MODULE 3

APPROXIMATE METHODS FOR SOLVING NONLINEAR EQUATIONS

In this module different approximate perturbation methods will be used to solve the nonlinear equations of motions derived in the previous module. Initially the straight forward expansion method will be used and the following listed methods will be discussed in this module.

- Straight forward Expansion
- Lindstedt Poincare' Method
- Modified Lindstedt-Poincare method
- Method of Multiple Scales
- Method of Averaging
- Harmonic Balance method
- Intrinsic Harmonic Balance method
- Generalized Harmonic Balance method
- Multiple time scale- Harmonic Balance

Module 3 Lecture 1

THE STRAIGHT FORWARD EXPANSION

In this method, one can consider the expansion of the response which is valid for a small but finite amplitude motions by introducing the book-keeping parameter ε. Let us use this method by taking the example of Duffing equation with quadratic and cubic nonlinearities which can be given by the following equation.

$$\ddot{x}+\omega_0^2 x+\alpha_2 x^2+\alpha_3 x^3=0 \tag{3.1.1}$$

Now using book-keeping parameter ε the response x can be expanded in the following form.

$$x(t;\varepsilon)=\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\varepsilon^3 x_3(t)+\ldots \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1) one obtains

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\varepsilon^3 x_3(t)+\ldots\right)+\omega_0^2\left(\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\varepsilon^3 x_3(t)+\ldots\right)\\&+\alpha_2\left(\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\varepsilon^3 x_3(t)+\ldots\right)^2+\alpha_3\left(\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\varepsilon^3 x_3(t)+\ldots\right)^3=0\end{aligned} \tag{3.1.3}$$

$$\varepsilon\left(\ddot{x}_1+\omega_0^2 x_1\right)+\varepsilon^2\left(\ddot{x}_2+\omega_0^2 x_2+\alpha_2 x_1^2\right)+\varepsilon^3\left(\ddot{x}+\omega_0^2 x_{3.}+2\alpha_2 x_1 x_2+\alpha_3 x_1^3\right)+o\left(\varepsilon^4\right)=0 \tag{3.1.4}$$

Considering the fact that $x_n, n=1,2,3,\cdots$ is independent of ε, one can set the coefficient of each power of ε equal to zero. This leads to the following set of equation:

Order of ε

$$\ddot{x}_1+\omega_0^2 x_1=0 \tag{3.1.5}$$

Order of ε^2

$$\ddot{x}_2+\omega_0^2 x_2=-\alpha_2 x_1^2 \tag{3.1.6}$$

Order of ε^3

$$\ddot{x}+\omega_0^2 x_{3.}=-2\alpha_2 x_1 x_2-\alpha_3 x_1^3 \tag{3.1.7}$$

Let us assume the initial conditions as $x(t=0)=u_0$ and $\dot{x}(t=0)=v_0$. (3.1.8)

In polar form it can be written as

$$x(t=0)=\varepsilon a_0\cos\beta_0 \text{ and } \dot{x}(t=0)=v_0=\varepsilon a_0\sin\beta_0 \tag{3.1.9}$$

Following Nayfeh and Mook (1979), there are two alternative ways to use the initial condition. In the first way one can substitute the assumed expansion (3.1.2) into the initial conditions and equate coefficients of like powers of ε. Then one determines the constant of integration.

So, $x(0;\varepsilon)=\varepsilon x_1(0)+\varepsilon^2 x_2(0)+\varepsilon^3 x_3(0)+\ldots=\varepsilon a_0\cos\beta_0+o\left(\varepsilon^2\right)$ (3.1.10)

Hence, $x_1(0)=a_0\cos\beta_0, \dot{x}_1(0)=v_0$ and $x_n(0)=0$ and $\dot{x}_n(0)=0$ for $n\geq 2$ (3.1.11)

Then one determines the constants of integration which satisfy (3.1.11).

In the second case, one can ignore the initial conditions and the homogeneous solution in all the x_n for $n \geq 2$, until the last step. Then, considering the constants of integration in x_1 to be function of ε, one expands the solution for x_1 in powers of ε and chooses the coefficients in the expansion such that the initial conditions are satisfied.

It is demonstrated in the book of Nayfeh and Mook (1979) that the two approaches are equivalent, yielding precisely the same result. The second method is preferred because there is much less algebra involved and, in many instances only the steady state responses are required which are independent of the initial conditions.

The general solution of (3.1.5) can be written in the form

$$x_1 = a\cos(\omega_0 t + \beta) \tag{3.1.12}$$

where a and β are constants. Following the first alternative, from Eq. (3.1.11) $a = a_0$ and $\beta = \beta_0$.

Following the second approach, we consider a and β to be functions of ε and at this point pay no regard to the initial conditions.

Substituting (3.1.12) into (3.1.6) yields

$$\ddot{x}_2 + \omega_0^2 x_2 = -\alpha_2 a^2 \cos^2(\omega_0 t + \beta) = -\frac{1}{2}\alpha_2 a^2 \left[1 + \cos(2\omega_0 t + 2\beta)\right] \tag{3.1.13}$$

Now, we have two choices for expressing x_2 as follows.

According to the first alternative considering both homogeneous part and particular integral one can write

$$x_2 = \frac{\alpha_2 a_0^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right] + a_2 \cos(\omega_0 t + \beta_2). \tag{3.1.14}$$

Here a_2 and β_2 are additional constants of integration, independent of ε, chosen such that (3.1.11) is satisfied.

According to second alternative one has to write only the particular integral part as

$$x_2 = \frac{\alpha_2 a^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right]. \tag{3.1.15}$$

Thus following the first alternative, we have

$$x = \varepsilon a_0 \cos(\omega_0 t + \beta_0) + \varepsilon^2 \left(\frac{a_0^2 \alpha_2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right] + a_2 \cos(\omega_0 t + \beta_2) \right) + o(\varepsilon^3) \tag{3.1.16}$$

Following the second alternative we have

$$x = \varepsilon a \cos(\omega_0 t + \beta) + \frac{\varepsilon^2 a^2 \alpha^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right] + o(\varepsilon^3) \tag{3.1.17}$$

Now substituting $\varepsilon a = \varepsilon A_1 + \varepsilon^2 A_2 + \ldots\ldots,$ and $\beta = B_0 + \varepsilon B_1 + \ldots.$ in Eq. (3.1.17) one can show that equation (3.1.17) and Eq. (3.1.16) are equivalent as follows.

$$x_1 = \varepsilon a\cos(\omega_0 t+\beta) = (\varepsilon A_1 + \varepsilon^2 A_2 + ...)\cos(\omega_0 t + B_0 + \varepsilon B_1 + ...)$$
$$= (\varepsilon A_1 + \varepsilon^2 A_2 + ...)\left[\cos(\omega_0 t + B_0)\cos(\varepsilon B_1 + ...) - \sin(\omega_0 t + B_0)\sin(\varepsilon B_1 + ...)\right] \quad (3.1.18)$$

Taking $\varepsilon B_1 + ..$ very small, one can write $\cos(\varepsilon B_1 + ...) = 1$ and $\sin(\varepsilon B_1 + ...) = \varepsilon B_1$. Hence, Eq. (3.1.18) can be written as

$$x_1 = \varepsilon A_1\cos(\omega_0 t + \beta_0) + \varepsilon^2\left(A_2^2 + A_1^2 B_1^2\right)^{1/2}\cos(\omega_0 t + \theta_2) + O(\varepsilon^3) \quad (3.1.19)$$

where $\theta_2 = B_0 + \tan^{-1}\left(\frac{A_1 B_1}{A_2}\right)$.

Similarly, $\frac{\varepsilon^2 a^2 \alpha^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right] + o(\varepsilon^3) = \frac{\varepsilon^2 A_1^2 \alpha^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2B_0) - 3\right] + o(\varepsilon^3)$ (3.1.20)

Choosing $A_1 = a_0, B_0 = \beta_0$ and A_2 and B_1 such that

$\left(A_2^2 + A_1^2 B_1^2\right)^{1/2} = a_2$ and $\beta_0 + \tan^{-1}\left(\frac{A_1 B_1}{A_2}\right) = \beta_2$ and using Eq. (3.1.18) and Eq. (3.1.19) in Eq. (3.1.17), the later equation reduces to that of equation (3.1.16). Thus one may use either of the alternatives.

Now substituting (3.1.12) and (3.1.15) in (3.1.7) yields

$$\ddot{x}_3 + \omega_0^2 x_3 = \frac{\alpha_2^2 a^3}{3\omega_0^2}\left[3\cos(\omega_0 t + \beta) - \cos(\omega_0 t + \beta)\cos(2\omega_0 t + 2\beta)\right] - \alpha_3 a^3\cos^3(\omega_0 t + \beta)$$
$$= \left(\frac{5\alpha_2^2}{6\omega_0^2} - \frac{3\alpha_3}{4}\right)a^3\cos(\omega_0 t + \beta) - \left(\frac{\alpha_3}{4} - \frac{\alpha_2^2}{6\omega_0^2}\right)a^3\cos(3\omega_0 t + 3\beta) \quad (3.1.21)$$

Due to the presence of the term $\cos(\omega_0 t + \beta)$ in the right hand side of the differential Eq. (3.1.21), the particular solution corresponding to this term can be written as

$$\left(\frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^3}\right)a^3 t\sin(\omega_0 t + \beta) \quad (3.1.22)$$

If the straightforward procedure is continued, terms containing the factors $t^m\cos(\omega_0 t + \beta)$ and $t^m\sin(\omega_0 t + \beta)$ will appear. Terms such as these are called secular terms.

Because of secular terms, expansion of (3.1.22) is not periodic and the solution grow without bound as t tends to infinity. Hence, x_3 does not provide a small correction to x_1 and x_2. One says that the expansion (3.1.22) is not uniformly valid as t increases.

Exercise problems:

1. Perform straightforward expansion for the (i) Duffing equation with cubic nonlinearity, (ii) van der Pol's equation considering 3 term expansion and compare your results by taking two term expansion. Write the disadvantage of this method. Develop a symbolic code to determine the response of the above mentioned systems using this method.

Module 3 Lecture 2

The lindstedt Poincare' method:

This method was developed by **Anders Lindstedt** (June 27, 1854 – May 16, 1939) and **Jules Henri Poincaré** (29 April 1854 – 17 July 1912) for uniformly approximating periodic solutions to ordinary differential equations when regular perturbation approaches fail. Here a new independent variable $\tau = \omega t$ is introduced where initially ω is an unspecified function of ε which is a book-keeping parameter ($\varepsilon \ll 1$). As the new governing equation contains ω in the coefficient of the second derivative, this permits the frequency and the amplitude to interact which a property is observed in nonlinear systems. One can choose the function ω in such a way as to eliminate the secular terms [Nayfeh and Mook, 1979]. This method is explained by taking the following ordinary differential equation of Duffing type.

$$\ddot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = 0 \tag{3.2.1}$$

By using $\tau = \omega t$ equation (3.2.1) becomes

$$\omega^2 \ddot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = 0 \tag{3.2.2}$$

Assuming the expansion for ω as

$$\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots.. \tag{3.2.3}$$

where ω_1, ω_2, … are unknown constants at this point. Moreover, similar to the straight forward expansion, x can be represented by an expansion having the form

$$x(t;\varepsilon) = \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \varepsilon^3 x_3(\tau) \tag{3.2.4}$$

where x_n $(n = 1, 2, 3\cdots)$ are independent of ε. Then (3.2.2) becomes

$$\begin{aligned}&\left(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2\right)^2 \frac{d^2}{d\tau^2}\left(\varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right) + \omega_0^2\left(\varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right)\\ &+\alpha_2\left(\varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right)^2 + \alpha_3\left(\varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3\right)^3 = 0\end{aligned} \tag{3.2.5}$$

Equating the coefficients of $\varepsilon, \varepsilon^2$ *and* ε^3 to zero one obtains

$$\frac{d^2 x_1}{d\tau^2} + x_1 = 0 \tag{3.2.6}$$

$$\omega_0^2\left(\frac{d^2 x_2}{d\tau^2} + x_2\right) = -2\omega_0\omega_1 \frac{d^2 x_1}{d\tau^2} - \alpha_2 x_1^2 \tag{3.2.7}$$

$$\omega_0^2\left(\frac{d^2 x_3}{d\tau^3} + x_3\right) = -2\omega_0\omega_1 \frac{d^2 x_1}{d\tau^2} - 2\alpha_2 x_1 x_2 - \left(\omega_1^2 + 2\omega_0\omega_2\right)\frac{d^2 x_1}{d\tau^2} \tag{3.2.8}$$

The general solution of Eq. (3.2.6) can be written in the form

$$x_1 = a\cos(\tau + \beta) \tag{3.2.9}$$

Here a and β are constants. Substituting (3.2.9) into (3.2.7) leads to

$$\omega_0^2\left(\frac{d^2x_2}{d\tau^2}+x_2\right)=\underbrace{2\omega_0\omega_1 a\cos(\tau+\beta)}_{\text{Secular Term}}-\frac{1}{2}\alpha_2 a^2\left[1+\cos 2(\tau+\beta)\right] \tag{3.2.10}$$

Due to the presence of the underlined term in equation (3.2.10), the response will be unbounded and x_2 will contain the secular term. Hence, this term must be eliminated which can be done by setting $\omega_1=0$. The solution of the remaining part of equation (3.2.10) can be written as follows.

$$x_2=-\frac{\alpha_2 a^2}{2\omega_0^2}\left[1-\frac{1}{3}\cos 2(\tau+\beta)\right] \tag{3.2.11}$$

Substituting the expression for x_1 and x_2 into (3.2.8) and recalling that $\omega_1=0$, one obtain

$$\omega_0^2\left(\frac{d^2x_3}{d\tau^3}+x_3\right)=\underbrace{2\left(\omega_0\omega_2 a-\frac{3}{8}\alpha_3 a^3+\frac{5}{12}\frac{\alpha_2^2 a^3}{\omega_0^2}\right)\cos(\tau+\beta)}_{\text{Secular term}}-\frac{1}{4}\left(\frac{2\alpha_2^2}{3\omega_0^2}+\alpha_3\right)a^3 \tag{3.2.12}$$

In equation (3.2.12) the underlined term will yield an unbounded solution and to eliminate this secular term from x_3, one must put

$$\left(\omega_0\omega_2 a-\frac{3}{8}\alpha_3 a^3+\frac{5}{12}\frac{\alpha_2^2 a^3}{\omega_0^2}\right)=0 \text{ or } \omega_2=\frac{\left(9\alpha_3\omega_0^2-10\alpha_2^2\right)a^2}{24\omega_0^3} \tag{3.2.13}$$

Hence from (3.2.3), (3.2.9) and (3.2.11) one obtains

$$x=\varepsilon a\cos(\omega t+\beta)-\frac{\varepsilon^2 a^2\alpha_2}{2\omega_0^2}\left[1-\frac{1}{3}\cos(2\omega t+2\beta)\right]+O\left(\varepsilon^3\right) \tag{3.2.14}$$

where

$$\omega=\omega_0\left[1+\frac{9\alpha_3\omega_0^2-10\alpha_2^2}{24\omega_0^4}\varepsilon^2 a^2\right]+O\left(\varepsilon^3\right) \tag{3.2.15}$$

Imposing the initial condition $x(t=0)=a_0\cos\beta_0$ and $\dot{x}(t=0)=-a_0\sin\beta_0$ from (3.2.14) one obtains

$$a_0\cos\beta_0=\varepsilon a\cos\beta-\frac{\varepsilon^2 a^2\alpha_2}{2\omega_0^2}\left[1-\frac{1}{3}\cos 2\beta\right] \tag{3.2.16}$$

$$-\omega_0 a_0\sin\beta_0=-\varepsilon a\omega\sin\beta-\frac{\varepsilon^2 a^2\alpha_2\omega}{3\omega_0^2}\sin 2\beta \tag{3.2.17}$$

One should solve these equations (3.2.16) and (3.2.17) to obtain a and β which will be used further in (3.2.14) to obtain the nonlinear response of the system.

Similar to the qualitative description of the motion, it may be noted that the Lindstedt-Poincare method produced (a) a periodic expression describing the motion of the system, (b) a frequency-amplitude relationship (c) higher harmonics in the higher order terms of the expression and (d) a drift or steady-streaming term $-\frac{1}{2}\varepsilon^2 a^2\alpha_2/\omega_0^2$. (Nayfeh and Mook 1979)

Example 3.2.1: Find the solution of the equation $\ddot{u}+u+0.1x^3=0$. Take initial conditions $t=0$, $x=0.001$ m and $\dot{x}=0.1$ m/s.

Solution: Here $\omega_0^2=1, \alpha_2=0, \alpha_3=1$ and ε=0.1

Substituting these parameters in equation (3.2.15),

$$\omega=\omega_0\left[1+\frac{9\alpha_3\omega_0^2-10\alpha_2^2}{24\omega_0^4}\varepsilon^2a^2\right]=1\left[1+\frac{9-10\times 0}{24}(0.1)^2a^2\right]=\left[1+\frac{3}{800}a^2\right]$$

Also, $x=\varepsilon a\cos(\omega t+\beta)-\frac{\varepsilon^2a^2\alpha_2}{2\omega_0^2}\left[1-\frac{1}{3}\cos(2\omega t+2\beta)\right]+O(\varepsilon^3)$

Now from initial condition

$$0.001=0.1a\cos\beta-\left(\frac{0.01a^2\times 0}{2}\right)\left[1-\frac{1}{3}\cos 2\beta\right]=0.1a\cos\beta$$

$$0.1=-0.1a\omega\sin\beta-\left(\frac{0.01a^2\omega\times 0}{3}\right)\sin 2\beta=-0.1a\omega\sin\beta$$

$$a^2=\frac{1}{0.01}\left(0.001^2+\frac{0.001}{\omega^2}\right)=0.0001+\frac{0.1}{\omega^2}$$

$$\omega=\left[1+\frac{3}{800}a^2\right]=\left[1+\frac{3}{800}\left(0.0001+\frac{0.1}{\omega^2}\right)\right]=1+3e-7+\frac{3}{8000\omega^2}$$

$$or, \omega-\frac{3}{8000\omega^2}=1.0000003$$

$$or, 8000\omega^3-8000.0024\omega^2-3=0$$

ω=1.0004. The other two roots are complex numbers.

So, $a=0.3266$

$$\tan\beta=-\frac{0.1}{0.01\omega}=-\frac{10}{\omega}$$

β = -1.4707.

So, $x=0.03226\cos(1.004t-1.4707)$.

Exercise problem:

1. Find the nonlinear response of a simple pendulum taking the equation of motion up to cubic order nonlinearities. Plot the phase portrait and compare this with that obtained from the qualitative analysis.

2. Use a symbolic software to derive and find the response of the system governed by equation $\ddot{x}+\sum_{n=1}^{N}\alpha_n x^n=0$ (N = 5, quintic nonlinearities) using L-P method use initial conditions $u(0)=a$, $\dot{u}(0)=0$.

Ref: Nayfeh and Mook 1979

Module 3 Lecture 3

Modified Lindstedt Poincare' technique

The Lindstedt-Poincare' (L-P) method described in previous lecture can be applied to weakly nonlinear systems. To apply this method to strongly nonlinear system, the L-P method has been modified by many researchers. Here the method proposed by Cheung et al. (1991) is discussed. In this modified Lindstedt-Poincare' method the coefficient of the nonlinear term α can be written as a function of the book keeping parameter ε and component of the expansion of the nonlinear frequency or the forcing frequency (ω_0, ω_1). Similar to L-P method here also nondimensional time $\tau = \omega t$ is used in the governing equation (3.4.1) to obtain the following equation.

$$\omega^2 \ddot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = 0 \tag{3.3.1}$$

or in general the equation can be written as

$$\omega^2 \ddot{x} + \omega_0^2 x + \varepsilon f(x) = 0 \tag{3.3.2}$$

Unlike in L-P method, here ε may not be small.

Following four steps have been proposed in this method.

1. In contrast to the standard L-P where expansion of ω is carried out, here it is proposed to expand ω^2.

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots.. \tag{3.3.3}$$

2. A new parameter α is introduced.

$$\alpha = \frac{\varepsilon\omega_1}{\omega_0^2 + \varepsilon\omega_1} \tag{3.3.4}$$

It may be noted that α is the ratio of the 2nd term to the first two terms in the expansion given in Eq. (3.3.3).

From Eq. (3.3.4) one can write

$$\varepsilon = \frac{\omega_0^2\alpha}{\omega_1(1-\alpha)} \tag{3.3.5}$$

and $\omega_0^2 + \varepsilon\omega_1 = \dfrac{\omega_0^2}{1-\alpha}$ (3.3.6)

So, $\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = \omega_0^2 + \varepsilon\omega_1\left(1 + \dfrac{\varepsilon^2\omega_2 + ...}{\omega_0^2 + \varepsilon\omega_1}\right) = \dfrac{\omega_0^2}{1-\alpha}\left(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots\right)$ (3.3.7)

Here, ω_1 and δ_i ($i = 2, 3, \cdots$) are unknown which will be obtained in the subsequent steps.

Substituting Eq. (3.3.6) and Eq. (3.3.7) in Eq. (3.3.2), one can write

$$\frac{\omega_0^2}{1-\alpha}\left(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots\right)\ddot{x} + \omega_0^2 x + \frac{\omega_0^2\alpha}{\omega_1(1-\alpha)} f(x) = 0 \qquad (3.3.8)$$

$$\text{Or, } \left(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots\right)\ddot{x} + (1-\alpha)x + \frac{\alpha}{\omega_1} f(x) = 0 \qquad (3.3.9)$$

From Eq. (3.3.5) it can be observed that as $\varepsilon\omega_1 \to 0, \alpha \to 0$. Also as $\varepsilon\omega_1 \to \infty, \alpha \to 1$. Hence irrespective of the value of $\varepsilon\omega_1$, α value is small. Hence by introducing this parameter α, one can reduce the strongly nonlinear system to a weakly nonlinear system on which the regular L-P or other perturbation method can be used.

3. Expand x into a power series using α

$$x(t;\alpha) = x_0 + \alpha x_1(\tau) + \alpha^2 x_2(\tau) + \alpha^3 x_3(\tau) + \cdots = \sum_{n=0}^{m} \alpha^n x_n \qquad (3.3.10)$$

Now substituting (3.3.10) in (3.3.9) and equating the coefficients of like power of α, one can obtain the following set of linear differential equations.

$$\frac{d^2 x_0}{d\tau^2} + x_0 = 0 \qquad (3.3.11)$$

$$\frac{d^2 x_1}{d\tau^2} + x_1 = x_0 - \frac{1}{\omega_1} f(x_0) \qquad (3.3.12)$$

$$\frac{d^2 x_2}{d\tau^2} + x_2 = -\delta_2\frac{d^2 x_0}{d\tau^2} + x_1 - \frac{1}{\omega_1}x_1(\text{terms of } f(x_0, x_1;\alpha) \text{ having power of } \alpha = 1) \qquad (3.3.13)$$

The usual steps in L-P method may be applied to solve these equations to obtain the solution of Eq. (3.3.2) to any desired order of α.

