,\Delta p_2,\Delta q_2\right\}^T \qquad (6.12.19)$$

Where T is the transpose and $[j_c]$ is the Jacobian matrix whose eigenvalues will determine the stability and bifurcation of the system.

The stability boundary for the linear system (i.e. the trivial state) can be obtained from the eigen values of the matrix $[J_c]$ by letting $p_1=p_2=q_1=q_2=0$.

The first order solution of the system in terms of $p_i,\ q_i\ (i=1,2)$ can be given by

$$u_1=p_1\cos\overline{\omega}_1\tau+q_1\sin\overline{\omega}_1\tau, \qquad (6.12.20a)$$

$$u_2=p_2\cos 3\overline{\omega}_1\tau+q_2\sin 3\overline{\omega}_1\tau\ , \qquad (6.12.20b)$$

where $\overline{\omega}_1=\omega_1+\frac{1}{2}\varepsilon\sigma_1$ (6.12.21)

If the external frequency $\Omega=\omega_m\pm\omega_n$ where ω_n is the nth natural frequency of the system one will obtain combination resonance of sum $\left(\Omega=\omega_m+\omega_n\right)$ or difference $\left(\Omega=\omega_m-\omega_n\right)$ type for which one may refer to the work of Dwivedy and Kar (1999). Also an exhaustive list of literature is given for the interested reader.

Numerical Results and Discussion

Following Zavodney and Nayfeh [7] and keeping internal resonance in view, a metallic beam is considered with the following properties:

L=125.4 mm, I=0.04851 mm^4, E=0.20936× N/mm^2, $Z_r=1\ mm$, $c=0.1\ N.s/mm^2$,
$\rho=0.03332\ g/mm$, $\mu=3.68979$, J=0.959, $\beta=0.25$

The roots of the characteristics equation are found numerically to be k_1=1.80097, k_2=3.2836 and the corresponding non-dimensional natural frequencies are ω_1=1 and ω_2=3.33179. The book keeping parameter ε and scaling factor λ are taken as 0.001 and 0.1, respectively. The coefficients of damping $\left(\xi_n\right)$, excitation $\left(f_{mm}\right)$ and non-linear terms $\left(\alpha_{klm}^n,\beta_{klm}^n,\gamma_{klm}^n\right)$ are found to be of the same order. The values of other required parameters expressed in the appendix (Kar and Dwivedy 1999) are calculated to be

α_{e11}=2.54149, α_{e12}=-12.2027,α_{e21}=-6.63699, α_{e22}=-195.55,Q_{12}=14.62282, Q_{21}=7.84674,

f_{11}^*=0.0655762, f_{12}^*=0.0122118, f_{21}^*=0.04249, f_{22}^*=0.1699298, ξ_1^*=0.0118963, ξ_2^*=0.0045865

Figure 6.12.2 shows the trivial state instability regions for the system with principal parametric resonance for different damping parameters. While the region bounded by the curves is unstable the regions outside the curves are stable. Clearly due to the presence of internal resonance, in addition to the main unstable region near ϕ =2, additional alternate zones of stable and unstable trivial branches exists. With increase in damping and forcing amplitude these additional zones get merged with the main unstable region. Here it may be noted that while with increase in damping the instability region decreases, with increase in forcing amplitude, the instability region increases.

