NONLINEAR VIBRATIONS

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Module 1 Lecture 1

In this lecture the course outline and the module and lecture wise breakup of the nonlinear vibration course are discussed. Also, the list of reference books, journals have been given.

Course Outline

Introduction: linear and nonlinear systems, conservative and non-conservative systems; potential well, Phase planes, types of forces and responses, fixed points, periodic, quasi-periodic and chaotic responses; Local and global stability; commonly observed nonlinear phenomena: multiple response, bifurcations, jump phenomena.

Development of nonlinear governing equation of motion of Mechanical systems, linearization techniques, ordering techniques; commonly used nonlinear equations: Duffing equation, Van der Pol's oscillator, Mathieu's and Hill's equations.

Analytical solution methods: Harmonic balance, perturbation techniques (Linstedt-Poincare', method of Multiple Scales, Averaging – Krylov-Bogoliubov-Mitropolsky), incremental harmonic balance, modified Lindstedt Poincare' techniques.

Stability and bifurcation analysis: static and dynamic bifurcations of fixed point and periodic response, different routes to chaotic response (period doubling, torus break down, attractor merging etc.), crisis.

Numerical techniques: time response, phase portrait, FFT, Poincare' maps, point attractors, limit cycles and their numerical computation, strange attractors and chaos; Lyapunov exponents and their determination, basin of attraction: point to point mapping and cell to cell mapping.

Application: Single degree of freedom systems: Free vibration-Duffing's oscillator; primary-, secondary-and multiple- resonances; Forced oscillations: Van der Pol's oscillator; parametric excitation: Mathieu's and Hill's equations, Floquet theory; effects of damping and nonlinearity. Multi degree of freedom and continuous systems.

Course Pre-requisites : Mechanical Vibration, Engineering Mechanics

Text/References

1. Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, Wiley-Interscience, 1979.
2. Hayashi, C. *Nonlinear Oscillations in Physical Systems*, McGraw-Hill, 1964.
3. Evan-Ivanowski, R. M., *Resonance Oscillations in Mechanical Systems*, Elsevier, 1976.
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5. Seydel, R., *From Equilibrium to Chaos: Practical Bifurcation and Stability Analysis*, Elsevier, 1988.
6. Moon, F. C., *Chaotic & Fractal Dynamics: An Introduction for Applied Scientists and Engineers*, Wiley, 1992.
7. Rao, J. S., *Advanced Theory of Vibration: Nonlinear Vibration and One-dimensional Structures*, New Age International, 1992.

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9. A. H. Nayfeh, Introduction to Perturbation Techniques, Wiley, 1981

10.Wanda Szemplinska-Stupnicka, The Behavior of Nonlinear Vibrating Systems, Vol 1 &2, Kluwer Academic Publishers, 1990

11. Matthew Cartmell, Introduction to Linear, Parametric and Nonlinear Vibrations, Chapman and Hall, 1990.

12. T. S. Parker and L. O. Chua: Practical Numerical Algorithms for Chaotic Systems, Springer-Verlag, 1989

13. A. H. Nayfeh, Method of Normal forms, Wiley, 1993.

Journals

- International Journal of Non-linear Mechanics (ELSEVIER)
- Nonlinear Dynamics (SPRINGER)
- Journal of Sound and Vibration (ELSEVIER)
- Journal of Vibration and Acoustics (ASME)
- Journal of Dynamical Systems, Measurements and Control (ASME)
- Physics D: Nonlinear Phenomena (ELSEVIER)
- Chaos, Solitons and Fractals (ELSEVIER)
- International Journal of Nonlinear Sciences and Numerical Simulation, (Freund Publishing House)
- Journal of Computational and Nonlinear Dynamics (ASME)

Detailed Course Plan : (Module wise / Lecture wise)