4. In the fourth and last step, the initial value (i.e., $x(t=0) = a$ and $\dot{x}(t=0) = 0$) are separated into two parts as follows.

$$x(0) = a + b \tag{3.3.14}$$

$$x_0(0) = a \text{ and } x_i(0) = b_i \quad (i = 1, 2 \cdots) \tag{3.3.15}$$

Where a is the initial value of the sum of all odd harmonic terms of x and b_i is the initial value of the sum of all even harmonic terms of x_i.

$$b = \sum_{i=1} b_i \alpha^i \tag{3.3.16}$$

For detailed application of this method one may refer the work by Cheung et al. (1991), Chen and Cheung (1996). Franciosi and Tomasiello (1998) used Mathematica to analyze strongly nonlinear two degree of freedom system using modified L-P method. Latif (2004) and Yang et al. (2004) also used this method. Amore and Aranda (2005) used an improved L-P method in which they applied linear delta expansion (LDE) to L-P method and it is shown that this method can be applied to a wider range of nonlinear equations and it converges to the exact solution more rapidly than the conventional L-P method. Chen et al. (2007) used multi-dimensional L-P method. Xu (2007), Öziş andYıldırım (2007) used He's modified L-P method for strongly nonlinear system. Pušenjak (2008) extended L-P method for nonstationary response of strongly nonlinear system.

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Exercise Problems:

Problem 1: Apply modified L-P method for the following systems

(i) $\ddot{u}+\omega_0^2 u+\varepsilon\alpha u^3=0$

(ii) $\ddot{u}+\omega_0^2 u+\varepsilon\alpha u^2=0$

(iii) $\ddot{u}+\omega_0^2 u+\varepsilon\zeta\dot{u}+\varepsilon\alpha u^3=\varepsilon f\cos\Omega t$

(N.B: These problems are addressed in Cheung et al. (1991).)

Module 3 Lecture 4

The method of multiple scales

In method of multiple scales, the original time is written in terms of different time scales which are considered to be multiple independent variables, or scales, instead of a single variable. Here, the new independent variables ($T_n, n = 1, 2, \cdots$) of time are written using the book-keeping parameter ε as

$$T_n = \varepsilon^n t \tag{3.4.1}$$

Hence, the derivatives with respect to t can be written in terms of the partial derivatives with respect to the T_n as follows.

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \cdots = D_0 + \varepsilon D_1 + \cdots \tag{3.4.2}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right) + ... \tag{3.4.3}$$

Let us apply this method to the Duffing equation with quadratic and cubic nonlinearities

Example 3.4.1:

$$\ddot{x} + \omega_0^2 x + \varepsilon\alpha_2 x^2 + \varepsilon\alpha_3 x^3 = 0 \tag{3.4.4}$$

Similar to previous method here, one may assume that the solution of (3.4.4) can be represented by an expansion having the form

$$x(t;\varepsilon) = \varepsilon x_1(T_0, T_1, T_2, ...) + \varepsilon^2 x_2(T_0, T_1, T_2, ...) + \varepsilon^3 x_3(T_0, T_1, T_2, ...) + ... \tag{3.4.5}$$

We note that the number of independent time scales needed depends on the order to which the expansion is carried out. For example for $O(\varepsilon^3)$, one may consider T_0, T_1, and T_2. Substituting (3.4.3) and (3.4.5) into (3.4.4) and equating the coefficients of $\varepsilon, \varepsilon^2$, and ε^3 to zero, one obtains the following sets of equations.

Order of ε^1

$$D_0^2 x_1 + \omega_0^2 x_1 = 0 \tag{3.4.6}$$

Order of ε^2

$$D_0^2 x_2 + \omega_0^2 x_2 = -2D_0 D_1 x_1 - \alpha_2 x_1^2 \tag{3.4.7}$$

Order of ε^3

$$D_0^2 x_3 + \omega_0^2 x_3 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3 \tag{3.4.8}$$

The solution of (3.4.6) can be written as

$$x_1 = A(T_1, T_2)\exp(i\omega_0 T_0) + \bar{A}\exp(-i\omega_0 T_0)\,. \tag{3.4.9}$$

Here A is an unknown complex function and $\bar{A}$ is the complex conjugate of A. Substituting (3.4.9) into (3.4.7) leads to

$$D_0^2 x_2 + \omega_0^2 x_2 = -\underbrace{2i\omega_0 D_1 A\exp(i\omega_0 T_0)}_{\text{Secular term}} - \alpha_2\left[A^2\exp(2i\omega_0 T_0) + A\bar{A}\right] + cc \tag{3.4.10}$$

Here cc denotes the complex conjugate of the preceding terms. The particular solution of (3.4.10) has a secular term containing the factor $T_0\exp(i\omega_0 T_0)$. To have a bounded solution this term has to be eliminated. Hence one can obtain

$$D_1 A = \frac{dA}{dT_1} = 0 \tag{3.4.11}$$

Therefore A must be independent of T_1. With $D_1 A = 0$ the particular solution of (3.4.10) can be written as

$$x_2 = \frac{\alpha_2 A^2}{3\omega_0^2}\exp(2i\omega_0 T_0) - \frac{\alpha_2}{\omega_0^2}A\bar{A} + cc \tag{3.4.12}$$

Substituting the expression for x_1 and x_2 from equation (3.4.9) and (3.4.12) into (3.4.8) and recalling that $D_1 A = 0$ we obtain

$$D_0^2 x_3 + \omega_0^2 x_3 = -\underbrace{\left[2i\omega_0 D_2 A - \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{3\omega_0^2}A^2\bar{A}\right]\exp(i\omega_0 T_0)}_{\text{Secular Term}} - \frac{3\alpha_3\omega_0^2 + 2\alpha_2^2}{3\omega_0^2}A^3\exp(3i\omega_0 T_0) + cc \tag{3.4.13}$$

To eliminate the secular terms from x_3, we must put

$$2i\omega_0 D_2 A + \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{3\omega_0^2}A^2\bar{A} = 0 \tag{3.4.14}$$

To solve Eq. (3.4.14), it is convenient to write A in the polar form as

$$A = \frac{1}{2}a\exp(i\beta) \tag{3.4.15}$$

where a and β are real function of T_2. Substituting (3.4.15) into (3.4.14) and separating the result
into real and imaginary parts, we obtain

$$\omega a' = 0 \text{ and } \omega_0 a\beta' + \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^2}a^3 = 0 \tag{3.4.16}$$

where the prime denotes the derivative with respect to T_2. As $a' = 0$, a is a constant and

$$\beta' = -\frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^2\omega_0 a}a^3 \quad \text{or } \beta = \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^3}a^2 T_2 + \beta_0 \tag{3.4.17}$$

Here β_0 is a constant. Now using $T_2 = \varepsilon^2 t$ from (3.4.15) we find that

$$A = \frac{1}{2} a \exp\left[i \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^3} \varepsilon^2 a^2 t + i\beta_0 \right] \tag{3.4.18}$$

Substituting Eq. (3.4.18) in the expressions for x_1 and x_2 in Eqs. (3.4.9), (3.4.12) and (3.4.5), one obtains

$$x = \varepsilon a \cos(\omega t + \beta_0) - \frac{\varepsilon^2 a^2 \alpha_2}{2\omega_0^2}\left[1 - \frac{1}{3}\cos(2\omega t + 2\beta_0)\right] + O(\varepsilon^3) \tag{3.4.19}$$

Here $\omega = \omega_0 \left[1 + \frac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^4} \varepsilon^2 a^2 \right] + O(\varepsilon^3)$ (3.4.20)

This solution is in good agreement with the solution obtained using the Lindstedt-Poincare' procedure. The method of multiple scales though a little more involved, has advantage over the Lindstedt-Poincare method, for example it can treat damped systems conveniently (Nayfeh and Mook 1979).

Example 3.4.2: Find the expression for the frequency-response curve for a nonconservative system using method of multiple scales.

Solution:

Consider the governing equation of motion of a nonconservative system which can be given by

$$\ddot{u} + \omega_0^2 u = \varepsilon f(u, \dot{u}) . \tag{3.4.21}$$

Following standard procedure of method of multiple scales one may write

$$u(t;\varepsilon) = u_0 + \varepsilon u_1(T_0, T_1, T_2, ...) + \varepsilon^2 u_2(T_0, T_1, T_2, ...) + \varepsilon^3 u_3(T_0, T_1, T_2, ...) + ... \tag{3.4.22}$$

Substituting (3.4.3) and (3.4.22) into (3.4.21) and equating the coefficients of $\varepsilon^0, \varepsilon^1$ and ε^2 to zero,

one obtains the following sets of equations.

$$D_0^2 u_0 + \omega_0^2 u_0 = 0 \tag{3.4.23}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 + f(u_0, D_0 u_0) \tag{3.4.24}$$

$$D_0^2 u_2 + \omega_0^2 u_2 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3 \tag{3.4.25}$$

$$D_0^2 u_n + \omega_0^2 u_n = F(u_0, u_1, \cdots u_{n-1}) \text{ for } n \geq 2 \tag{3.4.26}$$

$$D_0^2 u_2 + \omega_0^2 u_2 = D_0^2(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots) + 2\varepsilon D_0 D_1 (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots)$$

$$+ \varepsilon^6 (D_1^2 + 2D_0 D_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots) \qquad = \varepsilon f(u_0, D_0^{u_0})$$

$$+ \omega_0^2(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots)$$

The solution of Eq. (3.4.23) can be given by

$$u_0 = A(T_1, T_2 \cdots)\exp(i\omega_0 T_0) + \bar{A}\exp(-i\omega_0 T_0) \tag{3.4.27}$$

Substituting Eq. (3.4.27) in Eq. (3.4.24) following equation is obtained.

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 &= -2i\omega_0 D_1 A\exp(i\omega_0 T_0) + 2i\omega_0 D_1 \bar{A}\exp(-i\omega_0 T_0) + \\ &f\left[A\exp(i\omega_0 T_0) + \bar{A}\exp(-i\omega_0 T_0), 2i\omega_0\left(A\exp(i\omega_0 T_0) - \bar{A}\exp(-i\omega_0 T_0)\right)\right] \end{aligned} \tag{3.4.28}$$

One may use Fourier series to write the forcing function as follows.

$$f = \sum_{n=-\infty}^{\infty} f_n\left(A, \bar{A}\right)\exp(in\omega_0 T_0) \tag{3.4.29}$$

$$\text{where, } f_n\left(A, \bar{A}\right) = \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-in\omega_0 T_0)dT_0 \tag{3.4.30}$$

Hence to eliminate secular term from Eq. (3.4.28) one may write

$$2iD_1 A = \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\omega_0 T_0)dT_0 \tag{3.4.31}$$

For a first order approximation, one may consider A to be a function of T_1 only and can write A in its polar form as

$$A(T_1) = \frac{1}{2}a(T_1)\exp(i\beta(T_1)) \tag{3.4.32}$$

Substituting (3.4. 32) in (3.4.31) one can write

$$2iD_1\left(\frac{1}{2}a(T_1)\exp(i\beta(T_1))\right) = \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\omega_0 T_0)dT_0 \tag{3.4.33}$$

$$\text{Or, } i\frac{da}{dT_1}\exp(i\beta(T_1)) - a\frac{d\beta}{dT_1}\exp(i\beta(T_1)) = \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\omega_0 T_0)dT_0 \tag{3.4.34}$$

$$\begin{aligned} \text{Or, } i\frac{da}{dT_1} - a\frac{d\beta}{dT_1} &= \frac{1}{2\pi\exp(i\beta(T_1))}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\omega_0 T_0)dT_0 \\ &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\omega_0 T_0)\exp(-i\beta(T_1))dT_0 \\ &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp\left(-i\left(\omega_0 T_0 + \beta(T_1)\right)\right)dT_0 = \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\exp(-i\phi)dT_0 \\ &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\left(\cos\phi - i\sin\phi\right)dT_0 \quad \text{where } \phi = \omega_0 T_0 + \beta(T_1) \end{aligned} \tag{3.4.35}$$

Separating the real and imaginary parts one may write

$$\frac{da}{dT_1} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi} \sin\phi\, f\left(\cos\phi, -a\omega_0 \sin\phi\right) d\phi \tag{3.4.36}$$

$$\frac{d\beta}{dT_1} = -\frac{1}{2\pi\omega_0 a}\int_0^{2\pi} \cos\phi\, f\left(\cos\phi, -a\omega_0 \sin\phi\right) d\phi \tag{3.4.37}$$

The first order approximation solution can be written as

$$\begin{aligned} u(t;\varepsilon) = u_0 &= A(T_1, T_2 \cdots)\exp(i\omega_0 T_0) + \bar{A}\exp(-i\omega_0 T_0) \\ &= \frac{1}{2}a\exp(i\beta)\exp(i\omega_0 T_0) + \frac{1}{2}a\exp(-i\beta)\exp(-i\omega_0 T_0) \\ &= \frac{1}{2}a\left(\exp(i\omega_0 T_0 + i\beta) + \exp(-i\omega_0 T_0 - i\beta)\right) \\ &= \frac{1}{2}a\left(\exp(i\phi) + \exp(-i\phi)\right) = a\cos\phi = a\cos(\omega_0 t + \beta) + O(\varepsilon) \end{aligned} \tag{3.4.38}$$

Exercise Problems:

1. Derive the frequency-amplitude relation for the following systems using multiple scales

(a) $\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} = 0$

(b) $\ddot{u} + \omega_0^2 u + \varepsilon\alpha_2 u^2 = \varepsilon f \cos\Omega t$

(c) $\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha u^3 = \varepsilon f \cos\Omega t$

(d) $\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} + \varepsilon\alpha_2 u^2 + \varepsilon\alpha_3 u^3 = \varepsilon f \cos\Omega t$

(e) $\ddot{u} + \omega_0^2 u + 2\varepsilon\zeta\dot{u} + \varepsilon\alpha u^3 + \varepsilon f \cos\Omega t u = 0$

Module 3 Lecture 5

METHOD OF MULTIPLE SCALES APPLIED TO FORCED VIBRATION

In this lecture the method of multiple scales is applied to a forced vibration system. One may follow similar procedure as in the previous lecture. But in this case additional secular terms will arise which will give different resonance conditions. In the following example the primary resonance condition for the forced Duffing equation is illustrated. It may be noted that unlike linear system in case of nonlinear system multiple equilibrium solution will arise.

Example 3.5.1: Find the frequency-amplitude relation for primary resonance condition for the forced Duffing equation.

$$\ddot{u}+\omega_0^2 u+2\varepsilon\mu\dot{u}+\varepsilon\alpha u^3=\varepsilon K\cos\Omega t \tag{3.5.1}$$

Solution

For primary resonance condition, the frequency of external excitation Ω should be nearly equal to that of natural frequency ω_0 of the system. Hence, to show the nearness of Ω to ω_0, one may use a detuning parameter σ, and by using book-keeping parameter it can be written that

$$\Omega=\omega_0+\varepsilon\sigma \tag{3.5.2}$$

Now expanding u using the book-keeping parameter and different time scales one may write

$$u(t;\varepsilon)=u_0(T_0,T_1)+\varepsilon u_1(T_0,T_1)+....... \tag{3.5.3}$$

Now substituting Eqs.(3.4.3) and (3.5.3) in Eq. (3.5.1) and separating the like power of ε, following equations are obtained.

$$D_0^2u_0+\omega_0^2u_0=0 \tag{3.5.4}$$

$$D_0^2u_1+\omega_0^2u_1=-2D_0D_1u_0-2\mu D_0u_0-\alpha u_0^3+f\cos(\omega_0T_0+\sigma T_1) \tag{3.5.5}$$

The solution of Eq. (3.5.4) can be given by

$$u_0=A(T_1,T_2)\exp(i\omega_0T_0)+\bar{A}(T_1,T_2)\exp(-i\omega_0T_0). \tag{3.5.6}$$

Substituting (3.5.6) in Eq. (3.5.5) one obtains

$$D_0^2u_1+\omega_0^2u_1=-\left[2i\omega_0\left(D_1A\exp(i\omega_0T_0)+\mu A\exp(i\omega_0T_0)\right)+3\alpha A^2\bar{A}\exp(i\omega_0T_0)\right]$$
$$-\alpha A^3\exp(3i\omega_0T_0)+\frac{1}{2}f\exp\left[i(\omega_0T_0+\sigma T_1)\right]+cc \tag{3.5.7}$$

Or,

$$D_0^2u_1+\omega_0^2u_1=\underbrace{-\left[2i\omega_0\left(A'+\mu A\right)+3\alpha A^2\bar{A}\right]\exp(i\omega_0T_0)}_{\text{Secular term}}-\alpha A^3\exp(3i\omega_0T_0)$$
$$+\underbrace{\frac{1}{2}f\exp\left[i(\omega_0T_0+\sigma T_1)\right]}_{\text{Nearly secular term}}+cc \tag{3.5.8}$$

In Eq. (3.5.8), the term containing $\exp(i\omega_0 T_0)$ is a secular term and term containing $\exp[i(\omega_0 T_0 + \sigma T_1)]$ is a nearly secular term as it will approach to a secular term when $\sigma \to 0$. To have a bounded solution these two terms should be eliminated by imposing the following condition.

$$2i\omega_0(A' + \mu A) + 3\alpha A^2 \bar{A} - \frac{1}{2} f \exp(i\sigma T_1) = 0 \quad (3.5.9)$$

Substituting $A = \frac{1}{2} a \exp(i\beta)$ in Eq. (3.5.9) and separating the real and imaginary parts, the following first order differential equations are obtained.

$$a' = -\mu a + \frac{1}{2}\frac{f}{\omega_0}\sin(\sigma T_1 - \beta) \quad (3.5.10)$$

$$a\beta' = \frac{3}{8}\frac{\alpha}{\omega_0}a^3 - \frac{1}{2}\frac{f}{\omega_0}\cos(\sigma T_1 - \beta) \quad (3.5.11)$$

One may write these two equations in their autonomous form by substituting $\gamma = \sigma T_1 - \beta$. The resulting equations are

$$a' = -\mu a + \frac{1}{2}\frac{f}{\omega_0}\sin\gamma \quad (3.5.12)$$

$$a\gamma' = a\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}a^3 + \frac{1}{2}\frac{f}{\omega_0}\cos\gamma \quad (3.5.13)$$

Equations (3.5.12) and (3.5.13) are known as the reduced equations and can be used for finding the response and stability of the system. By analytically or numerically solving these equations one may obtain the amplitude and phase of the response of the system. The first order response of the system can be given by

$$u = a\cos(\omega_0 t + \beta) + O(\varepsilon) \quad (3.5.14)$$

It may be noted that for steady state, the amplitude and phase of the system do not depend on the time and hence the time derivative terms i.e., a' and γ' should be equal to zero. Hence, for steady state one can write

$$\mu a = \frac{1}{2}\frac{f}{\omega_0}\sin\gamma \quad (3.5.15)$$

$$a\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}a^3 = -\frac{1}{2}\frac{f}{\omega_0}\cos\gamma \quad (3.5.16)$$

Now squaring and adding Eqs. (3.5.15) and Eq. (3.5.16), the following closed form equation is obtained.

$$\left[\mu^2 + \left(\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}a^2\right)^2\right]a^2 = \frac{f^2}{4\omega_0^2} \quad (3.5.17)$$

It may be noted that this equation is a 6th order polynomial in terms of a and is quadratic in terms of detuning parameter σ. Hence, solving the quadratic equation, one can obtain the following relation for the frequency response curve.

$$\sigma = \frac{3}{8}\frac{\alpha}{\omega_0}a^2 \pm \left(\frac{f^2}{4\omega_0^2 a^2} - \mu^2\right)^{\frac{1}{2}} \tag{3.5.18}$$

Example 3.5.2: Apply method of multiple scales to the following nonlinear parametrically excited system

$$\ddot{q} + 2\varepsilon\zeta\dot{q} + q + \varepsilon\left(\alpha_1 q^3 + \alpha_2 q^2\ddot{q} + \alpha_3\dot{q}^2 q\right) - \varepsilon f_1\cos\left(2\bar{\omega}\tau\right)q - \varepsilon k_1\left(1+\cos\left(2\bar{\omega}\tau\right)\right)\dot{q}q^2 = 0 \tag{3.5.19}$$

This equation contains parametric term $f_1\cos\left(2\bar{\omega}\tau\right)q$ and nonlinear damping term $k_1\left(1+\cos\left(2\bar{\omega}\tau\right)\right)\dot{q}q^2$, along with cubic geometric ($\alpha_1 q^3$) and inertial ($\alpha_2 q^2\ddot{q} + \alpha_3\dot{q}^2 q$) nonlinear terms.

Solution: In this method the displacement q can be represented in terms of different time scales (T_0, T_1) and a book keeping parameter ε as follows.

$$q(\tau;\varepsilon) = q_0\left(T_0, T_1\right) + \varepsilon q_1\left(T_0, T_1\right) + O\left(\varepsilon^2\right). \tag{3.5.20}$$

Here, $T_0 = \tau$, and $T_1 = \varepsilon\tau$. The transformation of first and second time derivatives are given by

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + O(\varepsilon^2) \text{ and } \frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2).$$

where, $D_0 = \frac{\partial}{\partial T_0}$, and $D_1 = \frac{\partial}{\partial T_1}$.

Substituting Eqs. (3.5.20) in (3.5.19) and equating the coefficient of like powers of ε, yields the following equations.