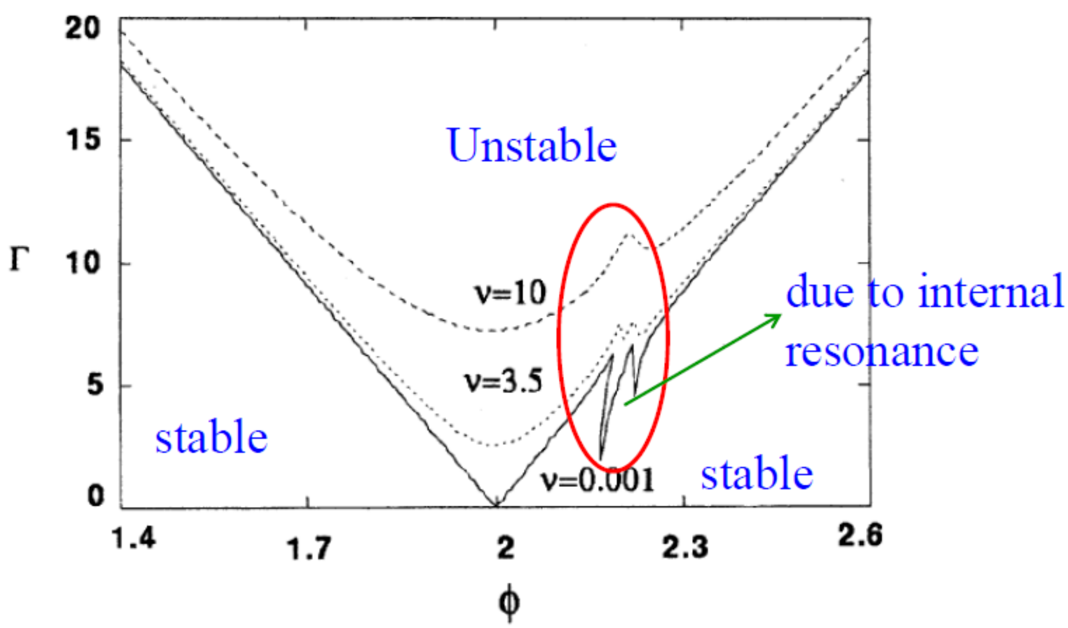

Fig. 6.12.2: A typical principal instability region

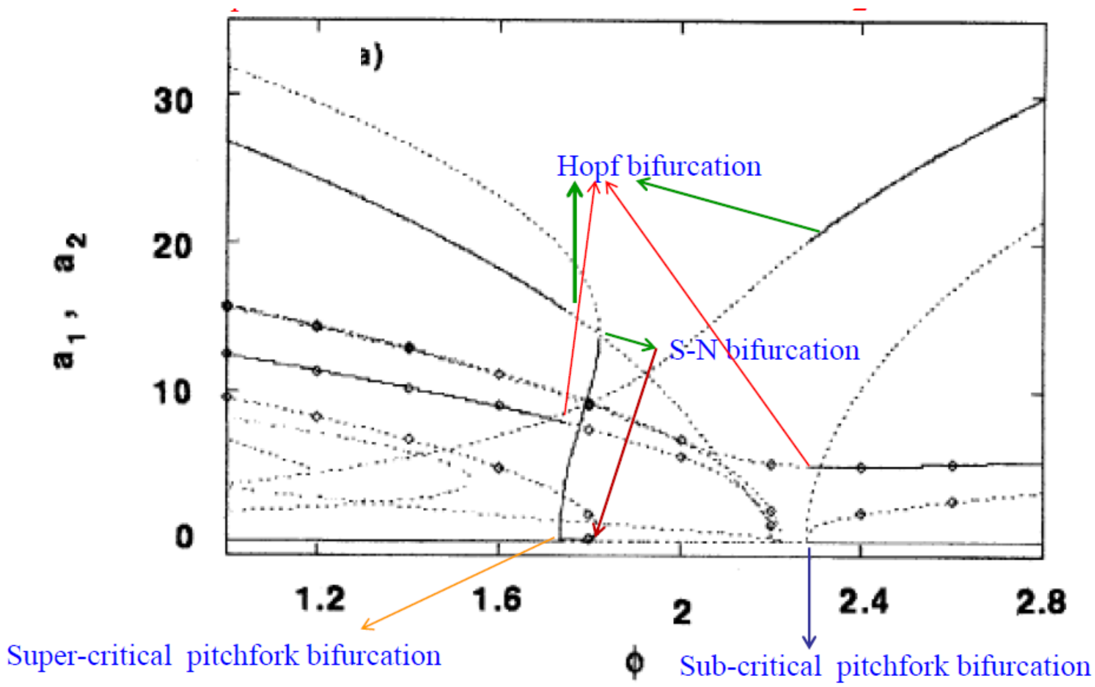

Fig. 6.12.3: Frequency response curve $\Gamma = 8, \nu = 0.001$

A typical frequency response curve is shown in Figure 6.12.3 for both the first mode (lines without bullet point) and the second mode (lines with bullet point). The stable branches are shown by solid lines and the unstable branches are shown by dotted line. One may observe multi stable regions for a wide range of frequency of the system. The nontrivial response amplitude of the first mode is observed to be larger than the second mode. While supercritical and subcritical pitchfork bifurcations are observed in the trivial state, both saddle node and Hopf bifurcations are observed in the nontrivial state. Due the presence of Hopf bifurcation stable periodic response occurs in the trivial unstable region. With decrease in forcing amplitude and damping parameter figure 6.12.4 shows the frequency response curve for both the modes for $\Gamma = 5, \nu = 0.01$. In addition to the other phenomenon described in the previous figure, here one may clearly observe (in the insert) the additional alternate stable and unstable trivial states near the main unstable region. While increasing the frequency one may observe jump up phenomena and while decreasing the frequency one may observe the jump down phenomena in the system.