Sl, No	Module	Lecture No	Content
1	1 Introduction	1	Mechanical vibration: Linear and nonlinear systems, types of forces and responses, Review of linear system: free vibration.
2		2	Review of Linear system: SDOF forced vibration, two degrees of freedom and continuous system,
3		3	Introduction to qualitative analysis of conservative systems, equilibrium points, potential well, centre, focus, saddle-point, cusp point Commonly observed nonlinear phenomena: multiple response, bifurcations, and jump phenomena and basin of attraction.
4	2 Derivation of nonlinear equation of motion	1	Force and moment based approach:single degree of freedom system
5		2	Force and moment based approach: multi degree of freedom system
6		3	d' Alembert's Principle: Continuous System
7		4	Extended Hamilton's Principle
8		5	Lagrange Principle
9		6	Development of temporal equation using Galerkin's method for

10		7	continuous system
10		7	Ordering techniques, scaling parameters, book-keeping parameter. Commonly used nonlinear equations
11	**3**	1	Straight forward expansions and sources of nonuniformity
12	**Approximate**	2	Linstedt-Poincare' method
13	**solution**	3	Modified Lindstedt-Poincare' Technique
14	**method**	4	Method of multiple scales
15		5	Method of multiple scales: Applied to forced vibration system Method of Averaging
16	**Perturbation**	6	Harmonic Balancing method
17	**analysis**	7	Method of Averaging
18	**method**	8	Generalized Method of Averaging
19		9	Method of normal form
20		10	Incremental Harmonic Balance method
21		**11**	INTRINSIC MULTIPLE SCALE HARMONIC BALA METHOD
22		**12**	HIGHER ORDER METHOD OF MULTIPLE SCALES
23	**4**	1	Stability analysis of fixed point response
24	**Stability and**	2	Bifurcation analysis of fixed point response
25	**Bifurcation**	3	Stability analysis of Periodic response
26	**Analysis**	4	Limit cycles and Bifurcation of Periodic Response
27		5	Quasi-periodic and Chaotic response
28	**5** **Numerical techniques**	1	Review of numerical solution of algebraic equations, solution of differential equations to obtain time response of nonlinear systems
29		2	Methods of model reduction and continuation techniques
30		3	Poincare section, basin of attraction and Liapunov exponent
31	**6** **Applications**	1	Single degree of freedom Nonlinear conservative systems with Cubic nonlinearities.
32		2	Single degree of freedom nonlinear conservative systems with quadratic and Cubic and nonlinearities.
33		3	Single degree of freedom non-conservative systems: viscous damping, quadratic and Coulomb damping
34		4	Non-conservative systems: Negative damping, van der Pol oscillator,simple pendulum with quadratic damping
37		5	Single degree of freedom Nonlinear systems with Cubic nonlinearities: Primary Resonance
36		6	Single degree of freedom nonlinear systems with Cubic nonlinearities: Nonresonant Hard excitation
35		7	Single degree of freedom Nonlinear systems with Cubic and quadratic nonlinearities and self sustained oscillations
38		8	Multi-degree of freedom nonlinear systems
39		9	Parametrically excited system: Floquet theory, Hill's infinite determinant
40		10	Parametric Instability region: sandwich beam vibration
41		11	Base excited magneto-elastic cantilever beam with tip mass
42		12	System with internal resonance: Two-mode interaction: Base excited cantilever beam with tip mass at arbitrary position

INTRODUCTION TO NONLINEAR MECHANICAL SYSTEMS

In this lecture the vibration of linear and nonlinear dynamical systems have been briefly discussed. Both inertia and energy based approaches have been introduced to derive the equation of motion. Review of linear single degree of freedom system free vibration is carried out.

Introduction

Any motion that repeats itself after an interval of time is called vibration or oscillation. The swinging of a pendulum (Fig.1.1.1) and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motion of bodies and forces associated with them.