Order ε^0 : $D_0^2 q_0 + q_0 = 0$, (3.5.21)

Order ε^1 : $D_0^2 q_1 + q_1 = -2D_0 D_1 q_0 - 2\zeta D_0 q_0 - \alpha_1 q_0^3 - \alpha_2\left(D_0^2 q_0\right)q_0 - \alpha_3\left(D_0 q_0\right)^2 q_0^2$

$$+ f_1\cos\left(2\bar{\omega}T_0\right)q_0 + k_1\left(1+\cos\left(2\bar{\omega}T_0\right)\right)\left(D_0 q_0\right)q_0^2. \tag{3.5.22}$$

General solutions of Eq. (3.5.21) can be written as

$$q_0 = A(T_1, T_2)\exp(iT_0) + \bar{A}(T_1, T_2)\exp(-iT_0). \tag{3.5.23}$$

Substituting Eq. (3.5.23) into Eq. (3.5.22) leads to

$$D_0^2 q_1 + q_1 = -\underbrace{\left(2i\,A' + 2i\zeta A + \left(3\alpha_1 - 3\alpha_2 + \alpha_3 - ik_1\right)A^2\bar{A}\right)}_{\text{secular term}}\exp(iT_0) + \nu A^3 \exp\left(3iT_0\right)$$

$$+ik_1 A^3 \exp\left(3iT_0\right) + \frac{f_1}{2}\left[\underbrace{\bar{A}\exp i(2\bar{\omega}-1)T_0}_{\text{nearly secular term}}\right]_0 + \frac{ik_1}{2}A^3 \exp i(2\bar{\omega}+3)T_0 + \frac{ik_1}{2}A^2\bar{A}\exp i(2\bar{\omega}+1)T_0 \quad (3.5.24)$$

$$-\underbrace{\frac{ik_1}{2}\bar{A}^2 A\exp i(2\bar{\omega}-1)T_0}_{\text{nearly secular term}} + \underbrace{\frac{ik_1}{2}A^3 \exp i(3-2\bar{\omega})T_0}_{\text{nearly secular term}} + cc$$

Here, $\nu = -\alpha_1 + \alpha_2 + \alpha_3$. One may observe that any solution of Eq. (3.5.24) will contain secular or small divisor terms when non-dimensional frequency of magnetic strength ($\bar{\varpi}$) is nearly equal to 1 which is known as simple resonance case. In this case, one may present detuning parameter σ to express the nearness of $\bar{\varpi}$ to 1, as

$$\bar{\varpi} = 1 + \varepsilon\sigma, \quad \text{and} \quad \sigma = O(1) \qquad (3.5.25)$$

Substituting Eq. (3.5.25) into Eq. (3.5.24), one may obtain the following secular or small divisor terms.

$$-2i\,A'\exp(iT_0) - 2i\zeta A\exp(iT_0) - \left(3\alpha_1 - 3\alpha_2 + \alpha_3 - ik_1\right)A^2\bar{A}\exp(iT_0)$$
$$+\frac{f_1}{2}\bar{A}\exp\left(2\sigma T_1\right) + i\frac{k_1}{2}A^3\exp\left(-2\sigma T_1\right) - i\frac{k_1}{2}\bar{A}^2 A\exp\left(2\sigma T_1\right) = 0. \qquad (3.5.26)$$

Putting A equal to $\frac{1}{2}a\left(T_1\right)e^{\left(i\beta(T_1)\right)}$ into Eq. (3.5.26) and separating the real and imaginary terms, one may find the reduced equations as given below.

$$\frac{da}{dT_2} = -\zeta a + \frac{k_1}{8}a^3 + \frac{f_1}{4}a\sin\gamma\,, \qquad (3.5.27)$$

$$a\frac{d\gamma}{dT_2} = 2a\left(\frac{\bar{\omega}-1}{\varepsilon}\right) - \frac{3}{4}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^3 + \frac{1}{4}a^3 k_1 \sin\gamma + \frac{f_1}{2}a\cos\gamma\,. \qquad (3.5.28)$$

For steady state as $\frac{da}{dT_2}, \frac{d\gamma}{dT_2}$ are equal to 0,the above equations reduces to

$$a\left(-\zeta + \frac{k_1}{8}a^2 + \frac{f_1}{4}\sin\gamma\right) = 0 \qquad (3.5.29)$$

$$2\left(\frac{\bar{\omega}-1}{\varepsilon}\right) - \frac{3}{4}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^2 + \frac{1}{4}a^2 k_1 \sin\gamma + \frac{f_1}{2}\cos\gamma = 0 \qquad (3.5.30)$$

One may observe from the Eq. (3.5.29) that, the system possesses both trivial ($a = 0$) and nontrivial ($a \neq 0$) responses. The nontrivial response can be obtained by numerically solving Eqs. (3.5.29) and (3.5.30). Later it can be studied that the stability of the steady state response can be determined by finding the eigenvalues of the Jacobian matrix obtained by perturbing Eqs. (3.5.27) and (3.5.28).

The first order non-trivial steady state response is given by

$$q = a\cos\left(\frac{1}{2}\left(\bar{\omega}\tau - \gamma\right)\right) \tag{3.5.31}$$

Exercise Problems

1. Find the expression for the transition curve for a parametrically excited system given by the following equation of motion.

$$\ddot{u} + \omega_0^2 u + 2\varepsilon\mu\dot{u} + \left(\varepsilon f \cos\Omega t\right)u = 0$$

2. The equation of motion for a bimaterial beam with alternating magnetic field and thermal loads (G. Y. Wu, Journal of Sound and Vibration 327(2009)197-210) can be given by

$$\Omega^2\ddot{q} + q + 2\Omega\left[k_1 + k_2\left(1 + \cos(2\tau)\right)q^2\right]\dot{q} - f_1\cos(2\tau)q = 0$$

Derive the expression for the frequency response by using method of multiple scales.

References for further reading

One may read higher order method of multiple scales from Rahman and Burton(1989) and Dwivedy and Kar (1999) .

1. Z. Rahman and T. D. Burton, On higher order method of multiple scales in nonlinear oscillations-periodic steady state response, Journal of Sound and Vibration **133,** 369-379, 1989.
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Module 3 Lecture 6

THE METHOD OF HARMONIC BALANCE

Harmonic balance method is the most commonly used method to study the nonlinear vibration problems. Here, the response of the system is assumed in terms of a Fourier series and using this expression in the governing differential equation and separating the coefficients of the harmonic terms one can obtain the unknown coefficients and frequency amplitude relation of the nonlinear system. One may assume the response in the following form.

$$x = \sum_{m=0}^{M} \widehat{A}_m \cos(m\omega t) + \widehat{B}_m \sin(m\omega t) = \sum_{m=0}^{M} A_m \cos(m\omega t + m\beta_0) \qquad (3.6.1)$$

Then substituting (3.6.1) in the governing equation and equating the coefficient of each of the lowest M + 1 harmonics to zero, one obtains a system of M + 1 algebraic equation relating ω and the A_m. Usually these equations are solved for $A_0, A_2, A_3, ..., A_m$ and ω in terms of A_1. The accuracy of the resulting periodic solution depends on the value of A_1 and the number of harmonics in the assumed solution. The method is illustrated using the following examples.

Example 3.6.1:

Find the expression for frequency amplitude relation for the single degree of freedom system with both quadratic and cubic nonlinearities using harmonic balance method by taking one , two and three terms in the expansion of the Fourier series.

$$\ddot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = 0 \qquad (3.6.2)$$

Solution :

Taking only one term expansion, from equation (3.6.1) one has

$$x = A_1 \cos(\omega t + \beta_0) = A_1 \cos\phi \qquad (3.6.3)$$

Substituting equation (3.6.3) into equation (3.6.2) yields,

$$-(\omega^2 - \omega_0^2) A_1 \cos\phi + \frac{1}{2}\alpha_2 A_1^2 \left[1 + \cos 2\phi\right] + \frac{1}{4}\alpha_3 A_1^3 \left[3\cos\phi + \cos 3\phi\right] = 0 \qquad (3.6.4)$$

Equating the co-efficient of $\cos\phi$ to zero, one obtains

$$\omega^2 = \omega_0^2 + \frac{3}{4}\alpha_3 A_1^2 \tag{3.6.5}$$

which for small A_1 becomes

$$\omega = \left[\omega_0^2 + \frac{3}{4}\alpha_3 A_1^2\right]^{1/2} \omega_0 \left[1 + \frac{3\alpha_3}{8\omega_0^2} A_1^2\right] \tag{3.6.6}$$

Comparing (3.6.6) with (3.2.14) we conclude that only part of the nonlinear correction to the frequency has been obtained.

Now taking two terms and following Nayfeh and Mook (1979) by putting

$$x = A_0 + A_1 \cos\phi \tag{3.6.7}$$

in (3.6.2) one obtains

$$\left[\omega_0^2 A_0 + \alpha_2 A_0^2 + \frac{1}{2}\alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2}\alpha_3 A_0 A_1^2\right] + \left[-\left(\omega^2 - \omega_0^2\right) A_1 + 2\alpha_2 A_0 A_1 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4}\alpha_3 A_1^3\right]\cos\phi$$
$$+\left[\frac{1}{2}\alpha_2 A_1^2 + \frac{3}{2}\alpha_3 A_0 A_1^2\right]\cos 2\phi + \frac{1}{4}\alpha_3 A_1^3 \cos 3\phi = 0 \tag{3.6.8}$$

Equating the constant term (terms with magenta colour) and the coefficient of $\cos\phi$ (terms with blue colour) to zero, one obtains the following equations.

$$\omega_0^2 A_0 + \alpha_2 A_0^2 + \frac{1}{2}\alpha_2 A_1^2 + \alpha_3 A_0^3 + \frac{3}{2}\alpha_3 A_0 A_1^2 = 0 \tag{3.6.9}$$

$$-\left(\omega^2 - \omega_0^2\right) + 2\alpha_2 A_0 + 3\alpha_3 A_0^2 + \frac{3}{4}\alpha_3 A_1^2 = 0 \tag{3.6.10}$$

When A_1 is small, neglecting terms containing A_0^2, A_1^2, A_0^3, from Eqs. (3.6.9) and (3.6.10) one can write

$$A_0 = \left[-\frac{1}{2}\frac{\alpha_2}{\omega_0^2} A_1^2 + O(A_1^4)\right] \tag{3.6.11}$$

$$\omega^2 = \omega_0^2 + \left(\frac{3}{4}\alpha_3 - \frac{\alpha_2^2}{\omega_0^2}\right) A_1^2 \tag{3.6.12}$$

Hence,

$$\omega = \omega_0 \left[1 + \frac{3\alpha_3\omega_0^2 - 4\alpha_2^2}{8\omega_0^4} A_1^2\right] \tag{3.6.13}$$

It may be noted that this expression for frequency is not same as that we obtained by using method of multiple scales or L-P method. Hence to obtain a consistent solution by using the method of harmonic balance, one need either to know about the solution *a priori* or one has to take many terms in the Fourier series and make a convergence analysis. Otherwise one might obtain an inaccurate approximation.

Using two harmonic terms

$$x = A_0 + A_1 \cos\phi + A_2 \cos 2\phi \tag{3.6.14}$$

where $\phi = \omega t + \beta_0$ and A_0 and $A_2 < A_1$. Substituting Eq. (3.6.14) in Eq. (3.6.2) and equating the coefficient of the constant part, coefficient of $\cos\phi$ and $\cos 2\phi$ equal to zero, one obtains the following equations.

Constant terms

$$\omega_0^2 A_0 + \alpha_2\left(A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2\right) + \alpha_3\left(A_0^3 + \frac{3}{2}A_0A_1^2 + \frac{3}{2}A_0A_2^2 + \frac{3}{4}A_1^2A_2\right) = 0 \quad (3.6.15)$$

Coefficient of $\cos\phi$

$$\left(\omega_0^2 - \omega^2\right) + 2\alpha_2 A_0 A_1 + \alpha_2 A_1 A_2 + 3\alpha_3 A_0^2 A_1 + \frac{3}{4}\alpha_3 A_1^3 + 3\alpha_3 A_0 A_1 A_2 + \frac{3}{2}\alpha_3 A_1 A_2^2 = 0 \quad (3.6.16)$$

Coefficient of $\cos 2\phi$

$$\left(\omega_0^2 - 4\omega^2\right)A_2 + \alpha_2\left(\frac{1}{2}A_1^2 + 2A_0A_2\right) + \frac{3}{4}\alpha_3\left(A_2^3 + 4A_0^2A_2 + 2A_1^2A_2 + 2A_0A_1^2\right) = 0 \quad (3.6.17)$$

Assuming A_1 to be small, one can observe from Eqs. (3.6.15-3.6.17) that A_0 and A_2 are of the order of A_1^2. So neglecting the terms of $O\left(A_1^4\right)$ and higher order terms one can write A_0 in terms of A_1 form Eq. (3.6.15) as follows.

$$A_0 = -\frac{1}{2}\frac{\alpha_2}{\omega_0^2}A_1^2 + O\left(A_1^4\right) \quad (3.6.18)$$

Now multiplying $4A_2$ in Eq.(3.6.16) and subtracting it from Eq.(3.6.17) one obtains the following equation.

$$\begin{aligned}
&4\left(\omega_0^2 - \omega^2\right)A_2 + 8\alpha_2 A_0 A_2 + 4\alpha_2 A_2^2 A_2 + 12\alpha_3 A_0^2 A_2 + 3\alpha_3 A_1^2 A_2 + 12\alpha_3 A_0 A_2^2 + 6\alpha_3 A_2^3 + \left(-\omega_0^2 + 4\omega^2\right)A_2 \\
&-\alpha_2\left(\frac{1}{2}A_1^2 + 2A_0A_2\right) - \alpha_3\left(\frac{3}{2}A_1^2\left(A_0 + A_2\right) + 3A_0^2A_2 + \frac{3}{4}A_2^3\right) = 0 \\
&or,\ 3\omega_0^2 A_2 - \frac{1}{2}\alpha_2 A_1^2 = 0 \\
&or,\ A_2 = \frac{1}{6\omega_0^2}\alpha_2 A_1^2 + O\left(A_1^4\right)
\end{aligned} \quad (3.6.19)$$

Substituting the expressions for A_0 and A_2 in Eq. (3.6.16) one obtains

$$\omega^2 = \omega_0^2 + 2\alpha_2 A_0 + \alpha_2 A_2 + 3\alpha_3 A_0^2 + \frac{3}{4}\alpha_3 A_1^2 + 3\alpha_3 A_0 A_2 + \frac{3}{2}\alpha_3 A_2^2$$

$$\text{Or, } \omega^2 = \omega_0^2 + 2\alpha_2\left(-\frac{1}{2}\frac{\alpha_2}{\omega_0^2}A_1^2\right) + \alpha_2\left(\frac{1}{6}\frac{\alpha_2}{\omega_0^2}A_1^2\right) + \frac{3}{4}\alpha_3 A_1^2 + O\left(A_1^4\right)$$

$$\text{Or, } \omega^2 = \omega_0^2 + \left(-\frac{5}{6}\frac{\alpha_2}{\alpha_1 = \omega_0^2} + \frac{3}{4}\alpha_3\right)A_1^2 + O\left(A_1^4\right)$$

$$\text{Or, } \omega^2 = \omega_0^2 + \left(\frac{18\alpha_3\omega_0^2 - 20\alpha_2}{24\omega_0^2}\right)A_1^2 + O\left(A_1^4\right) \tag{3.6.20}$$

$$\text{Or, } \omega = \omega_0\left[1 + \left(\frac{18\alpha_3\omega_0^2 - 20\alpha_2}{24\omega_0^4}\right)A_1^2\right]^{1/2}$$

$$\text{Or, } \omega = \omega_0\left[1 + \left(\frac{9\alpha_3\omega_0^2 - 10\alpha_2}{24\omega_0^4}\right)A_1^2\right]$$

By substituting $A_1 = \varepsilon a$, this expression is same as that obtained by applying method of multiple scales and Lindstedt Poincare' technique.

Now substituting the expression of A_0 and A_2 in Eq. (3.6.14) one obtains

$$x = A_1\cos\phi - \frac{1}{2}\frac{\alpha_2}{\omega_0^2}A_1^2\left[1 - \frac{1}{3}\cos 2\phi\right] \tag{3.6.21}$$

Though the harmonic balance method is the most commonly used method for analyzing the nonlinear structural vibration, it has several disadvantages. First the formulation is very tedious not only for a multi degree of freedom nonlinear system but also with higher harmonic terms taken into account. Second, to obtain a consistent solution, one needs to know *a priori* which harmonic terms to be included in the analysis. Third a separate analysis is required to study the stability of the system.

Exercise problems:

Determine the frequency response of a 3-degree of freedom system given by the following equation

$$\begin{aligned}
&\ddot{x}_1 + \omega_0^2 x_1 + c_{12}\left(x_1 - x_2\right) + \alpha\left(x_1 - x_2\right)^3 + 2\varsigma\left(\dot{x}_1 - \dot{x}_2\right) = P\cos\Omega t \\
&\ddot{x}_2 + c_{21}\left(x_2 - x_1\right) + c_{23}\left(x_2 - x_3\right) - \alpha\left(x_1 - x_2\right)^3 + 2\varsigma\left(\dot{x}_2 - \dot{x}_1\right) = 0 \\
&\ddot{x}_3 + c_{32}\left(x_3 - x_2\right) = 0
\end{aligned} \tag{3.6.22}$$

These equations represent that of a three mass system where the first mass is connected to a rigid support by a spring and subjected to a harmonic force $P\cos\Omega t$. The first and second mass are connected by a spring with cubic nonlinearity and a linear damper. The second and third mass is connected by a linear spring. (Refer Stupnicka (1990, volume 2, page 152-162)).

Materials for further reading

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- J. J. Wu. A generalized harmonic balance method for forced nonlinear oscillations: the subharmonic cases. Journal of Sound and Vibration, 159(3), 503-525,19

Module 3 Lecture 7

METHOD OF AVERAGING:

This is one of the techniques for variation of parameters and there are many techniques such as van der Pol's technique, Krylov-Bogoliubov, the generalized method of averaging, the Krylov-Bogoliubov-Mitropolsky technique, etc.(Nayfeh 1973). A detailed study of this method can be found in the book of Nayfeh (1973). Few of these techniques are described here with examples.

Vander Pol's Technique

Consider the equation

$$\frac{d^2u}{dt^2}+\omega_0^2u+\varepsilon\left(u^2-1\right)\frac{du}{dt}=\varepsilon f\Omega\cos\Omega t \tag{3.7.1}$$

Assuming ε to be small and the frequency of external excitation nearly equal to the natural frequency ω_0 which can be written by using a detuning parameter σ as follows

$$\Omega=\omega_0+\varepsilon\sigma \tag{3.7.2}$$

Initially the solution of Eq. (3.7.1) can be assumed to that of the equation considering ε equal to zero but with variable coefficient as given below.

$$u(t)=a_1(t)\cos\Omega t+a_2(t)\sin\Omega t \tag{3.7.3}$$

Here $a_1(t)$ and $a_2(t)$ are assumed to be slowly varying function of time. Hence, $\frac{da_i}{dt}=o(\varepsilon)$ and $\frac{d^2a_i}{dt^2}=o(\varepsilon^2)$. Differentiating Eq.(3.7.3) twice one obtains

$$\dot{u}=\left(\dot{a}+a_2\Omega\right)\cos\Omega t+\left(\dot{a}_2-a_1\Omega\right)\sin\Omega t \tag{3.7.4}$$

$$\ddot{u}=\left(-\Omega^2a_1+2\Omega\dot{a}_2+\ddot{a}_1\right)\cos\Omega t+\left(-2\Omega\dot{a}_1+\ddot{a}_2-\Omega^2a_2\right)\sin\Omega t \tag{3.7.5}$$

Substituting Eq. (3.7.3-3.7.5) in Eq. (3.7.1) one obtains

$$\begin{aligned}&\left(-\Omega^2a_1+2\Omega\dot{a}_2+\ddot{a}_1\right)\cos\Omega t+\left(-2\Omega\dot{a}_1+\ddot{a}_2-\Omega^2a_2\right)\sin\Omega t+\omega_0^2\left(a_1(t)\cos\Omega t+a_2(t)\sin\Omega t\right)\\&+\varepsilon\left(\left(a_1(t)\cos\Omega t+a_2(t)\sin\Omega t\right)^2-1\right)\left(\left(\dot{a}+a_2\Omega\right)\cos\Omega t+\left(\dot{a}_2-a_1\Omega\right)\sin\Omega t\right)=\varepsilon f\Omega\cos\Omega t\end{aligned} \tag{3.7.6}$$

Or,

$$\begin{aligned}&\left((-\Omega^2+\omega_0^2)a_1+2\Omega\dot{a}_2+\ddot{a}_1+\right)\cos\Omega t+\left(-2\Omega\dot{a}_1+\ddot{a}_2+(-\Omega^2+\omega_0^2)a_2\right)\sin\Omega t\\&+\varepsilon\Omega\begin{pmatrix}a_1^2a_2\cos^3\Omega t-a_2^2a_1\sin^3\Omega t+\left(a_2^3-2a_1^2a_2\right)\sin^2\Omega t\cos\Omega t-\\\left(a_1^3-2a_1a_2^2\right)\cos^2\Omega t\sin\Omega t-a_2\Omega\cos\Omega t+a_1\Omega\sin\Omega t\end{pmatrix}+hot=\varepsilon f\Omega\cos\Omega t\end{aligned} \tag{3.7.8}$$

Now using,

$\cos^3\Omega t = (\cos 3\Omega t + 3\cos\Omega t)/4$, $\sin^3\Omega t = (3\sin\Omega t - \sin 3\Omega t)/4$,

$\cos^2\Omega t \sin\Omega t = (\sin\Omega t - \sin 3\Omega t)/4$ and $\sin^2\Omega t\cos\Omega t = (\cos\Omega t - \cos 3\Omega t)/4$ (3.7.9)

in Eq. (3.7.8) and keeping in mind that $\ddot{a}_1$ and $\ddot{a}_2$ are $o(\varepsilon^2)$ and then equating the coefficient of $\cos\Omega t$ and $\sin\Omega t$ to zero one obtains the following two equations.

$$2\dot{a}_1 + \left(\frac{\Omega^2 - \omega_0^2}{\Omega}\right)a_2 - \varepsilon a_1\left(1 - \frac{a_1^2 + a_2^2}{4}\right) = 0 \tag{3.7.10}$$

$$2\dot{a}_2 - \left(\frac{\Omega^2 - \omega_0^2}{\Omega}\right)a_1 - \varepsilon a_2\left(1 - \frac{a_1^2 + a_2^2}{4}\right) = \varepsilon f \tag{3.7.11}$$

For steady state, $\dot{a}_1 = \dot{a}_2 = 0$. Using Eq. (3.7.2) in the above equations, one may obtain

$$\frac{\Omega^2 - \omega_0^2}{\Omega} = \frac{(\omega_0 + \varepsilon\sigma)^2 - \omega_0^2}{\Omega} = \frac{\omega_0^2 + 2\varepsilon\omega_0\sigma + \varepsilon^2\sigma^2 - \omega_0^2}{\Omega} \simeq 2\varepsilon\sigma\,. \tag{3.7.12}$$

Taking the equilibrium solution to be a_{10} and a_{20} and writing $\rho_0 = \frac{a_{10}^2 + a_{20}^2}{4}$, Eqs. (3.7.10) and (3.7.11) reduced to the following equations.

$$2\sigma a_{20} - a_{10}(1 - \rho_0) = 0 \tag{3.7.13}$$

$$-2\sigma a_{10} - a_{20}(1 - \rho_0) = f \tag{3.7.14}$$

squaring and adding Eqs. (3.7.13) and (3.7.14) one obtains

$$4\sigma^2\left(a_{10}^2 + a_{20}^2\right) + (1 - \rho_0)^2\left(a_{10}^2 + a_{20}^2\right) = f^2 \tag{3.7.15}$$

$$4\rho_0\left(4\sigma^2 + (1 - \rho_0)^2\right) = f^2 \tag{3.7.16}$$

This is the frequency response equation of the system governed by van der Pol's equation. For forcing function $f = 0$, Eq. (3.7.16) reduces to

$$4\sigma^2 + (1 - \rho_0)^2 = 0 \tag{3.7.17}$$

Krylov–Bogoliubov Technique

Let us consider a general equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \varepsilon f(x, \dot{x}) \tag{3.7.18}$$

According to this method, one may assume the solution of this equation same as the solution of the linear equation by substituting $\varepsilon = 0$, but in this case the constant terms are assumed as function of time. So the solution of this equation can be written as

$$x = a(t)\cos\left[\omega_0 t + \beta(t)\right] \tag{3.7.19}$$

Also it is assumed that

$$\dot{x} = -\omega_0 a(t)\sin\left[\omega_0 t + \beta\left(t\right)\right] \tag{3.7.20}$$