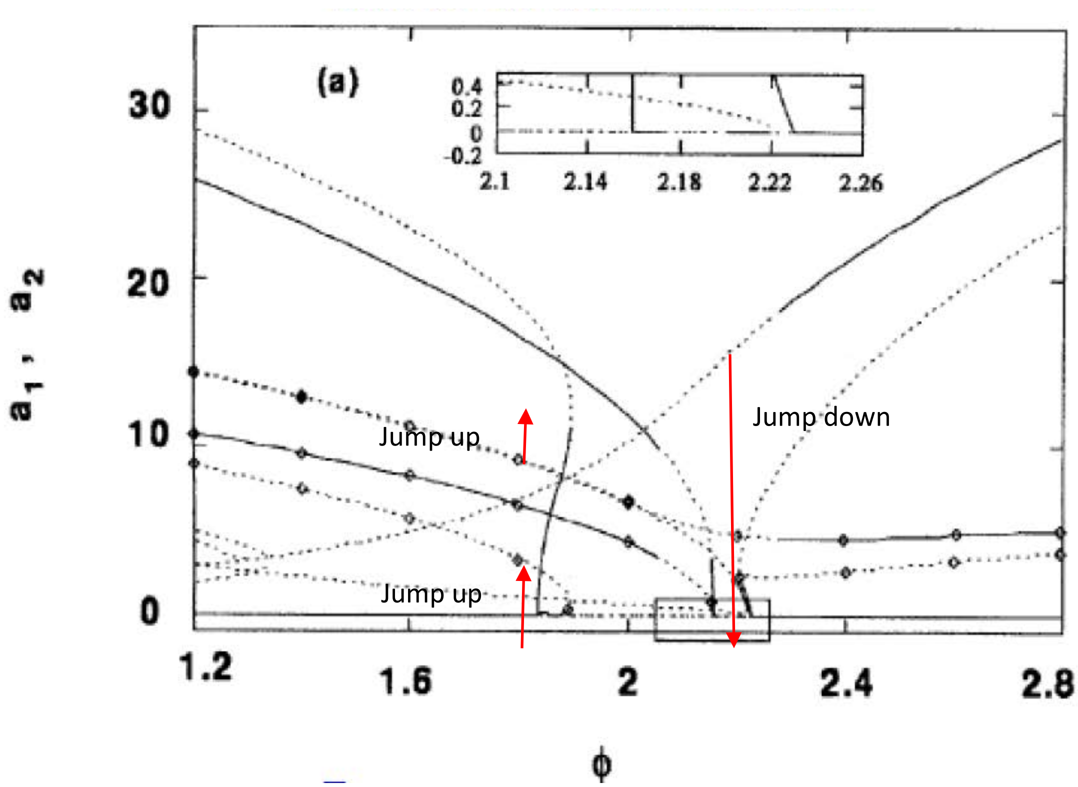

Fig. 6.12.4: Frequency response curve for $\Gamma = 5, \nu = 0.01$

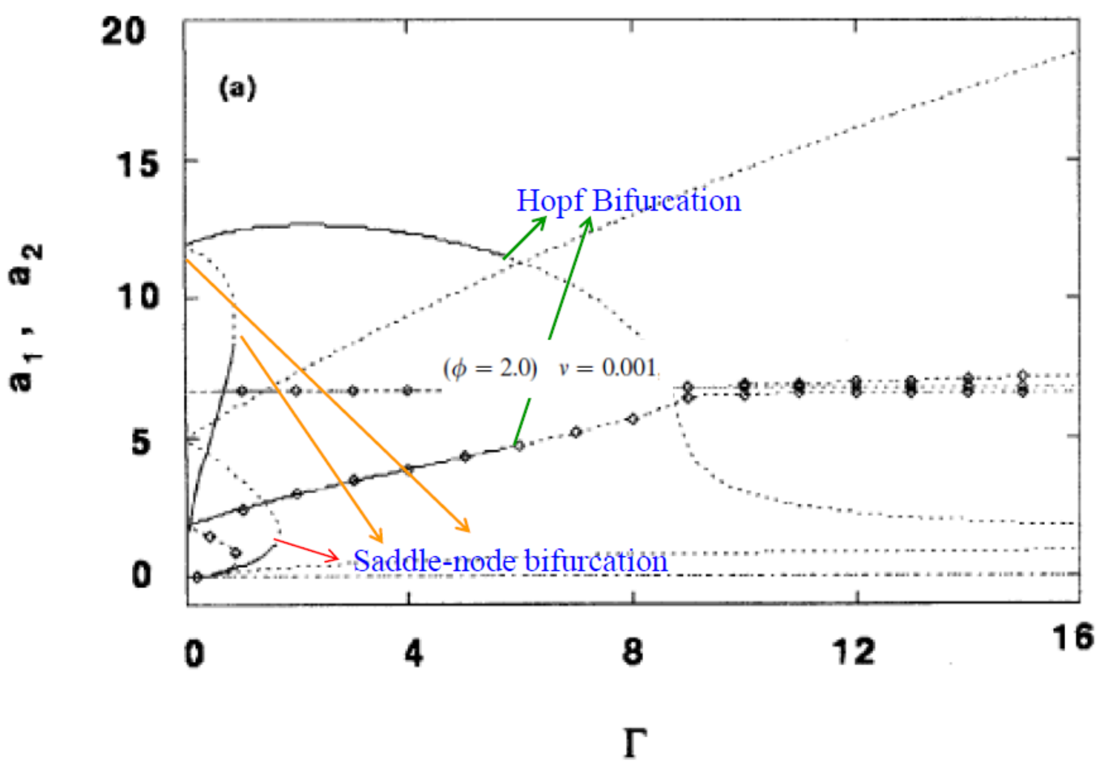