Elementary Parts of Vibrating system
- A means of storing potential energy (Spring or elasticity)
- A means of storing kinetic energy (Mass or inertia)
- A means by which energy is gradually lost (damper)

The forces acting on the systems are
- Disturbing forces
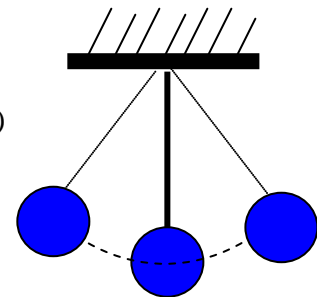- Restoring force
- Inertia force
- Damping force

Fig. 1.1.1: Swinging of a Pendulum

Degree of Freedom: The minimum number of independent coordinates required to determine completely the position of all parts of a system at any instant of time defines the degree of freedom of the system.

System with a finite number of degrees of freedom are called discrete or lumped parameter system, and those with an infinite number of degrees of freedom are called continuous or distributed systems.

Classification of Vibration:
- Free and forced
- Damped and undamped
- Linear and nonlinear
- Deterministic and Random

Free vibration: If a system after initial disturbance is left to vibrate on its own, the ensuing vibration is called free vibration.

Forced Vibration: If the system is subjected to an external force (often a repeating type of force) the resulting vibration is known as forced vibration

Damped and undamped: If damping is present, then the resulting vibration is damped vibration and when damping is absent it is undamped vibration. The damped vibration can again be classified as under-damped, critically-damped and over-damped system depending on the damping ratio of the system.

Linear vibration: If all the basic components of a vibratory system – the spring the mass and the damper behave linearly, the resulting vibration is known as linear vibration. Principle of superposition is valid in this case.

Nonlinear Vibration: If one or more basic components of a vibratory system are not linear then the system is nonlinear. All most all the system can be modelled as a nonlinear system.

Depending on excitation:

Deterministic: If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as deterministic vibration.

Random Vibration: In this cases the value of the excitation at any given time can not be predicted. Example. Wind velocity, road roughness and ground motion during earth quake etc.

2. Linear and Nonlinear systems

A system is said to be linear or nonlinear depending on the force response characteristic of the system. The block diagram relating to output *x(t)* and input *f(t)* of a dynamical system can be represented as shown in Fig 1.1.2(a) and mathematically represented as shown in Fig. 1.1.2(b).
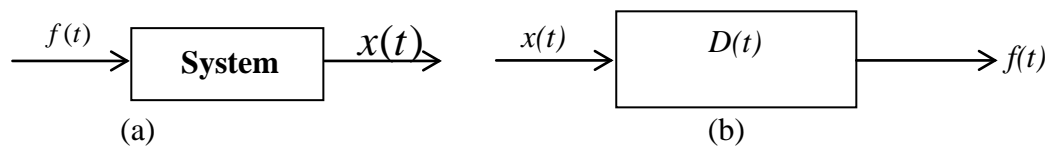

(a)   (b)

Fig.1.1. 2: (a) Block diagram showing the force-response and (b) mathematical representation of the input and the output through the operator *D(t)*.

A linear system may be of first or second order depending on the presence of the basic elements. A typical first order system with linear spring and viscous damping is shown in Fig. 1.1. 3(a) and that of a second order system is shown in Fig.1.1.3 (b) as they can be represented by $c\dot{x} + kx = F(t)$ and $m\ddot{x} + c\dot{x} + kx = F(t)$ respectively.

As shown in Fig. 1.1 2(b), a system can be represented by using a operator *D* such that *Dx(t) = f(t)*, where *D* is the differential operator, *x(t)* is the response and *f(t)* is the excitation input.

A system $Dx(t) = f(t)$ is said to be linear if it satisfies the following two conditions.