Differentiating (3.7.19) one may write

$$\dot{x} = -\left(\omega_0 + \frac{d\beta}{dt}\right)a(t)\sin\left[\omega_0 t + \beta\left(t\right)\right] + \frac{da}{dt}\cos\left[\omega_0 t + \beta(t)\right] \tag{3.7.21}$$

Differentiating (3.7.20) one may write

$$\ddot{x} = -\omega_0 \frac{da}{dt}\sin\left[\omega_0 t + \beta\left(t\right)\right] - \omega_0\left(\omega_0 + \frac{d\beta}{dt}\right)a(t)\cos\left[\omega_0 t + \beta\left(t\right)\right] \tag{3.7.22}$$

Substituting $\phi = \omega_0 t + \beta(t)$, compairing Eq. (3.7.20) and Eq. (3.7.21)

$$-\frac{d\beta}{dt}a\sin\phi + \frac{da}{dt}\cos\phi = 0 \tag{3.7.23}$$

Also, from Eq. (3.7.18) and Eq. (3.7.22)

$$\ddot{x} + \omega_0^2 x = -\omega_0 \frac{da}{dt}\sin\phi + \left(-\omega_0^2 - \omega_0\frac{d\beta}{dt}\right)a\cos\phi + \omega_0^2 a\cos\phi = \varepsilon f\left(a\cos\phi, -\omega_0 a\sin\phi\right) \tag{3.7.24}$$

Or, $-\omega_0 \frac{da}{dt}\sin\phi - \omega_0 a\frac{d\beta}{dt}\cos\phi = \varepsilon f\left(a\cos\phi, -\omega_0 a\sin\phi\right)$ (3.7.25)

From Eq. (3.7.23) and Eq. (3.7.25) one may write Carrying out the operation $\omega_0\cos\phi\times$ Eq. (3.7.23) - $\sin\phi\times$Eq. (3.7.25) yields

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega_0}\sin\phi f\left(a\cos\phi, -\omega_0 a\sin\phi\right) \tag{3.7.26}$$

Carrying out the operation $\omega_0\sin\phi\times$ Eq. (3.7.23) + $\cos\phi\times$Eq. (3.7.25) yields

$$\frac{d\beta}{dt} = -\frac{\varepsilon}{a\omega_0}\cos\phi f\left(a\cos\phi, -\omega_0 a\sin\phi\right) \tag{3.7.27}$$

For small ε, $\frac{da}{dt}$ and $\frac{d\beta}{dt}$ are small; hence a and β vary much more slowly with time t than $\phi = \omega_0 t + \beta$. In other words, a and β hardly change during the period of oscillation $T = \frac{2\pi}{\omega_0}$ of $\cos\phi$ and $\sin\phi$. Hence, one may average the equations (3.7.26) and (3.7.27) over the period T. Considering a, β, $\frac{da}{dt}$ and $\frac{d\beta}{dt}$ to be constant during this averaging one obtains the following equations.

$$\begin{aligned}\frac{da}{dt} &= -\frac{\varepsilon}{\omega_0 T}\int_0^T \sin\phi f\left(a\cos\phi, -\omega_0 a\sin\phi\right)dt \\ &= -\frac{\varepsilon\omega_0}{\omega_0 2\pi}\int_0^{2\pi}\sin\phi f\left(a\cos\phi, -\omega_0 a\sin\phi\right)\frac{d\phi}{\omega_0}\end{aligned} \tag{3.7.28}$$

Similarly

$$\frac{d\beta}{dt} = -\frac{\varepsilon}{a\omega_0 T}\int_0^T \cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)dt$$
$$= -\frac{\varepsilon\omega_0}{a\omega_0 2\pi}\int_0^{2\pi} \cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)\frac{d\phi}{\omega_0} \quad (3.7.29)$$

Hence, from equation (3.7.28) and (3.7.29) one can write the averaged equations as follows.

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega_0}\int_0^{2\pi} \sin\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)d\phi \quad (3.7.30)$$

$$a\frac{d\beta}{dt} = -\frac{\varepsilon}{2\pi\omega_0}\int_0^{2\pi} \cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)d\phi \quad (3.7.31)$$

It may be noted that the above two equations are obtained by multiplying $-\frac{\varepsilon}{2\pi\omega_0}\sin\phi$ and $-\frac{\varepsilon}{2\pi\omega_0}\cos\phi$ to the forcing function (f) and integrating it from 0 to 2π. But in the forcing function one should substitute $x = a\cos\phi$ and $\dot{x} = -\omega_0 a\sin\phi$.

Example 3.7.1:

Let us apply Krylov–Bogoliubov Technique to Duffing equation with cubic nonlinearity.

Solution:

Here the equation is given by

$$\ddot{x} + \omega_0^2 x = \varepsilon f\left(x, \dot{x}\right) = -\varepsilon x^3 \quad (3.7.32)$$

Hence, $\varepsilon f\left(x,\dot{x}\right) = -\varepsilon\left(a\cos\phi\right)^3 = -\varepsilon a^3\cos^3\phi = -\varepsilon a^3\left(\frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi\right)$ (3.7.33)

Using equation (3.7.30) and (3.7.31) one can write

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega_0}\int_0^{2\pi} \sin\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)d\phi$$
$$= \frac{\varepsilon a^3}{2\pi\omega_0}\int_0^{2\pi} \sin\phi\left(\frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi\right)d\phi = 0 \quad (3.7.34)$$

$$a\frac{d\beta}{dt} = \frac{\varepsilon a^3}{2\pi\omega_0}\int_0^{2\pi} \cos\phi\left(\frac{3}{4}\cos\phi + \frac{1}{4}\cos 3\phi\right)d\phi = \frac{3\pi\varepsilon a^3}{8\pi\omega_0} = \varepsilon\frac{3a^3}{8\omega_0} \quad (3.7.35)$$

One may use the following Matlab code to find the integration

```
syms p
int(cos(p)*(3*cos(p)+cos(3*p)),0,2*pi)

(ans =  3*pi)
```

Or instead of writing $\cos^3\phi$ in terms of $\cos\phi$ and $\cos 3\phi$, one may directly integrate $\cos\phi * \cos^3\phi$ symbolically using Matlab as follows.

```
syms p
int(cos(p)*(cos(p))^3,0,2*pi)

ans = (3*pi)/4
```

From Eq. (3.7.34) and (3.7.35) one may obtain

$a = \text{constant}$ and $\beta = \varepsilon \dfrac{3a^2}{8\omega_0} t + \beta_0$

Hence, using equation (3.7.19), the solution of this equation can be given by

$$x = a(t)\cos\left[\omega_0 t + \beta(t)\right] = a\cos\left(\omega_0 t + \varepsilon\frac{3a^2}{8\omega_0}t + \beta_0\right) = a\cos\left(\left(\omega_0 + \varepsilon\frac{3a^2}{8\omega_0}\right)t + \beta_0\right) \quad (3.7.36)$$

So the frequency of oscillation of the system is $\omega_0 + \varepsilon\dfrac{3a^3}{8\omega_0}$. But it may be noted that this frequency expression is not correct. Hence one has to use better approximation to obtain the accurate solution. In the next lecture generalized averaging and the KBM method will be described which give better results than the KB method.

Exercise Problems

1. Use Krylov–Bogoliubov Technique to find the response of a single degree of freedom system with (i) viscous damping, (ii) Coulomb damping, (iii) Negative damping, (iv) quadratic nonlinear damping and (v) hysteretic damping.

References

- A. H. Nayfeh, Perturbation Methods, John Wiley & Sons, New York, 1973.
- A. H. Nayfeh and D.T. Mook, Nonlinear oscillations, John Wiley & Sons, New York, 1979.

Module 3 Lecture 8

Generalized method of averaging

In this case instead of writing the reduced equations in terms of a and β, it is written in terms of a and ϕ as follows. This lecture is adopted from the book by Nayfeh (1973).

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega_0}\sin\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right) \tag{3.8.1}$$

As $\phi = \omega_0 t + \beta$ and $\frac{d\beta}{dt} = -\frac{\varepsilon}{a\omega_0}\cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)$ one may write

$$\frac{d\phi}{dt} = \omega_0 - \frac{\varepsilon}{a\omega_0}\cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right) \tag{3.8.2}$$

Unlike in the case of K-B method, instead of integrating Eq. (3.8.1) and (3.8.2) to get a and ϕ, a near-identity transform has been used in this method as follows.

$$a = \bar{a} + \varepsilon a_1\left(\bar{a}, \bar{\phi}\right) + \varepsilon^2 a_2\left(\bar{a}, \bar{\phi}\right) + \cdots \tag{3.8.3}$$

$$\phi = \bar{\phi} + \varepsilon \phi_1\left(\bar{a}, \bar{\phi}\right) + \varepsilon^2 \phi_2\left(\bar{a}, \bar{\phi}\right) + \cdots \tag{3.8.4}$$

Substituting Eq. (3.8.3) in Eq. (3.8.1) and Eq. (3.8.4) in Eq. (3.8.2), it can be written as

$$\frac{d\bar{a}}{dt} = \varepsilon A_1\left(\bar{a}\right) + \varepsilon^2 A_2\left(\bar{a}\right) + \cdots \tag{3.8.5}$$

$$\frac{d\bar{\phi}}{dt} = \omega_0 + \varepsilon B_1\left(\bar{a}\right) + \varepsilon^2 B_2\left(\bar{a}\right) + \cdots \tag{3.8.6}$$

with A_i and B_i independent of $\bar{\phi}$. Substituting Eqs. (3.8.3)-(3.8.6) in Eqs. (3.8.1) and (3.8.2), expanding and equating coefficients of like power of ε, one obtains equations in the following forms.

$$\omega_0 \frac{\partial a_n}{\partial \bar{\phi}} + A_n = F_n\left(\bar{a}, \bar{\phi}\right) \tag{3.8.7}$$

$$\omega_0 \frac{\partial \phi_n}{\partial \bar{\phi}} + B_n = G_n\left(\bar{a}, \bar{\phi}\right) \tag{3.8.8}$$

Here, $F_n\left(\bar{a}, \bar{\phi}\right)$ and $G_n\left(\bar{a}, \bar{\phi}\right)$ are known function of lower-order terms which contain short period terms and long-period terms. Denoting short-period and long period terms by superscript s and l, respectively, one may write

$$A_n = F_n^l, \qquad B_n = G_n^l \tag{3.8.9}$$

So, $$\omega_0 \frac{\partial a_n}{\partial \bar{\phi}} = F_n^s, \qquad \omega_0 \frac{\partial \phi_n}{\partial \bar{\phi}} = G_n^s \tag{3.8.10}$$

These equations can be solved to obtain the frequency response curve of the system.

Example 3.8.1

Let us consider the example of van der Pol oscillator in which one may write

$$\ddot{u}+u=f\left(u,\dot{u}\right)=\left(1-u^{2}\right)\dot{u}\ . \tag{3.8.11}$$

Using generalized method of averaging we have to find the frequency response relation.

Solution:

Here, $\omega_0=1$. So, Eq. (3.8.1) and (3.8.2) can be written as

$$\begin{aligned}\frac{da}{dt}&=\frac{1}{8}\varepsilon\left[a\left(4-a^{2}\right)-4a\cos 2\phi+a^{3}\cos 4\phi\right]\\ \frac{d\phi}{dt}&=1+\frac{1}{8}\varepsilon\left[2\left(2-a^{2}\right)\sin 2\phi-a^{2}\sin 4\phi\right]\end{aligned} \tag{3.8.12}$$

Substituting Eq. (3.8.3) and Eq. (3.8.4) in Eq. (3.8.12) it can be written as

$$\begin{aligned}\frac{d\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)}{dt}&=\frac{1}{8}\varepsilon\left[\begin{array}{l}\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\left(4-\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)^2\right)\\ -4\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\cos 2\left(\bar{\phi}+\varepsilon\phi_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2\phi_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\\ +\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)^3\cos 4\left(\bar{\phi}+\varepsilon\phi_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2\phi_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\end{array}\right]\\ \frac{d\left(\bar{\phi}+\varepsilon\phi_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2\phi_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)}{dt}&=1+\frac{1}{8}\varepsilon\left[\begin{array}{l}2\left(2-\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)^2\right)\sin 2\left(\bar{\phi}+\varepsilon\phi_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2\phi_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\\ -\left(\bar{a}+\varepsilon a_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2 a_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)^2\sin 4\left(\bar{\phi}+\varepsilon\phi_1\left(\bar{a},\bar{\phi}\right)+\varepsilon^2\phi_2\left(\bar{a},\bar{\phi}\right)+\cdots\right)\end{array}\right]\end{aligned} \tag{3.8.13}$$

Substituting Eq. (3.8.5)-(3.8.8) in (3.8.12) one obtains

Order ε

$$\begin{aligned}\frac{\partial a_1}{\partial\bar{\phi}}+A_1&=\underbrace{\frac{1}{8}\bar{a}\left(4-\bar{a}^2\right)}_{\text{term without}\,\bar{\phi}}-\frac{1}{2}\bar{a}\cos 2\bar{\phi}+\frac{1}{8}\bar{a}^3\cos 4\bar{\phi}\\ \frac{\partial\phi_1}{\partial\bar{\phi}}+B_1&=\frac{1}{4}\left(2-\bar{a}^2\right)\sin 2\bar{\phi}-\frac{1}{8}\bar{a}^2\sin 4\bar{\phi}\end{aligned} \tag{3.8.14}$$

Order ε^2

$$\frac{\partial a_2}{\partial \bar{\phi}} + A_2 = -\frac{\partial a_1}{\partial \bar{a}} A_1 - \frac{\partial a_1}{\partial \bar{\phi}} B_1 + \frac{1}{8} a_1 \left[4 - 3\bar{a}^2 - 4\cos 2\bar{\phi} + 3\bar{a}^2 \cos 4\bar{\phi} \right]$$
$$+ \frac{1}{2} \bar{a} \phi_1 \left[2\sin 2\bar{\phi} - \bar{a}^2 \sin 4\bar{\phi} \right]$$
$$\frac{\partial \phi_2}{\partial \bar{\phi}} + B_2 = -\frac{\partial \phi_1}{\partial \bar{a}} A_1 - \frac{\partial \phi_1}{\partial \bar{\phi}} B_1 - \frac{1}{4} \bar{a} a_1 \left(2\sin 2\bar{\phi} + \sin 4\bar{\phi} \right)$$
$$+ \frac{1}{2} \phi_1 \left[\left(2 - \bar{a}^2 \right) \cos 2\bar{\phi} - \bar{a}^2 \cos 4\bar{\phi} \right] \quad (3.8.15)$$

From Eq. (3.8.13) taking terms without $\bar{\phi}$, one may write

$$A_1 = \frac{1}{8} \bar{a} \left(4 - a^{-2} \right), \quad B_1 = 0 \quad (3.8.16)$$

So $$\frac{\partial a_1}{\partial \bar{\phi}} = -\frac{1}{2} \bar{a} \cos 2\bar{\phi} + \frac{1}{8} \bar{a}^3 \cos 4\bar{\phi}, \quad \frac{\partial \phi_1}{\partial \bar{\phi}} = \frac{1}{4} \left(2 - \bar{a}^2 \right) \sin 2\bar{\phi} - \frac{1}{8} \bar{a}^2 \sin 4\bar{\phi} \quad (3.8.17)$$

Solving Eq. (3.8.17) a_1 and ϕ_1 can be written as follows.

$$a_1 = -\frac{1}{4} \bar{a} \sin 2\bar{\phi} + \frac{1}{32} \bar{a}^3 \sin 4\bar{\phi}$$
$$\phi_1 = -\frac{1}{8} \left(2 - \bar{a}^2 \right) \cos 2\bar{\phi} + \frac{1}{32} \bar{a}^2 \cos 4\bar{\phi} \quad (3.8.18)$$

Now substituting Eq. (3.8.16) in (3.8.14) and (3.8.15) one obtains the following equations.

$$\frac{\partial a_2}{\partial \bar{\phi}} + A_2 = short - period \ \ terms$$
$$\frac{\partial \phi_2}{\partial \bar{\phi}} + B_2 = -\frac{1}{8} + \frac{3}{16} \bar{a}^2 - \frac{11}{256} \bar{a}^4 + short - period \ \ terms \quad (3.8.19)$$

So, $$A_2 = 0, \qquad B_2 = -\frac{1}{8} + \frac{3}{16} \bar{a}^2 - \frac{11}{256} \bar{a}^4 \quad (3.8.20)$$

The first order solution of the system can be given by

$$u = a \cos \phi \quad (3.8.21)$$

Where,

$$a = \bar{a} + \varepsilon a_1 = \bar{a} - \frac{1}{4} \varepsilon \bar{a} \left[\sin 2\bar{\phi} - \frac{1}{8} \bar{a}^2 \sin 4\bar{\phi} \right] + O\left(\varepsilon^2 \right)$$
$$\phi = \bar{\phi} + \varepsilon \phi_1 = \bar{\phi} - \frac{1}{8} \varepsilon \left[\left(2 - \bar{a}^2 \right) \cos 2\bar{\phi} - \frac{1}{4} \bar{a}^2 \cos 4\bar{\phi} \right] + O\left(\varepsilon^2 \right) \quad (3.8.22)$$

$$\frac{d\bar{a}}{dt} = \varepsilon A_1 + \varepsilon^2 A_2 = \frac{1}{8} \varepsilon \bar{a} \left(4 - \bar{a}^2 \right) + O\left(\varepsilon^3 \right)$$
$$\frac{d\bar{\phi}}{dt} = \varepsilon B_1 + \varepsilon^2 B_2 = 1 - \frac{1}{8} \varepsilon^2 \left[1 - \frac{3}{2} \bar{a}^2 + \frac{11}{32} \bar{a}^4 \right] + O\left(\varepsilon^3 \right) \quad (3.8.23)$$

Krylov–Bogoliubov-Mitropolski Technique

In this case the solution is assumed as an asymptotic expansion of the form

$$u = a\cos\theta + \sum_{n=1}^{N} \varepsilon^n u_n(a,\theta) + O\left(\varepsilon^{n+1}\right) \tag{3.8.24}$$

Also one may consider the following equations

$$\frac{da}{dt} = \sum_{n=1}^{N} \varepsilon^n A_n(a) + O(\varepsilon^{N+1}) \tag{3.8.25}$$

$$\frac{d\theta}{dt} = \omega_0 + \sum_{n=1}^{N} \varepsilon^n \theta_n\ (a) + O(\varepsilon^{N+1}) \tag{3.8.26}$$

$$\frac{d}{dt} = \frac{da}{dt}\frac{\partial}{\partial t} + \frac{d\theta}{dt}\frac{\partial}{\partial\theta} \tag{3.8.27}$$

$$\frac{d^2}{dt^2} = \left(\frac{da}{dt}\right)^2 \frac{\partial^2}{\partial t^2} + \frac{d^2a}{dt^2}\frac{\partial}{\partial a} + 2\frac{da}{dt}\frac{d\theta}{dt}\frac{\partial^2}{\partial a\partial\theta} + \left(\frac{d\theta}{dt}\right)^2 \frac{\partial^2}{\partial\theta^2} + \frac{d^2\theta}{dt^2}\frac{\partial}{\partial\theta} \tag{3.8.28}$$

$$\frac{d^2a}{dt^2} = \frac{d}{dt}\left(\frac{da}{dt}\right) = \frac{da}{dt}\frac{d}{da}\left(\frac{da}{dt}\right) = \frac{da}{dt}\sum_{n=1}^{N} \varepsilon^n \frac{dA_n}{da} = \varepsilon^2 A_1 \frac{dA_1}{da} + O(\varepsilon^3) \tag{3.8.29}$$

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{da}{dt}\frac{d}{da}\left(\frac{d\theta}{dt}\right) = \frac{da}{dt}\sum_{n=1}^{N} \varepsilon^n \frac{d\theta_n}{da} = \varepsilon^2 A_1 \frac{d\theta_1}{da} + O(\varepsilon^3) \tag{3.8.30}$$

Example 3.8.2:

Apply KBM method to the Duffing equation.