Figure 6.12.5 shows the force response curve $\phi = 2.0, \ \nu = 0.001$.

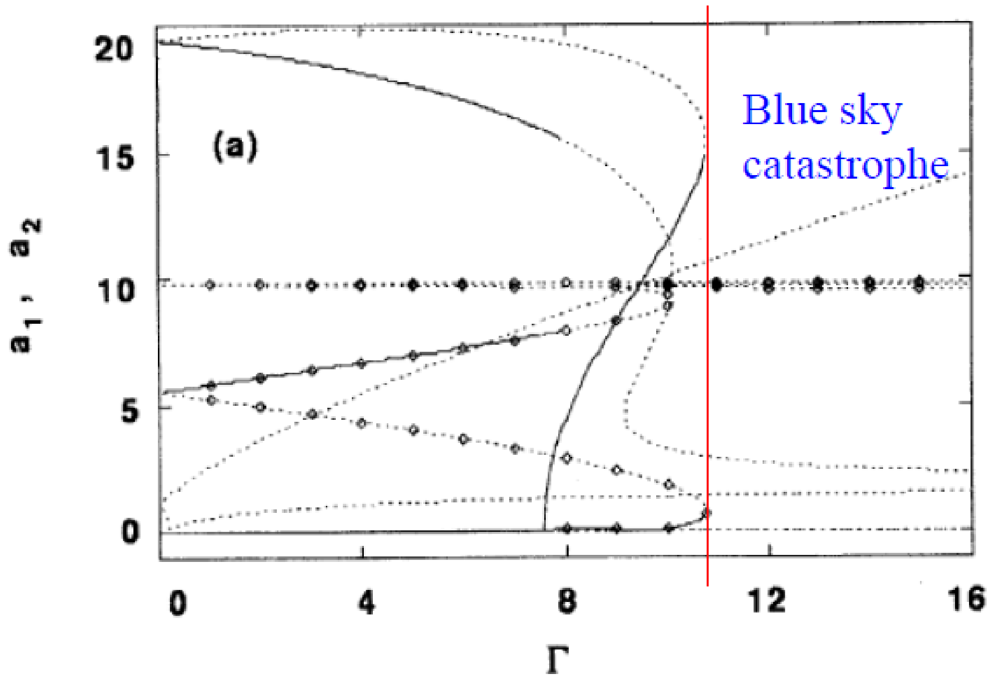

Figure 6.12.6 shows the force response curve $\phi = 1.75, \ \nu = 0.001$.