1. The response to $\alpha f(t)$ is $\alpha x(t)$, where α is a constant.
2. The response to $f_1(t) + f_2(t)$ is $x_1(t) + x_2(t)$ where the $x_1(t)$ is the response to $f_1(t)$ and $x_2(t)$ is the response to $f_2(t)$
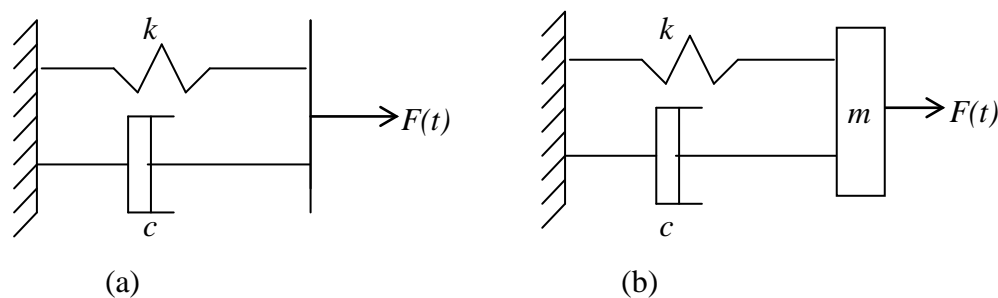

(a)   (b)

Fig 1.1.3(a) First order system (b) second order system

In case $D(\alpha\,x(t)) = \alpha\,Dx(t)$, the operator D and hence the system is said to posses homogeneity property and when $D\big[x_1(t) + x_2(t)\big] = Dx_1(t) + Dx_2(t)$ the system is said to posses *additive* property. If an operator D does not possess the homogeneity and additivity property the system is said to be nonlinear.

Example 1: Check whether system given by the following is linear or nonlinear

$$Dx(t) = a_0(t)\frac{d^2x(t)}{dt^2} + a_1(t)\frac{dx(t)}{dt} + a_2(t)\big[1 - \varepsilon x^2(t)\big]x(t) \qquad (1.1.1)$$

where, ε is a const ant
Solution: check the homogeneity

$$D\big[\alpha x(t)\big] = a_0(t)\alpha\frac{d^2x(t)}{dt^2} + a_1(t)\alpha\frac{dx(t)}{dt} + a_2(t)\alpha\big[1 - \varepsilon\alpha^2 x^2(t)\big]x(t) \neq \alpha\,Dx(t) \ (1.1.2)$$

Hence homogeneity condition is not satisfied
Similarly substituting $x(t) = x_1(t) + x_2(t)$ $\qquad\qquad\qquad\qquad (1.1.3)$
One obtains
$$D\big[x_1(t) + x_2(t)\big] \neq Dx_1t) + Dx_2(t). \qquad\qquad\qquad (1.1.4)$$

which does not satisfy additive property also. Hence the system is a nonlinear system. It may be noted that, the term containing ε causes the nonlinearity of the system. If $\varepsilon = 0$, the equation becomes linear by satisfying both homogeneity and additive properties.

Hence it may be observed that
1) A system is linear if the function $x(t)$ and its derivatives appear to the first (or zero) power only; otherwise the system is nonlinear.
2) A system is linear if a_0, a_1 and a_2 depend as time alone, or they are constant.

 Steps for Vibration Analysis
   - Convert the physical system to simplified mathematical model
   - Determine the equation of motion of the system
   - Solve the equation of motion to obtain the response
   - Interpretation of the result for the physical system.

To convert the physical system into simpler models one may use the concept of equivalent system. To determine the equation of motion basically one may use either the vector approach with the Newtonian approach or d'Alembert principle based on free body diagram or one may go for scalar approach using the energy concept. In scalar approach one may use Lagrange method, which is a differential procedure or extended Hamilton's principle based on integral procedure. Different methods/laws/principle used to determine the equations of motion of the vibrating systems are briefly introduced below. In module 2 they are described in detail.

Derivation of Equation of motion

Depending on coordinate:
 In Newtonian mechanics motions are measured relative to an inertial reference frame, i.e, a reference frame at rest or moving uniformly relatively to an average position of " fixed stars" and displacement, velocity and acceleration are absolute values.