Solution:

The Duffing equation is given by

$$\ddot{u} + \omega_0^2 u = -\varepsilon u^3 \tag{3.8.31}$$

Using Eq. (3.8.24) one may write

$$u = a\cos\theta + \varepsilon u_1(a,\theta) + \varepsilon^2 u_2(a,\theta) + \mathrm{O}\left(\varepsilon^3\right)\ . \tag{3.8.32}$$

Substituting Eq. (3.8.32) in Eq. (3.8.31)and using Eq. (3.8.25)- Eq. (3.8.31) the three terms of Eq.(3.8.31) can be written as follows.

$$
\begin{aligned}
\frac{d^2u}{dt^2} &= \left(\frac{da}{dt}\right)^2 \frac{\partial^2\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)}{\partial t^2}+\frac{d^2a}{dt^2}\frac{\partial\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)}{\partial a} \\
&+2\frac{da}{dt}\frac{d\theta}{dt}\frac{\partial^2\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)}{\partial a\partial\theta}+\left(\frac{d\theta}{dt}\right)^2\frac{\partial^2\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)}{\partial\theta^2} \\
&+\frac{d^2\theta}{dt^2}\frac{\partial\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)}{\partial\theta} \\
\omega_0^2 u &= \omega_0^2\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)+O(\varepsilon^3) \\
-\varepsilon u^3 &= -\varepsilon\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta\right)+O(\varepsilon^3)
\end{aligned}
\tag{3.8.33}
$$

Or,

$$
\begin{aligned}
&\left(\frac{da}{dt}\right)^2\left(-\left(\frac{da}{dt}\frac{d\theta}{dt}\sin\theta+a\cos\theta\left(\frac{d\theta}{dt}\right)^2+a\sin\theta\frac{d^2\theta}{dt^2}\right)+\frac{d^2a}{dt^2}\cos\theta-\frac{da}{dt}\frac{d\theta}{dt}\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2}\right) \\
&+\left(\frac{d^2a}{dt^2}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)+2\left(\frac{da}{dt}\right)\left(\frac{d\theta}{dt}\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right) \\
&+\left(\frac{d\theta}{dt}\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)+\left(\frac{d^2\theta}{dt^2}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)+ \\
&\omega_0^2\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)= \\
&-\varepsilon\left(a\cos\theta+\varepsilon u_1\left(a,\theta\right)+\varepsilon^2 u_2\left(a,\theta\right)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta\right)+\mathrm{O}(\varepsilon^3)
\end{aligned}
\tag{3.8.34}
$$

Substituting Eq. 3.8.25 to Eq. 3.8.30 in the above equations one may obtain the following expression

$$\text{Or, }\left(\varepsilon A_1+\varepsilon^2 A_2\right)^2\left(\begin{array}{l}-\left(\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+a\cos\theta\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2+a\sin\theta\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\right)\\+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\cos\theta-\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2}\end{array}\right)$$
$$+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)+2\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right)$$
$$+\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)+\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)+$$
$$\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)=$$
$$-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)^3=\left(\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta\right)+\mathrm{O}(\varepsilon^3)$$

(3.8.35)

$$\underbrace{\text{Or, }\left(\varepsilon A_1+\varepsilon^2 A_2\right)^2\left(\begin{array}{l}-\left(\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+a\cos\theta\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2+a\sin\theta\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\right)\\+\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\cos\theta-\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial t^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial t^2}\end{array}\right)}_{\varepsilon^2 A_1^2 a\omega_0^2\cos\theta}$$
$$+\underbrace{\left(\varepsilon^2 A_1\frac{dA_1}{da}\right)\left(\cos\theta+\varepsilon\frac{\partial u_1}{\partial a}+\varepsilon^2\frac{\partial u_2}{\partial a}\right)}_{\varepsilon^2 A_1\frac{dA_1}{da}\cos\theta}+\underbrace{2\left(\varepsilon A_1+\varepsilon^2 A_2\right)\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)\left(-\sin\theta+\varepsilon\frac{\partial^2 u_1}{\partial a\partial\theta}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\partial\theta}\right)}_{-2\varepsilon A_1\omega_0\sin\theta-2\varepsilon^2 A_1\theta_1\sin\theta+2\varepsilon^2\omega_0 A_2\sin\theta+2\varepsilon^2 A_1\omega_0\frac{\partial^2 u_1}{\partial a\partial\theta}}$$
$$+\underbrace{\left(\omega_0+\varepsilon\theta_1+\varepsilon^2\theta_2\right)^2\left(-a\cos\theta+\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}\right)}_{-a\omega_0^2\cos\theta+\omega_0^2\varepsilon\frac{\partial^2 u_1}{\partial\theta^2}+\omega_0^2\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2}-\varepsilon^2\theta_1^2 a\cos\theta-2\omega_0\varepsilon\theta_1 a\cos\theta+2\omega_0\varepsilon^2\theta_1\frac{\partial^2 u_1}{\partial\theta^2}-2\omega_0\varepsilon^2\theta_2 a\cos\theta}$$
$$+\underbrace{\left(\varepsilon^2 A_1\frac{d\theta_1}{da}\right)\left(-a\sin\theta+\varepsilon\frac{\partial u_1}{\partial\theta}+\varepsilon^2\frac{\partial u_2}{\partial\theta}\right)}_{-a\varepsilon^2 A_1\frac{d\theta_1}{da}\sin\theta}+\omega_0^2\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)$$
$$=\underbrace{-\varepsilon\left(a\cos\theta+\varepsilon u_1(a,\theta)+\varepsilon^2 u_2(a,\theta)\right)^3}_{\varepsilon a^3\cos^3\theta+3\varepsilon^2 u_1 a\cos\theta}$$

(3.8.36)

Or,

$$\varepsilon^2 A_1^2 a\omega_0^2 \cos\theta + \varepsilon^2 A_1 \frac{dA_1}{da}\cos\theta - 2\varepsilon A_1\omega_0 \sin\theta - 2\varepsilon^2 A_1\theta_1 \sin\theta + 2\varepsilon^2\omega_0 A_2 \sin\theta$$
$$+2\varepsilon^2 A_1\omega_0 \frac{\partial^2 u_1}{\partial a\partial\theta} - a\omega_0^2\cos\theta + \omega_0^2\varepsilon\frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2\varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2} - \varepsilon^2\theta_1^2 a\cos\theta - 2\omega_0\varepsilon\theta_1 a\cos\theta \tag{3.8.37}$$
$$+2\omega_0\theta_1\varepsilon^2\frac{\partial^2 u_1}{\partial\theta^2} - 2\omega_0\varepsilon^2\theta_2 a\cos\theta - a\varepsilon^2 A_1\frac{d\theta_1}{da}\sin\theta + \omega_0^2\left(a\cos\theta + \varepsilon u_1 + \varepsilon^2 u_2\right)$$
$$= -\left(\varepsilon a^3\cos^3\theta + 3\varepsilon^2 u_1 a\cos\theta\right)$$

Or,

$$a\omega_0^2\cos\theta - a\omega_0^2\cos\theta + \varepsilon\left(\omega_0^2\frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2 u_1 - 2\omega_0\varepsilon\theta_1 a\cos\theta - 2\varepsilon A_1\omega_0\sin\theta + a^3\cos^3\theta\right) \tag{3.8.38}$$
$$+\varepsilon^2\left(\begin{matrix}\omega_0^2\frac{\partial^2 u_2}{\partial\theta^2} + \omega_0^2 u_2 + 2\omega_0\theta_1\frac{\partial^2 u_1}{\partial\theta^2} + A_1^2 a\omega_0^2\cos\theta + A_1\frac{dA_1}{da}\cos\theta - 2\omega_0\theta_2 a\cos\theta - \theta_1^2 a\cos\theta \\ -2A_1\theta_1\sin\theta + 2\omega_0 A_2\sin\theta - aA_1\frac{d\theta_1}{da}\sin\theta + 2\omega_0 A_1\frac{\partial^2 u_1}{\partial a\partial\theta} + 3u_1 a^2\cos^2\theta\end{matrix}\right) = 0$$

Now collecting the terms with different order of ε, one obtains the following equations.

$$\omega_0^2\frac{\partial^2 u_1}{\partial\theta^2} + \omega_0^2 u_1 = 2\omega_0\theta_1 a\cos\theta + 2\omega_0 A_1\sin\theta - a^3\cos^3\theta \tag{3.8.39}$$

$$\omega_0^2\frac{\partial^2 u_2}{\partial\theta^2} + \omega_0^2 u_2 = \left[(2\omega_0\theta_2 + \theta_1^2)a - A_1\frac{dA_1}{da}\right]\cos\theta + \left[2(\omega_0 A_2 + A_1\theta_1) + aA_1\frac{\partial\theta_1}{\partial a}\right]\sin\theta$$
$$-3u_1 a^2\cos^2\theta - 2\omega_0\theta_1\frac{d^2 u_1}{d\theta^2} - 2\omega_0 A_1\frac{d^2 u_1}{\partial a\partial\theta} \tag{3.8.40}$$

To eliminate Secular term

$$A_1 = 0, \quad \theta_1 = \frac{3a^2}{8\omega_0} \tag{3.8.41}$$

$$u_1 = \frac{a^3}{32\omega_0^2}\cos 3\theta \tag{3.8.42}$$

$$\omega_0^2\frac{\partial^2 u_2}{\partial\theta^2} + \omega_0^2 u_2 = \left(2\omega_0\theta_2 + \frac{15a^4}{128\omega_0^2}\right)a\cos\theta + 2\omega_0 A_2\sin\theta + \frac{a^5}{128\omega_0^2}(21\cos 3\theta - 3\cos 5\theta) \tag{3.8.43}$$

$$A_2 = 0, \quad \theta_2 = -\frac{15a^4}{256\omega_0^3} \tag{3.8.44}$$

$$u_2 = -\frac{\mathrm{a}^5}{1024\omega_0^4}(21\cos 3\theta - \cos 5\theta) \tag{3.8.45}$$

$$u = a\cos\theta + \frac{\varepsilon a^3}{32\omega_0^2}\cos 3\theta - \frac{\varepsilon^2 a^5}{1024\omega_0^2}(21\cos 3\theta - \cos 5\theta) + \mathrm{O}(\varepsilon^3) \tag{3.8.46}$$

$$\frac{d\mathrm{a}}{d\mathrm{t}} = 0, \text{ or } \quad a = a_0 = \text{constant}$$

$$\frac{d\theta}{dt} = \omega_0 + \frac{3\varepsilon a^2}{8\omega_0^2} - \frac{15\varepsilon^2 a^4}{256\omega_0^3} \tag{3.8.47}$$

$$\theta = \omega_0\left[1 + \frac{3\varepsilon \mathrm{a}^2}{8\omega_0^2} - \frac{15\varepsilon^2 \mathrm{a}^4}{256\omega_0^4}\right]t + \theta_0 + \mathrm{O}(\varepsilon^3) \tag{3.8.48}$$

The van der Pol's Oscillator

$$\ddot{u} + u = \varepsilon(1 - u^2)\dot{u} \tag{3.8.49}$$

$$\frac{\partial^2 u_1}{\partial\theta^2} + u_1 = 2\theta_1 a\cos\theta + 2A_1\sin\theta - a\left(1 - \frac{1}{4}a^2\right)\sin\theta - \frac{1}{4}a^3\sin 3\theta \tag{3.8.50}$$

$$\frac{\partial^2 u_2}{\partial\theta^2} + u_2 = \underbrace{\left[(2\theta_2 + \theta_1^2)a - A_1\frac{dA_1}{da}\right]}_{\text{secular term}}\cos\theta + \underbrace{\left[2(A_2 + A_1\theta_1) + aA_1\frac{d\theta_1}{da}\right]}_{\text{secular term}}\sin\theta$$
$$-2\theta_1\frac{\partial^2 u_1}{\partial\theta^2} - 2A_1\frac{\partial^2 u_1}{\partial a\partial\theta} + \left(1 - \frac{a^2}{2}(1 + \cos 2\theta)\right)\left(A_1\cos\theta - a\theta_1\sin\theta + \frac{\partial u_1}{\partial\theta}\right) + u_1 a^2\sin 2\theta \tag{3.8.51}$$

Elimination of secular terms from the right-hand side

$$\theta_1 = 0, \quad A_1 = \frac{1}{2}a\left(1 - \frac{1}{4}a^2\right) \tag{3.8.52}$$

$$u_1 = -\frac{a^3}{32}\sin 3\theta \tag{3.8.53}$$

$$\frac{\partial^2 u_2}{\partial\theta^2} + u_2 = \left[2a\theta_2 - A_1\frac{dA_1}{da} + \left(1 - \frac{3}{4}a^2\right)A_1 + \frac{a^3}{128}\right]\cos\theta$$
$$+ 2A_2\sin\theta + \frac{a^3(a^2 + 8)}{128}\cos 3\theta + \frac{5a^5}{128}\cos 5\theta \tag{3.8.54}$$

$$A_2 = 0, \qquad \theta_2 = \frac{A_1}{2a}\left(\frac{dA_1}{da} - 1 + \frac{3}{4}a^2\right) - \frac{a^4}{256} \tag{3.8.55}$$

$$u_2 = -\frac{5a^5}{3072}\cos 5\theta - \frac{a^3(a^2+8)}{1024}\cos 3\theta \qquad (3.8.56)$$

$$u = a\cos\theta - \frac{\varepsilon a^3}{32}\sin 3\theta - \frac{\varepsilon^2 a^5}{1024}\left[\frac{5}{3}a^2\cos 5\theta + (a^2+8)\cos 3\theta\right] + O(\varepsilon^3) \qquad (3.8.57)$$

$$\frac{da}{dt} = \frac{\varepsilon a}{2}\left(1 - \frac{1}{4}a^2\right), \quad a^2 = \frac{4}{1+\left(\frac{4}{a^2}-1\right)e^{-\varepsilon t}} \qquad (3.8.58)$$

$$\frac{d\theta}{dt} = 1 + \varepsilon^2\left[\frac{A_1}{2a}\left(\frac{dA_1}{da} - 1 + \frac{3}{4}a^2\right) - \frac{a^4}{256}\right] \qquad (3.8.59)$$

$$\frac{d\theta}{dt} = 1 - \frac{\varepsilon^2}{16} - \frac{\varepsilon}{8a}\left(1 - \frac{7}{4}\mathrm{a}^2\right)\frac{da}{dt} \qquad (3.8.60)$$

$$\theta = t - \frac{\varepsilon^2}{16}t - \frac{\varepsilon}{8}\ln a + \frac{7\varepsilon}{64}a^2 + \varepsilon_0 \qquad (3.8.61)$$

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Module 3 Lecture 9

METHOD OF NORMAL FORM

In this lecture method of normal form will be used to determine the reduced equation which will be further used to find the response and stability of the nonlinear system. In case of normal form the solution of the linear equation with time varying coefficient is first considered. By substituting this solution with unknown coefficient in the governing equation of motion the normal form solution of the nonlinear equation has been obtained. This method is illustrated below using the nonlinear equation of a parametrically excited cantilever beam with axial load and magnetic field (Pratiher and Dwivedy, 2009).

Example 3.9.1: Find the normal form solution of the following equation.

$$\ddot{q}+q+2\,\varepsilon\zeta\dot{q}+\varepsilon\left(\alpha_1 q^3+\alpha_2 q^2\ddot{q}+\alpha_3\dot{q}^2 q\right)+ \varepsilon\left(\alpha_4\bar{\omega}_1^2\cos\left(\bar{\omega}_1\tau\right)q^2+\alpha_5\,\bar{\omega}_1^2\,\cos\left(\bar{\omega}_2\tau\right)+\alpha_6\cos\left(\bar{\omega}_2\tau\right)q\right)=0 \tag{3.9.1}$$

One may find that the non-dimensional temporal equation (3.9.1) has a linear forced term ($\alpha_5\bar{\omega}_1^2\,cos\,\bar{\omega}_1\tau$), a linear parametric term ($\alpha_6\cos\left(\bar{\omega}_2\tau\right)q$) and a nonlinear parametric excitation term ($\left(\alpha_4\bar{\omega}_1^2\cos\bar{\omega}_1\tau\right)q^2$) along with cubic geometric ($\alpha_1 q^3$) and inertial ($\alpha_2 q^2\ddot{q}+\alpha_3\dot{q}^2 q$) nonlinear terms. Here method of normal form Nayfeh (1993) is used which is described in the following section.

Solution:

To find the approximate solution of equation (3.9.1), one may use the method of normal form. In this method, one may transform the second order temporal equation of motion into a set of first order equations to determine the uniform expansions of the solutions of equation (3.9.1). The general solution of equation (3.9.1) by putting ϵ equal to zero is as follows.

$$q = A\exp(i\tau) + \bar{A}\exp(-i\tau), \tag{3.9.2}$$

Here, A is a complex number and $\bar{A}$ is the complex conjugate of A.
One may write the first time derivative of the q as

$$\dot{q} = i\left(A\exp(i\tau) - \bar{A}\exp(-i\tau)\right) \tag{3.9.3}$$

By replacing $A\exp(i\tau)$, and $\bar{A}\exp(-i\tau)$ in terms of ξ and $\bar{\xi}$, respectively into equations (3.9.2) and (3.9.3), yields the following expression.

$$q = \xi + \bar{\xi}, \quad \text{and} \quad \dot{q} = i\left(\xi - \bar{\xi}\right), \tag{3.9.4}$$

where, ξ is the complex number and $\bar{\xi}$ is the complex conjugate of ξ.

Substituting $z = \exp(i\bar{\omega}_1\tau)$, and $z_1 = \exp(i\bar{\omega}_2\tau)$, respectively into equation (3.9.4), results in the following equation.

$$\dot{\xi} = i\xi - \varepsilon\bar{\mu}\left(\xi - \bar{\xi}\right) + \frac{i}{2}\varepsilon\left[\alpha_1\left(\xi+\bar{\xi}\right)^3 + \alpha_2\left(\xi+\bar{\xi}\right)^2\left\{\left(\xi-\bar{\xi}\right)+2i\dot{\xi}\right\} - \alpha_3\left(\xi-\bar{\xi}\right)^2\left(\xi+\bar{\xi}\right)\right]$$
$$+\frac{i\,\bar{\omega}_1^2}{4}\varepsilon\left[\alpha_4\left(\xi+\bar{\xi}\right)^2\left(z+\bar{z}\right)+\alpha_5\left(z+\bar{z}\right)\right] + \frac{i}{4}\varepsilon\alpha_6\left(\xi+\bar{\xi}\right)\left(z_1+\bar{z}_1\right) \tag{3.9.5}$$

Here, introducing a nearly identify variable η, variable ξ may be written as

$\xi = \eta + \varepsilon h(\eta, \bar{\eta}, z, \bar{z}, z_1, \bar{z}_1) + O(\varepsilon^2)$, and

$$\dot{\xi} = \dot{\eta} + \varepsilon\left(\frac{\partial h}{\partial \eta}\dot{\eta} + \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{\partial h}{\partial z}\dot{z} + \frac{\partial h}{\partial \bar{z}}\dot{\bar{z}} + \frac{\partial h}{\partial z_1}\dot{z}_1 + \frac{\partial h}{\partial \bar{z}_1}\dot{\bar{z}}_1\right) + O(\varepsilon^2) \tag{3.9.6}$$

Substituting equation (3.9.6) into the equation (3.9.5), one may obtain

$$\dot{\eta} = i(\eta + \varepsilon h) - \varepsilon\bar{\mu}(\eta - \bar{\eta}) - \left(\frac{\partial h}{\partial \eta}\dot{\eta} + \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{\partial h}{\partial z}\dot{z} + \frac{\partial h}{\partial \bar{z}}\dot{\bar{z}} + \frac{\partial h}{\partial z_1}\dot{z}_1 + \frac{\partial h}{\partial \bar{z}_1}\dot{\bar{z}}_1\right)$$
$$+\frac{i}{2}\varepsilon\left[\alpha_1(\eta+\bar{\eta})^3 + \alpha_2(\eta+\bar{\eta})^2\left\{(\eta-\bar{\eta})+2i\dot{\eta}\right\} - \alpha_3(\eta-\bar{\eta})^2(\eta+\bar{\eta})\right]$$
$$+\frac{i\,\bar{\omega}_1^2}{4}\varepsilon\left[\alpha_4(\eta+\bar{\eta})^2(z+\bar{z}) + \alpha_4(z+\bar{z})\right] + \frac{i}{4}\varepsilon\alpha_6(\eta+\bar{\eta})(z_1+\bar{z}_1) + O(\varepsilon^2) \tag{3.9.7}$$

As the temporal equation contains cubic nonlinear terms, assuming h to be of third order in η and $\bar{\eta}$ one may write

$$h=\Delta_1\eta+\Delta_2\bar{\eta}+\Delta_3 z+\Delta_4\bar{z}+\Phi_1\eta z_1+\Phi_2\bar{\eta}z_1+\Phi_3\eta\bar{z}_1+\Phi_4\bar{\eta}\bar{z}_1+\Gamma_1\eta^2 z+\Gamma_2\eta\bar{\eta}z$$
$$+\Gamma_3\bar{\eta}^2 z+\Gamma_4\eta^2\bar{z}+\Gamma_5\eta\bar{\eta}\bar{z}+\Gamma_6\bar{\eta}^2\bar{z}+\Lambda_1\eta^3+\Lambda_2\eta^2\bar{\eta}+\Lambda_3\eta\bar{\eta}^2+\Lambda_4\bar{\eta}^3. \qquad (3.9.8)$$

From equation (3.9.7), the first order approximate solution may be written as

$$\dot{\eta}=i\eta, \text{ and } \dot{\bar{\eta}}=-i\bar{\eta} \qquad (3.9.9)$$

Substituting equations (3.9.9) and (3.9.8) into equation (3.9.7), one may get the following expression.

$$\dot{\eta}=i\eta-\varepsilon\bar{\mu}\eta+\varepsilon\left(\bar{\mu}+2\,i\Delta_2\right)\bar{\eta}+i\varepsilon\left(\Delta_4+\frac{1}{4}\alpha_5\bar{\omega}_1^2\right)\bar{z}+i\varepsilon\left(\Delta_3\left(1-\bar{\omega}\right)+\frac{1}{4}\alpha_5\bar{\omega}_1^2\right)z$$
$$+i\varepsilon\left(\frac{1}{2}\left(\alpha_1-\alpha_2-\alpha_3\right)-2\,\mathrm{A}_1\right)\eta^3+i\varepsilon\left(\frac{1}{2}\left(\alpha_1-\alpha_2-\alpha_3\right)+4\,\mathrm{A}_4\right)\bar{\eta}^3$$
$$+i\varepsilon\left(\frac{3}{2}\left(\alpha_1-\alpha_2+\alpha_3\right)+2\,\mathrm{A}_3\right)\eta\bar{\eta}^2+\frac{3}{2}i\varepsilon\left(\alpha_1-\alpha_2+\frac{\alpha_3}{3}\right)\eta^2\bar{\eta}$$
$$+i\varepsilon\left(\frac{1}{4}\alpha_4\bar{\omega}_1^2-\Gamma_1\left(1+\bar{\omega}_1\right)\right)\eta^2 z+i\varepsilon\left(\frac{1}{2}\alpha_4\bar{\omega}_1^2+\Gamma_2\left(1-\bar{\omega}_1\right)\right)\eta\bar{\eta}z$$
$$+\ i\varepsilon\left(\frac{1}{4}\alpha_4\bar{\omega}_1^2+\Gamma_3\left(3-\bar{\omega}_1\right)\right)\bar{\eta}^2 z+i\varepsilon\left(\frac{1}{4}\alpha_4\bar{\omega}_1^2-\Gamma_4\left(1-\bar{\omega}_1\right)\right)\eta^2\bar{z}$$
$$+i\varepsilon\left(\frac{1}{2}\alpha_4\bar{\omega}_1^2+\Gamma_5\left(1+\bar{\omega}_1\right)\right)\eta\bar{\eta}z+i\varepsilon\left(\frac{1}{4}\alpha_4\bar{\omega}_1^2+\Gamma_6\left(3+\bar{\omega}_1\right)\right)\bar{\eta}^2 z$$
$$+i\varepsilon\left(-\bar{\omega}_2\Phi_1+\frac{1}{4}\alpha_6\right)\eta z_1+i\varepsilon\left(\bar{\omega}_2\Phi_3+\frac{1}{4}\alpha_6\right)\eta\bar{z}_1$$
$$+\ i\varepsilon\left(\left\{2-\bar{\omega}_2\right\}\Phi_2+\frac{1}{4}\alpha_6\right)\bar{\eta}z_1+i\varepsilon\left(\left\{2+\bar{\omega}_2\right\}\Phi_4+\frac{1}{4}\alpha_6\right)\bar{\eta}\bar{z}_1\quad+\mathrm{O}\left(\varepsilon^2\right) \qquad (3.9.10)$$

It may be noted that the above equation (3.9.10) does not depend on Δ_1 and Λ_2; hence both are arbitrary. It is observed that the terms containing $\eta^2\bar{\eta}, \eta\bar{\eta}z, \eta^2\bar{z}, z, \bar{\eta}z_1$ have small divisor or secular terms for simple ($\bar{\omega}_1\approx 1$), sub-harmonic ($\bar{\omega}_1\approx 3$), principal parametric ($\bar{\omega}_2\approx 2$), and simultaneous (i.e $\bar{\omega}_1\approx 1$ and $\bar{\omega}_2\approx 2$ or, $\bar{\omega}_1\approx 3$ and $\bar{\omega}_2\approx 2$) resonance conditions. One may choose $\Delta_2,\Delta_4,\Lambda_1,\Lambda_3,\Lambda_4,\Gamma_1,\Gamma_5,$ and Γ_6 to eliminate the nonresonance terms as

$$\Delta_2=-\frac{\bar{\mu}}{2\,i},\quad \Delta_4=-\frac{1}{4}\alpha_5\bar{\omega}^2,\quad \Lambda_1=\frac{1}{4}\left(\alpha_1-\alpha_2-\alpha_3\right),\ \Lambda_3=-\frac{3}{4}\left(\alpha_1-\alpha_2+\alpha_3\right)$$

$$\Lambda_4=-\frac{1}{4}\left(\alpha_1-\alpha_2-\alpha_3\right),\ \Gamma_1=\frac{\frac{1}{4}\alpha_4\bar{\omega}_1^2}{\left(1+\bar{\omega}_1\right)},\ \Gamma_5=-\frac{\frac{1}{2}\alpha_4\bar{\omega}_1^2}{\left(1+\bar{\omega}_1\right)},\ \text{and}\ \Gamma_1=-\frac{\frac{1}{4}\alpha_4\bar{\omega}_1^2}{\left(3+\bar{\omega}_1\right)}. \qquad (3.9.11)$$

In the following sections, the simple resonance case i.e. when the nondimensional frequency of base excitation $\bar{\omega}_1$ is nearly equal to 1 and principal parametric resonance case i.e. when the nondimensional frequency of the axial load $\bar{\omega}_2$ is nearly equal to the 2 are studied. The simultaneous resonance case ($\bar{\omega}_1 \approx 1$ and $\bar{\omega}_2 \approx 2$), and the higher order resonance conditions i.e the sub harmonic ($\bar{\omega}_1 \approx 3$) and the simultaneous resonance conditions $\bar{\omega}_1 \approx 3$ and $\bar{\omega}_2 \approx 2$ have not been studied and left as an exercise problem.