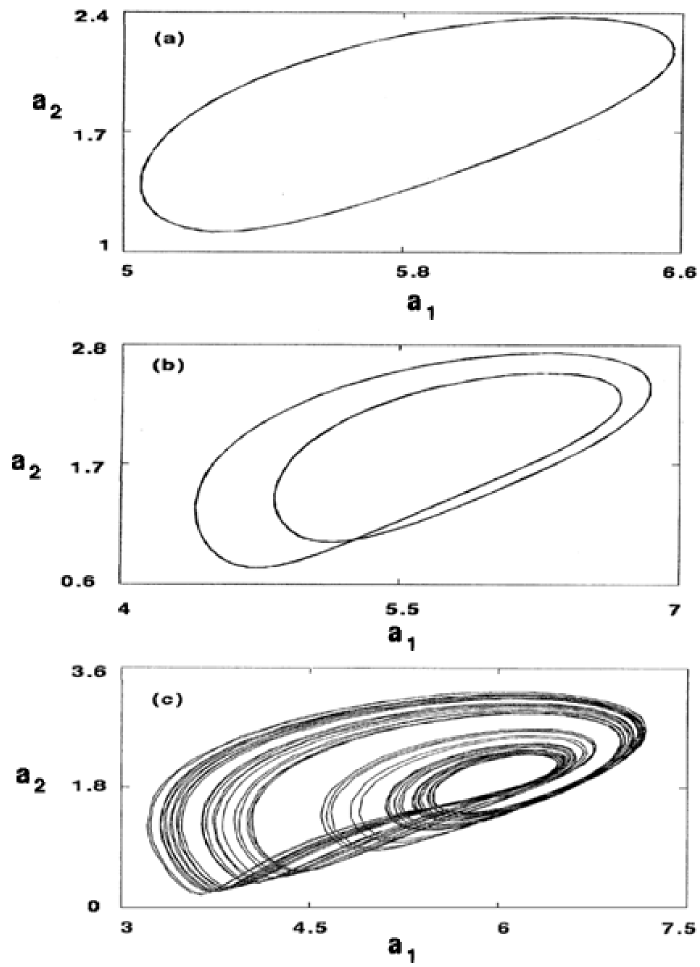

Figure 6.12.7 Phase portraits: cascade of period doubling leading to chaos for ϕ = 2.13, $\Gamma = 8.0$, (a) ν = 8.5 (periodic), (b) ν = 8.4(2T periodic), ν = 8.3 (chaotic orbits).

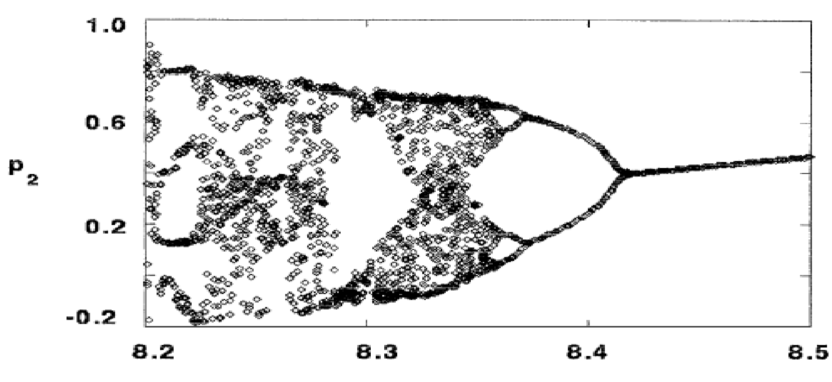

Figure 6.12.8 Poincare' showing period doubling route to chaos for ϕ = 2.13

Figure 6.12.5 shows the force response curve for the perfectly tuned system at ϕ = 2.0, ν = 0.001. Herethe nontrivial fixed point becomes unstable with saddle nodde bifurcation points at $\Gamma = 0.9$ and 1.6.And Hopf bifurcation points at $\Gamma = 5.75$. Here the trivial branch istotally unstable except at $\Gamma = 0$ .Similarly Figure 6.12.6 shows the force response curve of the system at ϕ = 1.75, ν = 0.001. Though the trivial response loses its stability at $\Gamma = 7.55$ through supercritical Pitchfork bifurcation, and a Hopf bifurcation is observed at $\Gamma = 7.95$, the system will fail through *blue sky catastrophe* if the amplitude of excitation is increased beyond the

turning point at $\Gamma = 10.75$. Figure 6.12.7(a) shows the periodic response originating from the Hopf bifurcation for ϕ = 2.13, $\Gamma = 8.0$, ν = 8.5. With decrease in the damping parameter ν to 8.42 one may observe a response with double period (Figure 6.12.7(b)). This period doubling phenomena continues with further decrease in damping parameter and finally a chaotic response (Figure 6.12.7(c)) is observed. Figure 6.12.8 shows the Poincare' section depicting cascade of period-doubling leading to chaos.

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