Generalized coordinate: These are a set of independent coordinates same in number as that of the vibrating system. For example, the motion of a double pendulum in planar motion can be represented completely either by θ_1, θ_2 the rotation of the first and second link respectively or by x_1, y_1, x_2, y_2 the Cartesian coordinates of first and second links. While in the later case 4 coordinates are required to represent completely the system, in the former case only 2 coordinates are required for the same. Hence, in this case θ_1, θ_2 is the generalized co-ordinate while x_1, y_1, x_2, y_2 are not the generalized one. One may note that these four coordinates are not independent and can be reduced to two by the use of length constraint.
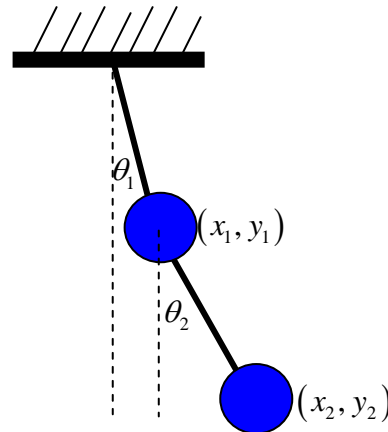
Fig. 1.1.4. Illustration of physical and generalized coordinates in a double pendulum

Newton's second law A particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.
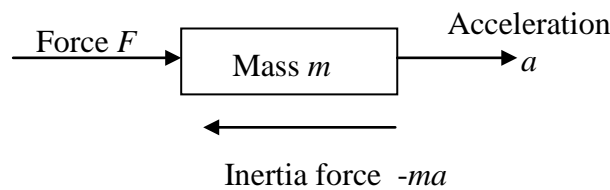

Inertia force $-ma$

Fig. 1.1.5 Application of Newton's second law

Taking, v_i -initial velocity, v_f -final velocity, and t time, according to Newton's 2nd law

$$F = m\left(\frac{v_f - v_i}{t}\right) = m\,a \tag{1.1.5}$$

Work energy principle

The work performed by a force F in moving a particle of mass M from position $\vec{r}_1$ to $\vec{r}_2$ is equal to the change in kinetic energy.

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}.d\vec{r} = \int_{r_1}^{r_2} d\left(\frac{1}{2}m\dot{\vec{r}}\ddot{\vec{r}}\right) = \frac{1}{2}m\dot{\vec{r}}_2.\dot{\vec{r}}_2 - \frac{1}{2}m\dot{\vec{r}}_1.\dot{\vec{r}}_1 = T_2 - T_1 \tag{1.1.6}$$

Here T$_1$ and T$_2$ are the Kinetic energy in position 1 and 2 respectively.
It can be shown that

- Force for which the work performed in moving a particle over a closed path is zero (considering all possible paths) are said to be *conservative force*.
- Work performed by a conservative force in moving a particle from $\vec{r}_1$ to $\vec{r}_2$ is equal to the negative of the change in potential energy from V_1 to V_2.
- Work performed by the nonconservative forces in carrying a particle from position r_1 to position r_2 is equal to the change in total energy

d'Alembert Principle The vectorial sum of the external forces and the inertia forces acting on a moving system is zero. Referring to Fig. 1.1.5 according to d'Alembert Principle $F + (-ma) = 0$ where $-ma$ is the inertia force.

Generalized Principle of d'Alembert:
The virtual work performed by the effective forces through infinitesimal virtual displacements compatible with the system constraints is zero.

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\,\ddot{\vec{r}}_i\right).\delta\vec{r}_i = 0 \qquad (1.1.7)$$

Extended Hamilton's Principle

For a system with Number of Particle one can conceive of a $3N$ dimensional space with the axes x_i, y_i, z_i and represent the position of the system of particles in that space and at any time t the position of a representative point P with coordinate $x_i(t)$, $y_i(t)$, $z_i(t)$ where i = 1,2,…N. The 3N dimensional space is known as the *Configuration Space.* As time unfolds, the representative point P traces a curve in the configuration space called the *true path, or the Newtonian path, or the dynamical path.* At the same time let us think of a different representative point P' resulting from imagining the system in a slightly different position defined by the virtual displacement δr_i (i = 1,2…N). As time changes the point P' traces a curve in the configuration space known as the *Varied Path.*
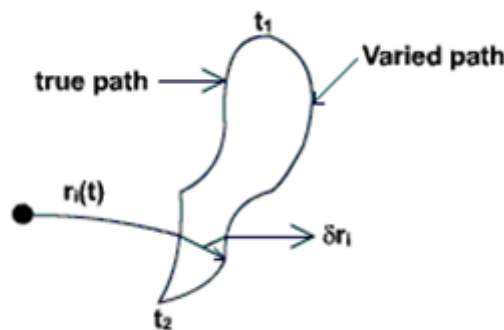