Simple resonance Case ($\bar{\omega}_1 \approx 1$ and $\bar{\omega}_2$ is away from 2)

For this simple resonance case, to express the nearness of $\bar{\omega}_1$ to 1, one introduces the detuning parameter σ as

$$\bar{\omega}_1 = 1 + \varepsilon\sigma, \quad \text{and} \quad \sigma = O(1) \tag{3.9.12}$$

Substituting equation (3.9.12) into equation (3.9.10) yields the following expression.

$$\dot{\eta} = i\eta - \varepsilon\bar{\mu}\eta + \frac{i\varepsilon}{2}(3\alpha_1 - 3\alpha_2 + \alpha_3)\eta^2\bar{\eta} + \frac{i\varepsilon\alpha_4}{2}\bar{\omega}^2\eta\bar{\eta}z + \frac{i\varepsilon\alpha_4}{4}\bar{\omega}^2\eta^2\bar{z} + \frac{i\varepsilon\alpha_5}{4}\bar{\omega}^2 z \tag{3.9.13}$$

Taking $\eta = \frac{1}{2}a\exp(i\beta)$ in equation (3.9.13) and separating the real and imaginary terms, one may find the following expression

$$\dot{a} = -\bar{\mu}a - \bar{\omega}^2\left(\frac{1}{8}\alpha_4 a^2 + \frac{1}{2}\alpha_5\right)\sin\gamma, \tag{3.9.14}$$

$$a\dot{\gamma} = a\sigma - \frac{3}{8}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^3 - \bar{\omega}^2\left(\frac{3}{8}\alpha_4 a^2 + \frac{1}{2}\alpha_5\right)\cos\gamma\ . \tag{3.9.15}$$

From equations (3.9.14)-(3.9.15), one may observe that the trivial response (i.e. $a = 0$) does not exist in this case. One may find the nontrivial response of the system by solving equations (3.9.14) and (3.9.15) simultaneously. For steady state solution, $\dot{a} = 0$ and $\dot{\gamma} = 0$.

To find the stability of the steady state responses, one may perturb the above equations (3.9.14) and (3.9.15) by substituting $a = a_o + a_1$ and $\gamma = \gamma_0 + \gamma_1$ where a_0, γ_0 are the equilibrium points, and then investigating the eigenvalues of the Jacobian matrix (J) which is given by

$$J=\begin{bmatrix} -\zeta+\dfrac{\frac{1}{2}\alpha_4 a_0^2\zeta}{\frac{1}{8}\alpha_2 a_0^2+\frac{1}{2}\alpha_5} & -\dfrac{\left(\frac{1}{8}\alpha_4 a_0^2+\frac{1}{2}\alpha_5\right)\left(\sigma a_0-\frac{3}{8}Ka_0^3\right)}{\frac{3}{8}\alpha_4 a_0^2+\frac{1}{2}\alpha_5} \\ -\dfrac{3}{4}\left(\alpha_1-\alpha_2+\dfrac{\alpha_3}{3}\right)a_0+\dfrac{\sigma a_0-\frac{3}{8}Ka_0^3}{a^2} & \\ -\dfrac{3}{4}\dfrac{\alpha_4\left(\sigma a_0-\frac{3}{8}\left(\alpha_1-\alpha_2+\frac{\alpha_3}{3}\right)a_0^3\right)}{\frac{3}{8}\alpha_4 a_0^2+\frac{1}{2}\alpha_5} & -\dfrac{\left(\frac{3}{8}\alpha_4 a_0^2+\frac{1}{2}\alpha_5\right)\zeta}{\frac{1}{8}\alpha_4 a_0^2+\frac{1}{2}\alpha_5} \end{bmatrix} \tag{3.9.16}$$

Principal parametric resonance Case ($\overline{\omega}_2 \approx 2$)

In this case, one may presents detuning parameter σ to express the nearness of $\overline{\omega}_2$ to 2, as

$$\overline{\omega}_2 = 2+2\varepsilon\sigma, \quad \text{and} \quad \sigma = O(1) \tag{3.9.17}$$

Substituting equation (3.9.17) into equation (3.9.10) yields the following expression.

$$\dot{\eta} = i\eta-\varepsilon\overline{\mu}\,\eta+\frac{i\varepsilon}{2}\left(3\alpha_1-3\alpha_2+\alpha_3\right)\eta^2\,\overline{\eta}+\frac{i\varepsilon\alpha_4}{2}\overline{\eta}\,z_1 \tag{3.9.18}$$

Putting $\eta=\frac{1}{2}a\exp(i\beta)$ in equation (3.9.18) and separating the real and imaginary terms, yield

$$\dot{a} = -\overline{\mu}\,a-\frac{\alpha_6}{4}a\sin\gamma \tag{3.9.19}$$

$$a\dot{\gamma} = 2a\sigma-\frac{6}{8}\left(\alpha_1-\alpha_2+\frac{\alpha_3}{3}\right)a^3-\frac{\alpha_6}{2}a\cos\gamma \tag{3.9.20}$$

By substituting $\dot{a}=0$ and $\dot{\gamma}=0$, one may note from the equation (3.9.19)-(3.9.20) that the system possess both trivial and nontrivial responses. Hence one may obtain the both responses by solving the equations (3.9.19)-(3.9.20) simultaneously.

In this case, to determine the stability of the steady state response system one may convert the polar form of modulations (i.e. equation (3.9.19) and (3.9.20)) into Cartesian form of modulation by letting $p = a\cos\gamma$ and $q = a\sin\gamma$. One may obtain following Cartesian form of modulations as

$$\dot{p} = -\mu p - 2\sigma q - \frac{1}{8}\left(p^2 + q^2\right)\left(\eta p - 6\kappa q\right) + \frac{1}{2}\alpha_4 \frac{pq}{\left(p^2 + q^2\right)^{\frac{1}{2}}} \quad (3.9.21)$$

$$\dot{q} = -\mu q - 2\sigma p - \frac{1}{8}\left(p^2 + q^2\right)\left(\eta q + 6\kappa p\right) - \frac{3}{4}\alpha_4 \frac{pq}{\left(p^2 + q^2\right)^{\frac{1}{2}}} \quad (3.9.22)$$

Hence, to obtain the stability of the steady state fixed-point response (p_0, q_0), one may disturb the equilibrium point (p_0, q_0) by substituting $p = p_0 + p_1$, and $q = q_0 + q_1$, in equations (3.9.21) and (3.9.22) and finding the eigenvalues of the resulting Jacobean matrix (J). One can express the Jacobian matrix as follows

$$J = \begin{bmatrix} -\mu - \frac{3}{8}\eta p_o^2 + \frac{3}{2} p_0 \kappa q_0 - \frac{1}{8}\eta q_0^2 & -2\sigma - \frac{1}{4} q_0 \eta p_0 + \frac{9}{4}\kappa q_0^2 + \frac{3}{4}\kappa p_0^2 \\ + \frac{\alpha_4 p_0^2 q_0}{2\sqrt{p_0^2 + q_0^2}} - \frac{\alpha_4 p_0^2 q_0}{2\left(p_0^2 + q_0^2\right)^{\frac{3}{2}}} & + \frac{\alpha_4 p_0^2 q_0}{2\sqrt{p_0^2 + q_0^2}} - \frac{\alpha_4 p_0^2 q_0}{2\left(p_0^2 + q_0^2\right)^{\frac{3}{2}}} \\ -\mu - \frac{3}{8}\eta p_0^2 - \frac{3}{2} p_0 \kappa q_0 - \frac{1}{8}\eta q_0^2 & -2\sigma - \frac{1}{4} q_0 \eta p_0 - \frac{9}{4}\kappa q_0^2 - \frac{3}{4}\kappa p_0^2 \\ - \frac{3\alpha_4 p_0^2 q_0}{4\sqrt{p_0^2 + q_0^2}} + \frac{3\alpha_4 p_0^2 q_0}{4\left(p_0^2 + q_0^2\right)^{\frac{3}{2}}} & - \frac{3\alpha_4 p_0^2 q_0}{4\sqrt{p_0^2 + q_0^2}} + \frac{3\alpha_4 p_0^2 q_0}{4\left(p_0^2 + q_0^2\right)^{\frac{3}{2}}} \end{bmatrix} \quad (3.9.23)$$

In this resonance condition, the response of the system will be stable if and only if the real part of all the eigenvalues are negative.

Exercise problem:

1. Use method of normal form to find the frequency response equations for the Duffing equation with cubic nonlinearity (refer book by Nayfeh 1993).
2. Use method of normal form to find the frequency response equations for the Duffing equation with quadratic and cubic nonlinearity (refer book by Nayfeh 1993).
3. Use method of normal form to find the frequency response equations for the van der Pol's equation (refer book by Nayfeh 1993).
4. Find the simultaneous resonance case ($\overline{\omega}_1 \approx 1$ and $\overline{\omega}_2 \approx 2$), and the higher order resonance conditions i.e. the sub harmonic ($\overline{\omega}_1 \approx 3$) and the simultaneous resonance conditions $\overline{\omega}_1 \approx 3$ and $\overline{\omega}_2 \approx 2$ of the system discussed in example 3.9.1.

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Module 3 Lecture 10

INCREMENTAL HARMONIC BALANCE METHOD

Lau and Cheung (1981) developed incremental harmonic balance method. A practical weakness of perturbation methods is that carrying out the expansion to higher order is very cumbersome, especially for multiple degree of freedom systems. In practice it is difficult to go beyond the third order unless the algebraic manipulations are performed by a computer (Cheung et al. 1990). In Incremental Harmonic Balance Method (IHB) one can deal with strongly non linear systems to any desired accuracy. This method is a combination of the incremental method (Newton-Raphson procedure) with the harmonic balance method (Ritz and Galerkin's averaging method). It is exactly equivalent to a Galerkin procedure followed by a Newton-Raphson method.

The method possesses advantages in studying systems with severe nonlinearities and is easily applied to systems with harmonic (or, more generally, periodic) excitation. Some insight into the solution method is lost, however, since the problem of solving the original governing differential equations is replaced with that of solving a second "simpler" set of equations involving increments in the motion, exciting force and/or frequency of excitation. Ferri (1986) shown that the IHB method is exactly equivalent to the Harmonic Balance Newton Raphson Method (HBNR). Here this method is illustrated by taking the example of a multi degree of freedom nonlinear system.

For a multi degree of freedom system with cubic non linearities, the non linear equations of motion in general can be written as

$$\sum_{j=1}^{n} M_{ij}\frac{d^2 q_j}{dt^2}+\sum_{j=1}^{n} C_{ij}\frac{dq_j}{dt}+\sum_{j=1}^{n} K_{ij}q_j+\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\alpha_{ijkl}\, q_j q_k q_l = f_i\cos(2m-1)\omega t,\quad i=1,2,\ldots,n. \tag{3.10.1}$$

by substituting $\tau=\omega t$ one may write (3.10.1) as

$$\omega^2 \sum_{j=1}^{n} M_{ij}\ddot{q}_j + \omega \sum_{j=1}^{n} C_{ij}\dot{q}_j + \sum_{j=1}^{n} K_{ij}q_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_{ijkl}\, q_j q_k q_l = f_i \cos(2m-1)\tau, \quad i = 1, 2, \ldots, n. \tag{3.10.2}$$

The q_j are the unknowns of the system, the dots denote derivatives with respect to the dimensionless time τ, and $M_{ij}, C_{ij}, K_{ij}, \alpha_{ijkl}, f_i$ and ω are coefficients of the mass, damping, linear stiffness, cubic stiffness, and excitation amplitude and excitation frequency respectively. Equation (3.10.2) can be written in the matrix form as

$$\omega^2 \overline{\boldsymbol{M}}\ddot{\boldsymbol{q}} + \omega \overline{\boldsymbol{C}}\dot{\boldsymbol{q}} + (\overline{\boldsymbol{K}} + \overline{\boldsymbol{K}}_{\boldsymbol{n}})\boldsymbol{q} = \overline{\boldsymbol{F}} \cos(2\boldsymbol{m}-1)\tau \tag{3.10.3}$$

where $\mathbf{q} = [q_1, q_2, \ldots\ldots, q_n]^T, \overline{\mathbf{F}} = [f_1, f_2, \ldots., f_n]^T, \overline{\mathbf{M}}, \overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ are mass, damping, linear stiffness matrices, with elements are denoted by $\overline{M}_{ij}, \overline{C}_{ij}$ and $\overline{K}_{ij}$ respectively, and $\overline{K}_n$ is the cubic non-linear stiffness matrix, its element α_{ijkl} being taken in the form

$$\overline{K}_{nij} = \sum_{k=1}^{n}\sum_{l=1}^{n} K_{ijkl} q_k q_l \tag{3.10.3}$$

The first step of the IHB Method is a Newton-Raphson procedure. Let $q_{jo}, f_{io} and \omega_o$ denote a state of vibration; the neighboring state can be expressed by adding the corresponding increments to them as follows:

$$q_j = q_{jo} + \Delta q_j,\ j = 1, 2, \ldots., n, \tag{3.10.4}$$

$$f_i = f_{io} + \Delta f_i, i = 1, 2, \ldots, n, \tag{3.10.5}$$

$$\omega = \omega_0 + \Delta\omega. \tag{3.10.6}$$

Substituting expansions (3.10.4)-(3.10.6) into equation (3.10.2) and neglecting small terms of higher order, one obtains the following linearized incremental equation in matrix form:

$$\omega_0^2 \overline{M}\Delta\ddot{q} + \omega_0 \overline{C}\Delta\dot{q} + (\overline{K} + 3\overline{K}_n)\Delta q = \overline{R} - (2\omega_0 \overline{M}\ddot{q}_0 + \overline{C}\dot{q}_0)\Delta\omega + \cos(2m-1)\tau\Delta F, \tag{3.10.7}$$

$$\overline{R} = \overline{F_0} \cos(2m-1)\tau - (\omega_0^2 \overline{M}\ddot{q}_0 + \omega_0 \overline{C}\dot{q}_0 + \overline{K}q_0 + \overline{K}_n q_0), \tag{3.10.8}$$

in which $q_0, \Delta q, \overline{F}_0, \Delta F$ and $\overline{K}_{nij}$ are given below.

$q_0 = [q_{10}, q_{20}, \ldots\ldots, q_{n0}]^T$, $\Delta q = [\Delta q_1, \Delta q_2, \ldots\ldots, \Delta q_n]^T$, $F_0 = [f_{10}, f_{20}, \ldots\ldots, f_{n0}]^T$,

$\Delta F = [\Delta f_1, \Delta f_2, \ldots\ldots, \Delta f_n]^T$ and $\overline{K}_{nij} = \sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_{ijkl} q_{ko} q_{10}$.

$\overline{R}$ is a corrective vector which goes to zero when the solution is reached.

The second step of the IHB method is the Galerkin's procedure. Because equation (3.10.2) is odd and the excitation force is periodic, one can assume for steady state response,

$$q_{j0} = \sum_{k=1}^{N_c} a_{jk} \cos(2k-1)\tau + \sum_{k=1}^{N_s} b_{jk} \sin(2k-1)\tau = C_s A_j, \tag{3.10.9}$$

$$\Delta q_j = \sum_{k=1}^{N_c} \Delta a_{jk} \cos(2k-1)\tau + \sum_{k=1}^{N_s} \Delta b_{jk} \sin(2k-1)\tau = C_s \Delta A_j, \tag{3.10.10}$$

Where

$$C_s = [\cos\tau, \cos 3\tau, ..., \cos(2N_c-1)\tau, \sin\tau, \sin 3\tau, ..., \sin(2N_s-1)\tau],$$

$$A_j = [a_{j1}, a_{j2}, ..., a_{jN_c}, b_{j1}, b_{j2}, ..., b_{jN_s}]^T,$$

$$\Delta A_j = [\Delta a_{j1}, \Delta a_{j2}, ..., \Delta a_{jN_c}, \Delta b_{j1}, \Delta b_{j2}, ..., \Delta b_{jN_s}]^T.$$

Hence the vectors of unknowns and their increments can be expressed by the Fourier coefficients vector A and its increment ΔA as follows:

$$q_0 = SA, \tag{3.10.11}$$

$$\Delta q = S\Delta A \tag{3.10.12}$$

where *S*, *A* and *ΔA* are given as follows.

$$S = \begin{bmatrix} C_s & & 0 \\ & & \\ 0 & & C_s \end{bmatrix}, \; A = [A_1, A_2, ..., A_n]^T, \text{and } \Delta A = [\Delta A_1, \Delta A_2, ..., \Delta A_n]^T,$$

$$M = \int_0^{2\pi} S^T \overline{M}\ddot{S} d\tau, \;\; C = \int_0^{2\pi} S^T \overline{C}\dot{S} d\tau, \;\; K = \int_0^{2\pi} S^T \overline{K} S d\tau,$$

$$K^{(3)} = \int_0^{2\pi} S^T \overline{K}^{(3)} S d\tau, \;\; F = \int_0^{2\pi} S^T \overline{F}_0 \cos(2m-1)\tau d\tau, \;\; R_f = \int_0^{2\pi} S^T \cos(2m-1)\tau d\tau,$$

Substituting equations (3.10.11) and (3.10.12) into equation (3.10.7) and using the Galerkin's procedure gives

$$\begin{aligned} &\int_0^{2\pi} \delta(\Delta q)^T [\omega_0^2 \overline{M} \Delta\ddot{q} + \omega_0 \overline{C} \Delta\dot{q} + (\overline{K} + 3\overline{K}^{(3)}) \Delta q] d\tau \\ &= \int_0^{2\pi} \delta(\Delta q)^T [\overline{R} - (2\omega_0 \overline{M} \ddot{q}_0 + \overline{C}\dot{q}_0) \Delta\omega + \cos(2m-1)\tau \Delta F] d\tau. \end{aligned} \tag{3.10.13}$$

One can easily obtain a set of linear equations in terms of ΔA, $\Delta\omega$ and ΔF ,

$$K_{mc} \Delta A = R - R_{mc} \Delta\omega + R_f \Delta F, \tag{3.10.14}$$

in which

$$K_{mc} = \omega_0^2 M + \omega_0 C + 3K_n \tag{3.10.15}$$

$$R = F - (\omega_0^2 M + \omega_0 C + K + K^{(3)}) A, \tag{3.10.16}$$

$$R_{mc} = (2\omega_0 M + C) A, \tag{3.10.17}$$

It is worth mentioning that in equation (3.10.14) the number of incremental unknowns is greater than the number of equations available due to the existence of ΔF and $\Delta\omega$. Since one is primarily interested in the frequency-response curves of the system for a constant level, F is fixed as a parameter vector, which implies $\Delta F = 0$. Hence equation (3.10.14) is reduced to

$$K_{mc}\Delta A = R - R_{mc}\Delta\omega. \qquad (3.10.18)$$

The solution process starts from a suggested solution (in general, from a corresponding known linear solution), and then the non-linear amplitude frequency response is solved point by point by incrementing frequency ω or incrementing one component of the amplitudes A. The Newton-Raphson iteration can be applied within an incremental step. In the incremental process, an increment which is prescribed *a priori* is called a control or active increment. If $\Delta\omega$ is specified as a control increment, then ω remains constant through the iterative process: i.e. $\Delta\omega = 0$, while other increments are solved from the equation

$$K_{mc}\Delta A = R \qquad (3.10.19)$$

The process is repeated until the magnitude of the corrected vector R is acceptably small-in which case a solution is obtained. This process is called *iteration.* The value of ω is then augmented an increment $\Delta\omega$ artificially, and a new iteration is repeated with the new value of ω until a new solution is obtained. The above process is called an *augmentation.* The whole solution process is an alternative application of augmentation and iteration.

The above incremental process in which $\Delta\omega$ is taken as active increment is called *ω-incrementation.* Similarly, it is equally possible to have amplitude incrementation. In this case, one component of ΔA, say Δa_{jk}, is specified as the control increment; then a_{jk} remains constant. $\Delta a_{jk} = 0$ through the iteration and one has to solve equation (3.10.18) to obtain other increments of ΔA and $\Delta\omega$. After the amplitude of R has reached the desired accuracy, the iteration is terminated and a new augmentation can be started by adding an increment on a_{jk}. This process is called a_{jk} *incrementation*. In practice, the active increment is chosen as the one that varies faster and therefore the ω-incrementation or the a_{jk} incrementation can be adopted along the response curves.

If one is interested in the forcing amplitude response curves of the system for a constant frequency level, then $\Delta\omega = 0$ and $\Delta f_i = 0$, $j \neq i$, and hence equation (3.10.14) is reduced to

$$K_{mc}\Delta A = R + R_f\Delta F \qquad (3.10.20)$$

Example 3.10.1:

Consider a two degree of freedom system consisting of two point masses and two springs with a linear damper, under a harmonic excitation shown in Figure 3.10.1. Find the solution of the system using incremental harmonic balance method.

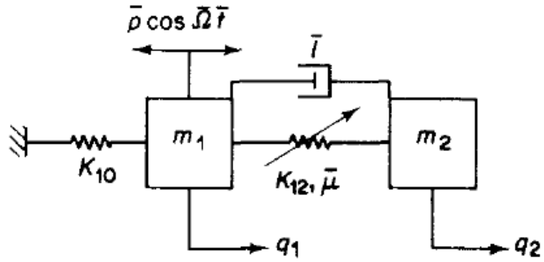

Fig 3.10.1: Schematic diagram of a two degree of freedom system with cubic nonlinear spring.