Fig1.1.6: True and Varied path

Of all the possible varied path, now consider only those that coincide with the true path at the two instants t_1 and t_2 as shown in Fig.1.1.6. the *Extended Hamilton's Equation* in terms of Physical coordinates q can be given by

$$\int_{t_1}^{t_2}\left(\delta T + \delta\bar{W}\right)dt = 0,\ \delta r_1(t_1) = \delta r_2(t_2) = 0, i = 1,2,....N \qquad (1.1.8)$$

where δT is the variation in kinetic energy and $\delta \bar{W}$ is the variation in the work done. But in many cases it is desirable to work with generalized coordinates. As δT and $\delta \bar{W}$ are independent of coordinates so one can write

$$\int_{t_1}^{t_2} \left(\delta T + \delta \bar{W} \right) dt = 0, \ \delta q_k(t) = \delta q_k(t) = 0 \tag{1.1.9}$$

where $k = 1, 2,…n$, n = no of dof of the system. The extended Hamilton's principle is very general and can be used for a large variety of systems. The only limitation is that the virtual displacement must be reversible which implies that the constraint forces must perform no work. Principle cannot be used for system with friction forces.

In general $\delta \bar{W} = \delta \bar{W}_c + \delta \bar{W}_{nc}$ (subscript c refers to conservative and nc refers to the nonconservative). Also, $\delta \bar{W}_c = -\delta V$. Now introducing Lagrangian $L = T - V$, the extended Hamilton's principle can be written as

$$\int_{t_1}^{t_2} \left(\delta L + \delta \bar{W}_{nc} \right) dt = 0, \quad \delta q_k(t_1) = \delta q_k(t_2) = 0, \tag{1.1.10}$$

where $k = 1, 2,…..n$

For conservative system $\delta \bar{W}_{nc} = 0$, Eq. (1.1.10) reduces to

$$\int_{t_1}^{t_2} \delta L dt = 0, \ \delta q_k(t_1) = \delta q_k(t_2) = 0 \tag{1.1.11}$$

which is known as the Hamilton's Principle.

Lagrange Principle

The Lagrange principle for a damped system can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) = Q_k \tag{1.1.12}$$

where

$$Q_k = \sum_i F_i . \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i M_i . \frac{\partial \omega_i}{\partial q_k}, k = 1, 2, ….n \tag{1.1.13}$$

where L is the Lagrangian given by $L=T-U$, T is the kinetic energy and U is the potential energy of the system. D is the dissipation energy and Q_k is the generalized force. F_i and M_i are the vector representation of the externally applied forces and moments respectively, the index k indicates which external force or moment is being considered, r_i is the position vector to the location where the force is applied, and ω_i is the system angular velocity about the axis along which the considered moment is applied.

Modeling of the system

- Single degree of freedom system

- Two degree of freedom system

- Multi-degree of freedom system

- Continuous system

Review of Linear system

Single Degree of Freedom Systems
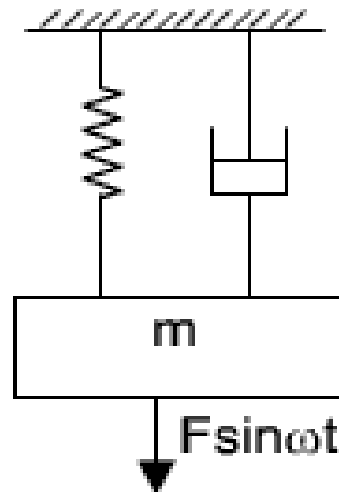
Steady state response due to Harmonic Oscillation

Fig 1.1.7 Spring –mass- damper system subjected to harmonic excitation

Equation of motion of the system with mass m, stiffness k, damping factor c and forcing amplitude F and forcing frequency ω can be given by the following equation.

$$m\ddot{x} + c\dot{x} + kx = F\sin\omega t \tag{1.1.14}$$

This can also be written as

$$\ddot{x} + \omega_n^2 x + 2\zeta\omega_n\dot{x} = (F/m)\sin\omega t \tag{1.1.15}$$

Here, $\omega_n \left(=\sqrt{k/m}\right)$ is the natural frequency, $\zeta \left(=\dfrac{c}{2\sqrt{mk}}=\dfrac{c}{2m\omega_n}\right)$ is the damping ratio

of the system.

For free vibration of the system the forcing term can be made zero and the equation can be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.1.16}$$

Free Vibration response of the system

- For undamped system ($\zeta = 0$)

$$x(t) = a\cos\omega_n t + b\sin\omega_n t \tag{1.1.17}$$

Here a and b are constants to be determined from the initial conditions. The system will vibrate with natural frequency and the response amplitude depends on the initial displacement and velocity of the system.

- For under damped system $\left(\zeta < 0\right)$

$$x(t) = X_1 e^{-\zeta \omega_n t} \sin(\sqrt{1-\zeta^2}\, \omega_n t + \psi)$$

(1.118)

Where X and ψ are constants to be determined from initial coditions

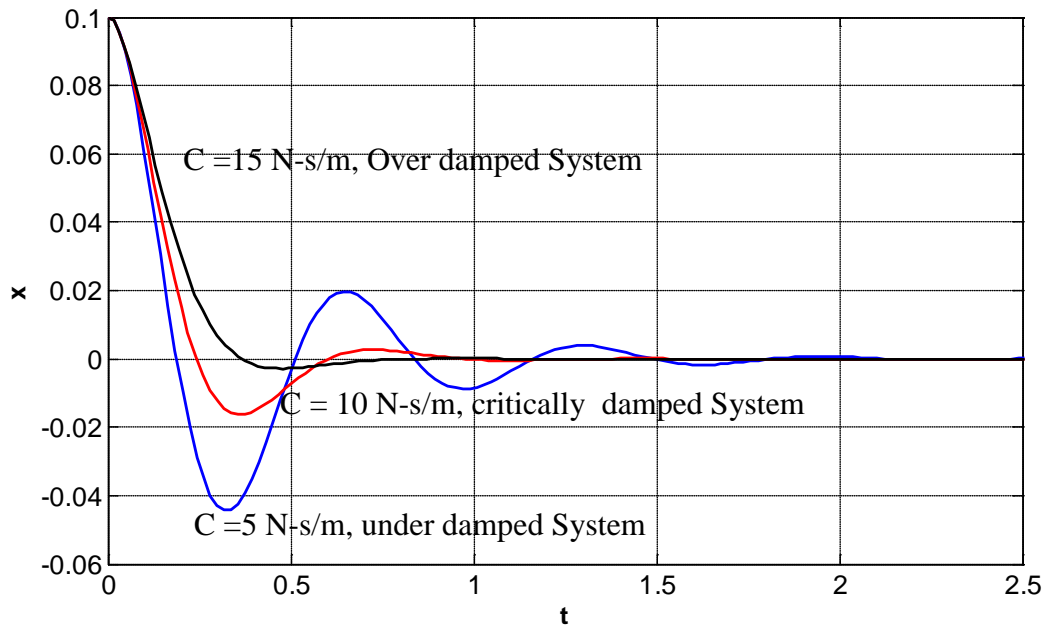- For critically damped system $\zeta = 1$
- For over damped system $\zeta > 1$

Fig. 1.1.8: Time response showing under-damped, over-damped and critically damped system. Here, $m = 1\,\text{kg}, k = 100$ N/m

Fig 1.1.8 shows the time response of a system with Mass m =1, stiffness K=1000w/m for 3 different values of damping representing under damped, critically damped and over damped system.