Solution:

One of the springs is linear with the stiffness coefficient k_{10} and the other has a cubic non linearity. Its restoring force is defined as

$$f_{12} = k_{12}(q_1 - q_2) + \bar{\mu}(q_1 - q_2)^3 \quad (3.10.21)$$

The differential equations of motion of the system can be written in non dimensional form as

$$\ddot{q}_1 + k^2 q_1 + \gamma(q_1 - q_2) + \mu\gamma l(\dot{q}_1 - \dot{q}_2) + \mu\gamma(q_1 - q_2)^3 = p\cos\Omega t, \quad (3.10.22)$$

$$\ddot{q}_2 + (q_2 - q_1) - \mu l(\dot{q}_1 - \dot{q}_2) - \mu\gamma(q_1 - q_2)^3 = 0, \quad (3.10.23)$$

Where

$\gamma = m_1/m_2,\ t = \bar{t}\sqrt{(k_{12}/m_2)},\ k^2 = k_{10}\gamma/k_{12},\ l = \bar{l}\sqrt{(k_{12}/m_2)},$

$\dot{q} = dq/dt,\ \mu = \bar{\mu}/k_{12},$ and $p = \bar{p}\gamma/k_{12}$.

q_1 , q_2 are displacements of point masses, t is time and m_1, m_2, k_{10}, k_{12}, μ, l , Ω and p are the masses of the system, coefficient of linear stiffness, coefficient of non linear stiffness, coefficient of damping, excitation frequency and excitation amplitude respectively.

In the solution process, the number of harmonic terms is taken as $N_c = N_s = 2$:

$$q_1 = a_{11}\cos\tau + a_{12}\cos 3\tau + b_{11}\sin\tau + b_{12}\sin 3\tau = A_{11}\cos(\tau + \phi_{11}) + A_{12}\cos(3\tau + \phi_{12}), \quad (3.10.24)$$

$$q_2 = a_{12}\cos\tau + a_{22}\cos 3\tau + b_{21}\sin\tau + b_{22}\sin 3\tau = A_{21}\cos(\tau + \phi_{11}) + A_{22}\cos(3\tau + \phi_{22}), \quad (3.10.25)$$

Where

$A_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2}, \phi_{ij} = \tan^{-1}(-b_{ij}/a_{ij}), i = 1, 2, j = 1, 2.$

There exist two types of non trivial solutions:

a) Fundamental resonance only, i.e. $q_1 = A_{12}\cos(3\tau + \phi_{12}),\ q_2 = A_{22}\cos(3\tau + \phi_{22}),$

b) Both fundamental resonance and sub harmonic resonance occur simultaneously: i.e. q_1 and q_2 take the form of the equations (3.10.24) and (3.10.25).

Exercise problems:

1. Use incremental harmonic balance method to find the frequency response equations for the Duffing equation with cubic nonlinearity.

2. Use incremental harmonic balance method to find the frequency response equations for the Duffing equation with quadratic and cubic nonlinearity.

3. Use incremental harmonic balance method to find the frequency response equations for the van der Pol's equation.

4. The equation of motion of a bimaterial beam with alternating magnetic field and thermal loads can be given by following equation. Use incremental harmonic balance method to solve this equation (refer Wu, 2009).

$$m\frac{\partial^2 v}{\partial t^2}+C_d\frac{\partial v}{\partial t}+\left(E_t I_t+E_l I_l\right)\frac{\partial^4 v}{\partial x^4}+\frac{\partial c}{\partial x}+\frac{\partial}{\partial x}\left[\left(\int_0^x p\,d\xi\right)\frac{\partial v}{\partial x}\right]+\left[A_t\gamma_t\left(\Delta T\right)+A_l\gamma_l\left(\Delta T\right)\right]\frac{\partial^2 v}{\partial x^2}=0$$

The equation in its temporal form can be written as

$$\Omega^2\frac{d^2 w}{d\tau^2}+2\Omega\left[k_1+k_2\left(1+\cos 2\tau\right)w^2\right]\frac{dw}{d\tau}+\left(1-2\varphi\cos 2\tau\right)w=0$$

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Module 3 Lecture 11

INTRINSIC MULTIPLE SCALE HARMONIC BALANCE METHOD

In this lecture both the method of multiple scale and harmonic balance method will be combined to obtain the solution of the nonlinear system. This method is explained with the help of free vibration of a system with cubic and quadratic nonlinearities of Duffing type. Consider the following non linear system

$$\ddot{u}+\omega_0^2 u+\alpha_2 u^2+\alpha_3 u^3=0 \tag{3.11.1}$$

Here the dot '.' denotes differentiation with respect to time. An intrinsic multiple-scale harmonic balancing method (IMSHB) can be applied to system (3.11.1) as follows. Similar to method of multiple scales, one may consider different time scales $T_0,\ T_1,\ T_2,\ T_3,\ \cdots$ as given below.

$$T_n=\varepsilon^n t, n=0,1,2,........ \tag{3.11.2}$$

So one can write

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2+...... \tag{3.11.3}$$

and $$\frac{d^2}{dt^2}=D_0{}^2+2\varepsilon D_0 D_1+\varepsilon^2(2D_0 D_2+D_1{}^2)+........... \tag{3.11.4}$$

where $D_n^m=\dfrac{\partial^m}{\partial T_n^m}$. To separate the linear and nonlinear terms one may introduce the scaling $u=\varepsilon x$ and write Eq. (3.11.1) as

$$\ddot{x}+\omega_0^2 x+\varepsilon\alpha_2 x^2+\varepsilon^2\alpha_3 x^3=0 \tag{3.11.5}$$

Now using Eqs. (3.11.2-4) in Eq. (3.11.5) one obtains the following equation.

$$D_0^2 x+2\varepsilon D_0 D_1 x+\varepsilon^2(2D_0 D_2+D_1^2)x+.......+\omega_0^2 x+\varepsilon\alpha_2 x^2+\varepsilon^2\alpha_3 x^3=0 \tag{3.11.6}$$

Now let the solution be expressed in the parametric form as

$$x=x(T_0,T_1,T_2;\varepsilon). \tag{3.11.7}$$

Substituting Eq. (3.11.7) in (3.11.6) and putting $\varepsilon=0$ one will obtain the zeroth order perturbation equation as follows.

Order of ε^0, $D_0^2 x+\omega_0^2 x=0,$ (3.11.8)

To obtain n^{th} order perturbation equations, it is proposed to differentiate Eq. (3.11.6) n times with respect to ε and set $\varepsilon=0$. So one will obtained the following perturbation equation of order ε^1 and ε^2.

Order of $\varepsilon^1:(D_0^2 x)'+2(D_0 D_1 x)+\omega_0^2 x'+\alpha_2 x^2=0,$ (3.11.9)

Order of $\varepsilon^2:(D_0^2 x)''+4(D_0 D_1 x)'+2(2D_0 D_2+D_1^2)x+\omega_0^2 x''+2\alpha_2(x^2)'+2\alpha_3 x^3=0$ (3.11.10)

Here $()'$ represent differentiation with respect to ε.

One may assume a general solution of two time scale expansions in the following form

$$x=\sum_{m=0}^{M}\left[a_m(\varepsilon;T_1,T_2)\cos m(\omega_0 T_0+\theta(\varepsilon;T_1,T_2))+b_m(\varepsilon;T_1,T_2)\sin m(\omega_0 T_0+\theta(\varepsilon;T_1,T_2))\right]$$

(3.11.11)

The amplitudes and phases are given in the form

$$a_m=a_m^0(T_1,T_2)+\varepsilon a_m^1(T_1,T_2)+\varepsilon^2 a_m^2(T_1,T_2)+..... \tag{3.11.12}$$

$$b_m=b_m^0(T_1,T_2)+\varepsilon b_m^1(T_1,T_2)+\varepsilon^2 b_m^2(T_1,T_2)+....... \tag{3.11.13}$$

$$\theta=\theta^0(T_1,T_2)+\varepsilon\theta^1(T_1,T_2)+\varepsilon^2\theta^2(T_1,T_2)+....... \tag{3.11.14}$$

In these expansions $a_m^0,a_m^1,......;b_m^0,b_m^1,....;$ and θ^0, $\theta^1,.....$ are to be determined through steps of perturbations.

Introducing expression (3.11.11) into the zero order perturbation equation gives

$$\left(D_0^2+\omega_0^2\right)\sum_{m=0}^{M}\begin{bmatrix}a_m(\varepsilon;T_1,T_2)\cos m(\omega_0 T_0+\theta(\varepsilon;T_1,T_2))\\ +b_m(\varepsilon;T_1,T_2)\sin m(\omega_0 T_0+\theta(\varepsilon;T_1,T_2))\end{bmatrix}=0 \tag{3.11.15}$$

$$\sum_{m=0}^{M}(m^2-1)\omega_0^2 a_m^0\cos m(\omega_0 T_0+\theta^0)=0 \tag{3.11.16}$$

and $\sum_{m=0}^{M}(m^2-1)\omega_0^2 b_m^0\sin m(\omega_0 T_0+\theta^0)=0;$ (3.11.17)

Hence, for $m=0$, $a_0^0=b_0^0=0$. Also for $m\geq 2$, $a_m^0=b_m^0=0$. (3.11.18)

Since the system is autonomous one can assume

$$b_1(\varepsilon; T_1, T_2) \equiv 0 \qquad (3.11.19)$$

In the IHB Method the process is simplified if the perturbation parameter is selected as one of the appropriate amplitudes (e.g. a_1). In the analogy with this approach, it is assumed here that a_1 is not a function of ε; i.e.

$$a_1(\varepsilon; T_1, T_2) \triangleq a(T_1, T_2). \qquad (3.11.20)$$

Substituting Eqs. (3.11.11) - (3.11.14) and Eqs. (3.11.18) - (3.11.20) in Eq.(3.11.10) one obtains

$$x = a_1(\varepsilon; T_1, T_2)\cos(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)) \qquad (3.11.21)$$

As $a_1 = a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2) + ...$ (3.11.22)

So, $x = \left(a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2)\right)\cos\left(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)\right)$ (3.11.23)

Now substituting (3.11.23) in the first order perturbation equation, the term by term expansion is given below.

$$D_0 x = \left(a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2)\right)\left(-\omega_0\right)\sin\left(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)\right) \qquad (3.11.24)$$

$$D_0^2 x = -\omega_0^2\left(a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2)\right)\cos\left(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)\right) \qquad (3.11.25)$$

$$\begin{aligned}(D_0^2 x)' &= -\omega_0^2\left(a_1^1(T_1,T_2) + 2\varepsilon a_1^2(T_1,T_2)\right)\cos(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)) \\ &+ \left(\theta^1 + 2\varepsilon\theta^2\right)\omega_0^2\left(a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2)\right)\sin(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2))\end{aligned} \qquad (3.11.26)$$

$$\begin{aligned}&2(D_0 D_1 x) = 2D_1(D_0 x) = \\ &-2\omega_0\begin{pmatrix}\left(D_1 a_1^0(T_1,T_2) + \varepsilon D_1 a_1^1(T_1,T_2) + \varepsilon^2 D_1 a_1^2(T_1,T_2)\right)\sin(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)) + \\ \left(D_1\theta^0 + \varepsilon D_1\theta^1 + \varepsilon^2 D_1\theta^2\right)\begin{pmatrix}a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \\ \varepsilon^2 a_1^2(T_1,T_2)\end{pmatrix}\cos(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2))\end{pmatrix}\end{aligned} \qquad (3.11.27)$$

$$\begin{aligned}\omega_0^2 x' &= \omega_0^2\left(a_1^1(T_1,T_2) + 2\varepsilon a_1^2(T_1,T_2)\right)\cos(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)) \\ &+ \omega_0^2\left(\theta^1 + 2\varepsilon\theta^2\right)\left(a_1^0(T_1,T_2) + \varepsilon a_1^1(T_1,T_2) + \varepsilon^2 a_1^2(T_1,T_2)\right)\sin(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2))\end{aligned} \qquad (3.11.28)$$

$$\alpha_2 x^2 = \frac{\alpha_2}{2}\left(\left(a_1^0\right)^2 + 2\varepsilon a_1^0 a_1^1 + \varepsilon^2\left(a_1^1\right)^2 + 2\varepsilon^2 a_1^0 a_1^2\right)\left(1 + \cos 2\left(\omega_0 T_0 + \theta(\varepsilon; T_1, T_2)\right)\right) \qquad (3.11.29)$$

$$\begin{aligned}\theta &= \theta^0(T_1,T_2) + \varepsilon\theta^1(T_1,T_2) + \varepsilon^2\theta^2(T_1,T_2) + ... \\ \theta' &= D_1\theta = D_1\left(\theta^0(T_1,T_2) + \varepsilon\theta^1(T_1,T_2) + \varepsilon^2\theta^2(T_1,T_2) + ...\right) = D_1\theta^0 + \varepsilon D_1\theta^1 + \varepsilon^2 D_1\theta^2\end{aligned} \qquad (3.11.30)$$

So, balancing the harmonics in the first order perturbation equation gives

$$\begin{aligned}&\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(a_1^1\right)^2+2\varepsilon^2 a_1^0 a_1^2\right)\\&+\begin{bmatrix}-\omega_0^2\left(a_1^1+2\varepsilon a_1^2\right)-2\omega_0\left(D_1\theta^0+\varepsilon D_1\theta^1+\varepsilon^2 D_1\theta^2\right)\left(a_1^0+\varepsilon a_1^1+\varepsilon^2 a_1^2\right)\\+\omega_0^2\left(a_1^1+2\varepsilon a_1^2\right)\end{bmatrix}\cos\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\\&+\begin{bmatrix}\omega_0^2\left(\theta^1+2\varepsilon\theta^2\right)\left(a_1^0+\varepsilon a_1^1+\varepsilon^2 a_1^2\right)-2\omega_0\left(D_1 a_1^0+\varepsilon D_1 a_1^1+\varepsilon^2 D_1 a_1^2\right)\\+\omega_0^2\left(\theta^1+2\varepsilon\theta^2\right)\left(a_1^0+\varepsilon a_1^1+\varepsilon^2 a_1^2\right)\end{bmatrix}\sin\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\\&+\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(a_1^1\right)^2+2\varepsilon^2 a_1^0 a_1^2\right)\cos 2\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\end{aligned}\tag{3.11.31}$$

$$\begin{aligned}&\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(\left(a_1^1\right)^2+2a_1^0 a_1^2\right)\right)\\&+\begin{bmatrix}\left(-\omega_0^2 a_1^1-2\omega_0 D_1\theta^0 a_1^0+\omega_0^2 a_1^1\right)\\+\varepsilon\left(-2\omega_0^2 a_1^2-2\omega_0\left(a_1^1 D_1\theta^0+a_1^0 D_1\theta^1\right)+2\omega_0^2 a_1^2\right)\\+\varepsilon^2\left(-2\omega_0\left(a_1^2 D_1\theta^0+a_1^0 D_1\theta^2+D_1\theta^1 a_1^1\right)\right)\end{bmatrix}\cos\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\\&+\begin{bmatrix}\omega_0^2\theta^1 a_1^0-2\omega_0 D_1 a_1^0+\omega_0^2\theta^1 a_1^0+\\\varepsilon\left(\omega_0^2\theta^1 a_1^1+2\omega_0^2\theta^2 a_1^0-2\omega_0 D_1 a_1^1+2\omega_0^2\theta^2 a_1^0+\omega_0^2\theta^1 a_1^1\right)\\+\varepsilon^2\left(2\omega_0^2\left(\theta^1 a_1^2+2\theta^2 a_1^1\right)-2\omega_0 D_1 a_1^2\right)\end{bmatrix}\sin\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\\&+\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(a_1^1\right)^2+2\varepsilon^2 a_1^0 a_1^2\right)\cos 2\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)=0\end{aligned}\tag{3.11.32}$$

$$\begin{aligned}&\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(\left(a_1^1\right)^2+2a_1^0 a_1^2\right)\right)+\\&\begin{bmatrix}\left(-2\omega_0 a_1^0 D_1\theta^0\right)+\varepsilon\left(-2\omega_0\left(a_1^1 D_1\theta^0+a_1^0 D_1\theta^1\right)\right)\\+\varepsilon^2\left(-2\omega_0\left(a_1^2 D_1\theta^0+a_1^0 D_1\theta^2+D_1\theta^1 a_1^1\right)\right)\end{bmatrix}\cos\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)+\\&\begin{bmatrix}2\omega_0^2\theta^1 a_1^0-2\omega_0 D_1 a_1^0+\varepsilon\left(2\omega_0^2\theta^1 a_1^1+4\omega_0^2\theta^2 a_1^0-2\omega_0 D_1 a_1^1\right)+\\+\varepsilon^2\left(2\omega_0^2\left(\theta^1 a_1^2+2\theta^2 a_1^1\right)-2\omega_0 D_1 a_1^2\right)\end{bmatrix}\sin\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)\\&+\frac{\alpha_2}{2}\left(\left(a_1^0\right)^2+2\varepsilon a_1^0 a_1^1+\varepsilon^2\left(a_1^1\right)^2+2\varepsilon^2 a_1^0 a_1^2\right)\cos 2\left(\omega_0 T_0+\theta(\varepsilon;T_1,T_2)\right)=0\end{aligned}\tag{3.11.33}$$

$$-\omega_c D_1 a_1^0 = 0, -2a_1^0 \omega_c D_1 \theta_1^0 = 0, \tag{3.11.34}$$

$$a_0^1 = -\alpha_2 a^2 / (2\omega_0^2), a_2^1 = \alpha_2 a^2 / (6\omega_0^2), \tag{3.11.35}$$

$$a_1^1 = a_3^1 = a_4^1 = ... = 0, b_0^1 = b_1^1 = b_2^1 = ... = 0, \theta^1 = 0 \tag{3.11.36}$$

The substitution of solution (3.11.26) and expressions (3.11.33) - (3.11.36) into the second order perturbation yields

$$\begin{aligned} &2(1-m^2)\omega_0^2 a_m^2 c_m - 2(1-m^2)\omega_0^2 a_1^0 \theta^2 s_1 + 2(1-m^2)\omega_0^2 b_m^2 s_m \\ &-4D_1 a_m^1 m\omega_0 s_m - 4D_2 a_1^0 \omega_0 s_1 - 4a_1^0 \omega_0 D_2 \theta^0 c_1 + 4\alpha_2 a_1^0 a_m^1 c_1 c_m \\ &-4\alpha_2 (a_1^0)^2 \theta^1 c_1 s_1 + 1.5\alpha_3 (a_1^0)^3 c_1 + 0.5\alpha_3 (a_1^0)^3 c_3 = 0 \end{aligned} \tag{3.11.37}$$

Where

$s_1 : D_2 a_1^0 = 0$

$c_1 = \cos(\omega_0 T_0 + \theta^0), s_1 = \sin(\omega_0 T_0 + \theta^0), c_m = \cos m(\omega_0 T_0 + \theta^0)$ and $s_m = \sin m(\omega_0 T_0 + \theta^0)$.

Balancing various harmonics in equation (3.11.37) gives

$$s_1 : D_2 a_1^0 = 0 \tag{3.11.38}$$

$$c_1 = 4a_1^0 \omega_0 D_2 \theta^0 = 4\alpha_2 a_1^0 a_0^1 + 2\alpha_2 a_1^0 a_2^1 + 1.5\alpha_3 (a_1^0)^3. \tag{3.11.39}$$

Substituting Eq. (3.11.35) in Eq. (3.11.39) one obtains

$$D_2 \theta^0 = \frac{3\alpha_3 a^2}{8\omega_0} - \frac{5\alpha_2^2 a^2}{12\omega_0^3} \tag{3.11.40}$$

Using (3.11.3) the differential equation of phase θ can be derived as

$$\frac{d\theta}{dt} = \left(D_0 + \varepsilon D_1 + \varepsilon^2 D_2\right)\theta = \varepsilon^2 \frac{3\alpha_3 a^2}{8\omega_0} - \varepsilon^2 \frac{5\alpha_2^2 a^2}{12\omega_0^3} + O(\varepsilon^3). \tag{3.11.41}$$

Therefore the amplitude frequency relation can be given by

$$\omega = \omega_c + \frac{3\alpha_3 \varepsilon^2 a^2}{8\omega_c} - \frac{5\alpha_2^2 \varepsilon^2 a^2}{12\omega_c^2} + O(\varepsilon^3) \tag{3.11.42}$$

Thus the above expression is in full agreement with the following equation

$$\omega = \sqrt{\alpha_1}[1 + \{(9\alpha_3\alpha_1 - 10\alpha_2^2)/(24\alpha_1^2)\}A_1^2] + ... \tag{3.11.43}$$

which was obtained from the conventional harmonic balance method.

Exercise problem:

1. Use intrinsic multiple scale harmonic balance method to find the frequency response equations for primary resonance of the Duffing equation with cubic nonlinearity and a weak forcing function. Write a Matlab code and plot the frequency response curves.

2. Use intrinsic multiple scale harmonic balance method to find the frequency response equations for the Duffing equation with quadratic and cubic nonlinearity. Use any symbolic software (Maple/Mathmatica) to derive the equations.

3. Use intrinsic multiple scale harmonic balance method to find the frequency response equations for the van der Pol's equation. Use any symbolic software (Maple/Mathmatica) to derive the equations. Also plot the time response and phase portrait to show the limit cycle.

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Module 3 Lecture 12

HIGHER ORDER METHOD OF MULTIPLE SCALES

In this lecture higher order method of multiple scales proposed by Rahman and Burton [1] will be discussed with the help of a example of parametrically excited system. The obtained equations will be compared with the commonly used method of multiple scales.

A uniform cantilever beam of length *L* carrying a mass *m* at an arbitrary position *d* from the fixed end and subjected to base motion is considered as an example of a parametrically excited system. Similar system has been considered by Zavodney and Nayfeh [7] and Dwivedy and Kar [3]. When the system is given a base motion $z(t) = Z_0 \cos \Omega t,$ the temporal equation of the motion of the beam is given by

$$\ddot{u} + 2\xi_0 \dot{u} + \left\{ \omega^2 - f_0 \cos \phi t \right\} u + \left\{ \alpha_0 u^3 + \beta_0 u \dot{u}^2 + \Gamma_0 u^2 \ddot{u} \right\} = 0 \tag{3.12.1}$$

Here *u* is the non dimensional transverse displacement of the beam, ξ_0 and f_0 are the damping and forcing parameters and ϕ is the non dimensional frequency of external excitation. The coefficient of geometrical non linear term (α_0) and inertia non linear terms (β_0, Γ_0) are introduced in the system due to the large transverse deflection during base excitation. Introducing the new time parameter $\tau(\tau = \phi t)$ and taking into account the smallness of damping, forcing and nonlinear terms through the bookkeeping parameter ε, Eq. (3.12.1) reduces to the non dimensional form

$$\phi^2 \ddot{u} + 2\varepsilon \xi \phi \dot{u} + \left\{ \omega^2 - \varepsilon f \cos \tau \right\} u + \varepsilon \left\{ \alpha u^3 + \phi^2 \left(\beta u \dot{u}^2 + \Gamma u^2 \ddot{u} \right) \right\} = 0 \tag{3.12.2}$$

Where $\left(\dot{}\right) = d\left(\ \right)/d\tau$, $\xi = \xi_0/\varepsilon, \alpha = \alpha_0/\varepsilon, \beta = \beta_0/\varepsilon$ and $\Gamma = \Gamma_0/\varepsilon$.

Method of Multiple Scales: Version II

Following [1-3], the displacement u, the external excitation ϕ, the damping ξ, the new time scale $T_n\left(n = 0,1,2,...\right)$, and the time derivatives are expanded as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2, \tag{3.12.3}$$

$$\phi^2 = 4\omega^2 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2, \tag{3.12.4}$$

$$\phi\xi = \xi_1 + \varepsilon\xi_2, \tag{3.12.5}$$

$$T_n = \varepsilon^n \tau \tag{3.12.6}$$

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \tag{3.12.7}$$

$$\frac{d}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right), \tag{3.12.8}$$

Where $D_n = \dfrac{\partial}{\partial T_n}$

Substituting the above in Eq. (3.12.2), collecting the coefficients of ε^n and equating them to zero, one obtains for

order of ε^0

$$4\omega^2 D_0^2 u_0 + \omega^2 u_0 = 0 \tag{3.12.9}$$

order of ε^1

$$\begin{aligned}&4\omega^2 D_0^2 u_1 + \omega^2 u_1 + \sigma_1 D_0^2 u_0 + 2\xi_1 d_0 U_0 - f u_0 \cos\tau \\ &+8\omega^2 D_0 D_1 u_0 + \alpha u_0^3 + 4\omega^2\left\{\beta\left(D_0 u_0\right)^2 u_0 + \Gamma u_0^2 D_0^2 u_0\right\} = 0\end{aligned} \tag{3.12.10}$$

order of ε^2

$$\begin{aligned}&4\omega^2 D_0^2 u_2 + \omega^2 u_2 + \sigma_1 D_0^2 u_1 + 2\sigma_1 D_0 D_1 u_0 + 8\omega^2 D_0 D_1 u_1 + 4\omega^2 D_1^2 u_0 \\ &+8\omega^2 D_0 D_2 u_0 + \sigma_2 D_0^2 u_0 + 2\xi_2 D_0 u_0 + 2\xi_1\left(D_1 u_0 + D_0 u_1\right) - f u_1 \cos\tau + 3\alpha u_0^2 \\ &+4\omega^2\left[\beta\left\{2u_0\left(D_0 u_0\right)\left(D_0 u_1 + \left(D_1 u_0\right) + \left(D_1 u_0\right)^2 u_1\right)\right\} + \Gamma\left\{u_0^2\left(D_0^2 u_1 + 2D_0 D_1 u_0\right) + 2u_0 u_1 D_0^2 u_0\right\}\right] \\ &+\sigma_1\left\{\beta\left(D_0 u_0\right)^2 u_0 + \Gamma u_0^2\left(D_0^2 u_0\right)\right\} = 0\end{aligned} \tag{3.12.11}$$

The solution of Eq. (3.12.9) is given by

$$u_0 = A\left(T_1, T_2\right)\exp\left(iT_0/2\right) + cc \tag{3.12.12}$$

Where $i = \sqrt{-1}$ and ‘cc’ indicates the complex conjugate of the preceding terms. Substituting the above equation in Eq. (3.12.10) we get

$$D_0^2 u_1 + \frac{1}{4} u_1 = \begin{bmatrix} \left\{ \frac{i\xi_1}{4\omega^2} A - \frac{\sigma_1}{16\omega^2} A + 2 f_c \bar{A} + iD_1 A + \alpha_{e1} A^2 \bar{A} \right\} \exp(iT_0/2) \\ +2\left\{ \alpha_{e2} A^3 + f_c A \right\} \exp(3iT_0/2) \end{bmatrix} + cc \qquad (3.12.13)$$

Where

$$\alpha_{e1} = \left(3\alpha/4\omega^2\right) + \beta/4 - 3\Gamma/4 \qquad (3.12.14)$$

$$\alpha_{e2} = \frac{1}{8}\left(\alpha/\omega^2 - \beta - \Gamma\right) \qquad (3.12.15)$$

$$f_c = -f/16\omega^2 \qquad (3.12.16)$$

To eliminate the secular terms from Eq. (3.12.13)

$$iD_1 A + \left\{ \frac{i\xi_1}{4\omega^2} A - \frac{\sigma_1}{16\omega^2} A + 2 f_c \bar{A} + \alpha_{e1} A^2 \bar{A} \right\} = 0 \qquad (3.12.17)$$

Hence from eq^n (3.12.13) one may write

$$u_1 = \left\{ f_c A + \alpha_{e2} A^3 \right\} \exp(3iT_0/2) + cc \qquad (3.12.18)$$

Substituting the expressions for u_1 in to Eq. (3.12.11) and eliminating the secular terms, one obtains

$$\begin{aligned} &4\omega^2 \left\{ iD_2 A + D_1^2 A + i(\beta - \Gamma)\bar{A} \right\} + i(\sigma_1 + 2\xi_1) D_1 A \\ &+ \left\{ i\xi_2 - \frac{1}{4}\sigma_2 \frac{1}{2} ff_c \right\} A - \frac{1}{2} f \alpha_{e2} A^3 \\ &+ \frac{1}{4}\sigma_1 (3\Gamma - \beta) A^2 \bar{A} + \alpha_{e3} \left(f_c A + \alpha_{e2} A^3 \right) \bar{A}^2 = 0 \end{aligned} \qquad (3.12.19)$$

Where $\alpha_{e3} = 3\alpha + \omega^2 (5\beta - 11\Gamma)$

In the above equation, the terms containing D_1 vanish as they are independent of the T_2 time scale [1-4].

Now, Eq. (3.12.17) and (3.12.19) can be combined to describe the modulation of the complex amplitude to the second non linear order with respect to the original time scale τ using

$$\frac{dA}{d\tau} = \varepsilon D_1 A + \varepsilon^2 D_2 A \qquad (3.12.20)$$

Hence, one has

$$-i\frac{dA}{d\tau} = -i\xi_{o1} A + \phi_{o1} A + \phi_{o2} \bar{A} + \alpha_{f1} A^2 \bar{A} + \alpha_{f2} A \bar{A}^2 + \alpha_{f3} A^3 + \alpha_{f4} A^3 \bar{A}^2 \qquad (3.12.21)$$

Where

$$\xi_{o1}=\frac{\xi_0}{4\omega^2},\ \phi_{o1}=\frac{\phi^2-4\omega^2}{16\omega^2}-\frac{f_0^2}{128\omega^2},\ \phi_{o2}=\frac{f_0^2}{8\omega^2},$$
$$\alpha_{f1}=-\frac{3\alpha_0}{4\omega^2}+\frac{\phi^2}{16\omega^2}\left(-\beta_0+3\Gamma_0\right),\ \alpha_{f2}=\frac{f_0}{64\omega^4}\left\{3\alpha_0+\omega^2\left(\beta_0+11\Gamma_0\right)\right\}, \tag{3.12.22}$$
$$\alpha_{f2}=\frac{f_0}{64\omega^4}\left\{3\alpha_0+\omega^2\left(\beta_0+11\Gamma_0\right)\right\},\ \alpha_{f3}=\frac{f_0}{64\omega^4}\left\{\alpha_0-\omega^2\left(\beta_0+11\Gamma_0\right)\right\},$$
$$\alpha_{f47}=-\left\{3\alpha_0+\omega^2\left(5\beta_0-11\Gamma_0\right)\right\}\left\{\alpha_0-\omega^2\left(\beta_0+\Gamma_0\right)\right\}/\left(32\omega^4\right)$$

Here, all the expansion terms recombine in to the original expression. Substituting the complex amplitude $A=(1/2)a\exp(i\theta)$ (where a and θ are real), in Eq. (3.12.21) and separating the real and imaginary parts, one obtains

$$\dot{a}=-\xi_{o1}a+\left\{\frac{1}{4}\left(\alpha_{f3}-\alpha_{f2}\right)a^3-\phi_{02}a\right\}\sin\left(2\phi\right) \tag{3.12.23}$$

$$a\dot{\theta}=-\left\{\phi_{o1}a+\frac{1}{4}\alpha_{f1}a^3+\frac{1}{16}\alpha_{f4}a^5\right\}-\left\{\frac{1}{4}\left(\alpha_{f3}+\alpha_{f2}\right)a^3+\phi_{o2}a\right\}\cos\left(2\theta\right) \tag{3.12.24}$$

Steady-state responses can be determined by setting the time derivatives to zero. Use of the trigonometric identity $\sin^2(2\theta)+\cos^2(2\theta)=1$, yields

$$k_7a^{12}+k_6a^{10}+k_5a^8+k_4a^6+k_3a^4+k_2a^2+k_1=0 \tag{3.12.25}$$

whose solution will give rise to the non linear response of the system. The coefficients k_1, k_2, … k_7 are defined in Appendix . this equation is solved numerically to find the six roots of a^2, out of which only two roots are real and the other roots are either negative or complex.

Now, the displacement u can be expressed as

$$u=a\cos(\theta+\tau/2)+f_{c0}a\cos(\theta+3\tau/2)+0.25\alpha_{e20}a^3\cos(3\theta+3\tau/2) \tag{3.12.26}$$

Where $f_{c0}=-f_0/16\omega^2$ and $\alpha_{e20}=\left(\alpha_0/\omega^2-\beta_0\Gamma_0\right)/8$

The stability of the system is studied in the usual manner by finding the egen values of the Jacobian matrix obtained by perturbing Eq. (3.12.23) and (3.12.24).

Method of Multiple Scales: Version I (original method)

Here, instead of expanding the detuning up to the second non linear order of ε, the detuning in the excitation is introduced as

$$\phi^2=4\omega^2+\varepsilon\sigma_1 \tag{3.12.27}$$

Also, substituting in Eq. (3.12.2) the same expressions for time scales T_0, T_1, T_2 and displacement u as in the case of MMS version II, and equating the coefficients of $\varepsilon^n(n=0,1,2,...)$ to zero , one gets

order of ε^0

$$4\omega^2D_0^2u_0+\omega^2u_0=0 \tag{3.12.28}$$

order of ε^1

$$4\omega^2\left(D_0^2u_1+2D_0D_1u_0\right)\omega^2u_1+2\xi_1D_0u_0+\sigma_1D_0^2u_0$$
$$-fu_0\cos\tau+\alpha u_0^3+4\omega^2\left\{\beta\left(D_0u_0\right)^2u_0+\Gamma u_0^2D_0^2u_0\right\}=0 \tag{3.12.29}$$

order of ε^2

$$\begin{aligned}&4\omega^2D_0^2u_2+\omega^2u_2+\sigma_1D_0^2u_1+2\sigma_1D_0D_1u_0+8\omega^2D_0D_1u_1+4\omega^2D_1^2u_0\\&+8\omega^2D_0D_2u_0+2\xi_1\left(D_1u_0+D_0u_1\right)-fu_1\cos\tau+3\alpha u_0^2u_1\\&+4\omega^2\left[\begin{array}{l}\beta\left\{2u_0\left(D_0u_0\right)\left(D_0u_1+\left(D_1u_0\right)+\left(D_1u_0\right)^2u_1\right)\right\}\\+\Gamma\left\{u_0^2\left(D_0^2u_1+2D_0D_1u_0\right)+2u_0u_1D_0^2u_0\right\}\end{array}\right]\\&+\sigma_1\left\{\beta\left(D_0u_0\right)^2u_0+\Gamma u_0^2\left(D_0^2u_0\right)\right\}=0\end{aligned} \tag{3.12.30}$$

where $\xi_1=\phi\xi$.

One may note that Eq. (3.12.28) and (3.12.29) are identical to Eq. (3.12.10) and (3.12.11), respectively. However, the detuning used in both cases are different. Hence, in the case of MMS version I

$$u_0=A(T_1,T_2)\exp(iT_0/2)+cc \tag{3.12.31}$$

$$u_1=\left\{f_cA+\alpha_{e2}A^3\right\}\exp(3iT_0/2)+cc \tag{3.12.32}$$

$$iDS_1A+\left\{\frac{i\xi_1}{4\omega^2}A-\frac{\sigma_1}{16\omega^2}A+2f_c\bar{A}+\alpha_{e1}A^2\bar{A}\right\}=0 \tag{3.12.33}$$

where $f_c=-f/\left(16\omega^2\right)$, $\alpha_{e1}=\left(3\alpha+\omega^2\beta-3\omega^2\Gamma\right)/\left(4\omega^2\right)$ and $\alpha_{e2}=\left(\alpha-\omega^2\beta-\omega^2\Gamma\right)/\left(8\omega^2\right)$ as in the previous version.

Substituting of the expressions for u_0 and u_1 in Eq. (3.12.30) and elimination of the secular terms yield.

$$\begin{aligned}&4\omega^2\left(iD_2A+D_1^2A\right)+\left(i\sigma_1+2\xi_1+8\omega^2\Gamma A\bar{A}\right)D_1A+4\omega^2i\left(\beta-\Gamma\right)A^2D_1\bar{A}\\&-\frac{1}{2}f\left(f_cA+\alpha_{e2}A^3\right)+\left(f_cA+\alpha_{e2}A^3\right)\left(3\alpha+5\omega^2\beta11\omega^2\Gamma\right)\bar{A}^2+\frac{\sigma_1}{4}\left(\beta-3\Gamma\right)A^2\bar{A}=0\end{aligned} \tag{3.12.34}$$

Inserting the expressions for D_1A from Eq. (3.12.33) in Eq. (3.12.34) and using

$$\frac{dA}{d\tau}=\varepsilon D_1A+\varepsilon^2D_2A \tag{3.12.35}$$

One obtains

$$\begin{aligned}\dot{A}=&\left\{\left(\frac{-\varepsilon\xi_1}{4\omega^2}+\frac{\varepsilon^2\sigma_1\xi_1}{16\omega^4}\right)+i\left(\frac{-\varepsilon\xi_1}{16\omega^2}+\frac{3\varepsilon^2\sigma_1^2}{256\omega^4}-\frac{\varepsilon^2\xi_1^2}{16\omega^4}+\frac{3\varepsilon^2f^2}{128\omega^4}\right)\right\}A\\&+i\left\{\frac{\varepsilon f}{8\omega^2}+\frac{\varepsilon^2\sigma_1 f}{32\omega^4}\right\}\bar{A}+\left\{\varepsilon^2\frac{\xi_1}{4\omega^2}\alpha_{e4}+i\left(\varepsilon\alpha_{e1}-\varepsilon^2\frac{\alpha\sigma_{e1}}{8\omega^2}\right)\right\}A^2\bar{A}\\&+i\varepsilon^2\left\{\left(\frac{f\alpha_{e5}}{16\omega^2}\right)A\bar{A}^2+\left(\frac{f\alpha_{e5}}{16\omega^2}\right)A^3+\left(\alpha_{e1}\alpha_{e7}+\alpha_{e2}\alpha_{e3}\right)A^3\bar{A}^2\right\}\end{aligned} \tag{3.12.36}$$

Where

$$\alpha_{e4}=\frac{3\alpha}{2\omega^2}+\frac{3\beta}{2}-\frac{\Gamma}{2},\quad \alpha_{e5}=\frac{3\alpha}{4\omega^2}-\frac{3\beta}{4}+21\frac{\Gamma}{4}$$
$$\alpha_{e6}=\frac{-7\alpha}{8\omega^2}-\frac{9\beta}{8}+\frac{15\Gamma}{8},\quad \alpha_{e7}=\frac{-3\alpha}{4\omega^2}+\frac{3\beta}{4}-\frac{9\Gamma}{4} \tag{3.12.37}$$

Now, to find the nonlinear response, as in the previous case, substituting $A=(1/2)a\exp(i\theta)$ (where a and θ are real), and separating the real and imaginary parts, one arrives at

$$\dot{a}=K_{11}a+K_{31}a^3+\left\{K_{22}a+\left(K_{42}-K_{52}\right)a^3\right\}\sin(2\theta) \tag{3.12.38}$$

$$a\dot{\theta}=K_{12}a+K_{32}a^3+K_{62}a^5+\left\{K_{22}a+\left(K_{42}+K_{52}\right)a^3\right\}\cos(2\theta) \tag{3.12.39}$$

For a steady-state response, $\dot{a}=\dot{\theta}=0$. Using the trigonometric identity $\sin^2(2\theta)+\cos^2(2\theta)=1$, one has the following polynomial expressions for amplitude a

$$B_7a^{12}+B_6a^{10}+B_5a^8+B_4a^6+B_3a^4+B_2a^2+B_1=0 \tag{3.12.40}$$

Where the expressions for $K_{11},....,K_{62}$ and $B_1,....,B_7$ are given in Appendix . The steady state response of the system is found by numerically solving the above equation. Out of the six roots obtained for a^2, only two roots have physical significance as the other roots are either complex or negative real numbers. The stability of the steady-state response is studied by perturbing Eq. (3.12.38) and (3.12.39) and finding the egenvalues of the resulting Jacobian matrix.

Appendix:

$$c_1=4\xi_{01},\ c_2=\alpha_{f3}-\alpha_{f2},\ c_3=4\phi_{02},\ c_4=4\phi_{01},\ c_5=\alpha_{f1},\ c_6=0.25\alpha_{f4},\ c_7=\alpha_{f2}+\alpha_{f3}$$

$$k_1=c_3^2\left(c_1^2+c_4^2-c_3^2\right),\ k_2=2c_1^2c_3c_7-2c_4^2c_3c_2+2c_3^2\left(c_4c_5+c_2c_3-c_3c_7\right)$$

$$k_3=c_1^2c_7^2+c_4^2c_2^2+2c_3^2c_4c_6+c_3^2c_{5.}^2-4c_3c_2c_4c_5-c_3^2\left(c_2^2c_7^2\right)+4c_2c_3^2c_7$$

$$k_4=2c_3^2c_5c_6+2c_4c_5c_2^2-4c_3c_2c_4c_6-2c_3c_2c_5^2+2c_2c_3c_7^2-2c_3c_7c_2^2$$

$$k_5=c_3^2c_6^2+c_5^2c_2^2+2c_4c_6c_2^2-4c_3c_2c_5c_6-c_2^2c_7^2,\ k_6=-2c_3c_2c_6^2+2c_5c_6c_2^2,\ k_7=\left(c_2c_6\right)^2$$

$$k_{11}=\frac{-\xi_0}{4\omega^2}+\frac{\left(\phi^2-4\omega^2\right)\xi_0}{16\omega^4},\ k_{12}=\frac{\left(\phi^2-4\omega^2\right)}{16\omega^2}+\frac{3\left(\phi^2-4\omega^2\right)^2}{256\omega^4}+\frac{\xi_0^2}{16\omega^4}+\frac{3f_0^2}{128\omega^4}$$

$$k_{22}=\frac{f_0}{8\omega^2\left\{1+\dfrac{\phi^2-4\omega^2}{4\omega^2}\right\}},\ k_{31}=\varepsilon\frac{\xi_0}{16\omega^2}\alpha_{e4},\ k_{32}=\frac{\varepsilon}{4}\left(\alpha_{e1}-\frac{\left(\phi^2-4\omega^2\right)\alpha_{e1}}{8\omega^2}\right)$$

$$k_{42}=\varepsilon\left(\frac{f_0\alpha_{e5}}{64\omega^2}\right),\ k_{52}=\varepsilon\left(\frac{f_0\alpha_{e6}}{32\omega^2}\right),\ k_{62}=\varepsilon^2\left(\alpha_{e1}\alpha_{e7}+\alpha_{e2}\alpha_{e3}\right)/16$$

$$B_1=K_{22}^2\left(K_{11}^2+k_{12}^2-K_{22}^2\right)$$

$$B_2=2K_{11}^2\left(K_{42}+K_{52}\right)K_{22}+K_{12}^2K_{22}\left(K_{42}-K_{52}\right)+2K_{22}^2\left(K_{11}K_{31}+K_{32}K_{12}-2K_{22}K_{42}\right)$$

$$B_3 = K_{22}^2\left(K_{31}^2 + K_{32}^2 + 2K_{12}K_{62} - 6K_{42}^2 + 2K_{52}^2\right) + 4K_{22}K_{32}K_{12}\left(K_{42} - K_{52}\right) + K_{11}^2\left(K_{42} + K_{52}\right)^2$$
$$+ 4K_{11}K_{31}K_{22}\left(K_{42} + K_{52}\right) + K_{12}^2\left(K_{42} - K_{52}\right)^2$$

$$B_4 = -2K_{31}^2K_{22}\left(K_{42} + K_{52}\right) + 2K_{11}K_{31}\left(K_{42} + K_{52}\right)^2 + 2K_{22}^2K_{32}K_{62}$$
$$+ 2\left(K_{32}^2 + 2K_{12}K_{62}\right)K_{22}\left(K_{42} + K_{52}\right)^2 + 2K_{12}K_{32}\left(K_{42} - K_{52}\right)^2 - 4K_{22}K_{42}\left(K_{42}^2 - K_{52}^2\right)$$

$$B_5 = K_{31}^2\left(K_{42} + K_{52}\right)^2 + \left(K_{62}K_{22}\right)^2 + 4K_{32}K_{62}K_{22}\left(K_{42} - K_{52}\right)$$
$$+ \left(K_{32}^2 + 2K_{12}K_{62}\right)\left(K_{42} - K_{52}\right)^2 - \left(K_{42}^2 - K_{52}^2\right)^2$$

$$B_6 = 2K_{62}\left(K_{42}K_{52}\right)\left\{K_{62}K_{22} + K_{32}\left(K_{42} - K_{52}\right)\right\}$$

$$B_7 = \left\{K_{62}\left(K_{42} - K_{52}\right)\right\}^2$$

Exercise problem:

1. Use second order method of multiple scale (version II) to find the frequency response equations for primary resonance of the Duffing equation with cubic nonlinearity and a weak forcing function.

$$\ddot{u} + 2\varepsilon\xi\dot{u} + \omega^2 u + \varepsilon\alpha u^3 = \varepsilon f \cos\tau$$

Taking $\xi = 0.2$, $\omega = 1$, $\alpha = 0.8$ and $f = 0.5$, write a Matlab code to plot the frequency response curves for different values of book-keeping parameter.

2. Use second order method of multiple scale (version II) method to find the frequency response equations for the Duffing equation with quadratic and cubic nonlinearity.

3. Use second order method of multiple scale (version II) method to find the equations for frequency response for the Rayleigh equation. Use any symbolic software (Maple/Mathematica) to derive the equations. Also plot the time response and phase portrait to show the limit cycle.

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For further study of the interested reader:

Here a list of references is provided which will give an idea about the other methods used to solve the nonlinear differential equations for the vibrating system. Some reliable methods for obtaining exact solutions of nonlinear differential equations are given below.

- The inverse scattering transform
- the Hirota linear method
- the Bäcklund transformation
- the homogeneous balance method
- the exp-function method
- the Jacobi elliptic function expansion method
- the F-expansion method
- the auxiliary equation method
- the tanh method
- the simplest equation method

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