Forced vibration response

For an under damped system the solution of the () equation is given below.

$$x(t) = \underbrace{x_1 e^{-\zeta \omega_n t} \sin(\sqrt{1-\zeta^2}\, \omega_n t + \psi)}_{\text{Transient part}} + \underbrace{\frac{F}{\sqrt{(k-M\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \phi)}_{\text{Steady state response}}$$

(1.1.19)

Exercise Problem

1. Using principle of superposition check whether the following systems are linear or nonlinear.
(a) $2\ddot{x} + 100x + 10\dot{x} = 0.2\cos 2t + 0.5\sin 2t$
(b) $2\ddot{x} + 10\dot{x} + 100x + 10x^2 = 0.2\cos 2t$
(c) $2\ddot{x} + 10\dot{x} + 100x + 10x^3 = 0.2\cos 2t$

(d) $2\ddot{x} + 10\dot{x} + 100x + (2\cos 2t)x = 0$

(e) $2\ddot{x} + 10\dot{x} + 100x + (2\cos 2t)x + 5x^3 = 0$

2. Write a Matlab code to plot the response of a spring-mass-damper system subjected a harmonic force $F\sin\omega t$. Take
$m = 1\,\text{kg}, k = 100\,\text{N/m}$, c=100 N-s/m, F=1 N, (a) ω=2rad/s, (b) ω=10rad/s, (c)ω= 20 rad/s

Module 1 Lecture 2

Review on Linear Vibrating Systems

In this lecture review of the linear Vibrating system has been carried out. Here Problem related single, two and multi degree of freedom system have been discussed for both free and forced vibrant response. Also, analyzing of continuous system has been carried out.

Review of SDOF system with Harmonic forcing

Example 1.2.1:
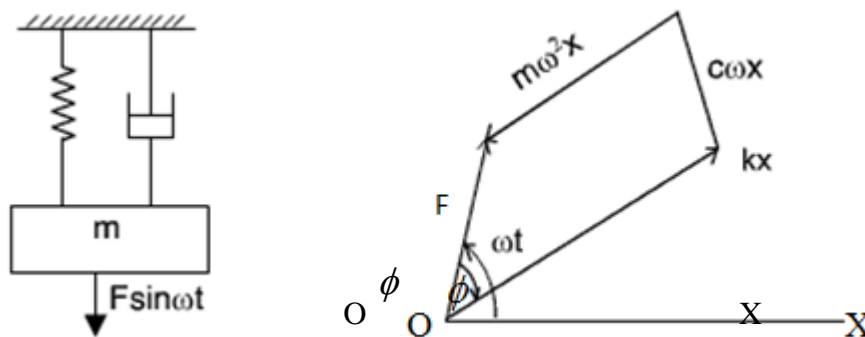Find the response of single degree of freedom systems with harmonic forcing.
Solution

Figure 1.2.1 (a) Spring-mass damper system (b) Force polygon

Figure 1.2.1(a) show a spring-mass-damper system subjected to harmonic forcing $F\sin(\omega t)$. Let m, c, k represent mass, stiffness and damping factor of the system. The equation of motion of the system can be given as

$$m\ddot{x} + c\dot{x} + kx = F\sin\omega t \qquad (1.2.1)$$

Taking OX as the reference line, the force polygon shows all the forces viz., spring force(kx), damping force($c\omega x$), inertia force($m\omega^2 x$) and external force(F). From the figure, it is clear that the angle between the external force and the displacement vector is ϕ. The steady state response of the system can be given by

$$x = X\sin(\omega t - \phi) \qquad (1.2.2)$$

Here, $X = \dfrac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$ (1.2.3)

and $\phi = \tan^{-1}\dfrac{c\omega}{k - m\omega^2}$ (1.2.4)

Recalling,

$$\boxed{\begin{aligned} &\omega_n = \sqrt{\dfrac{k}{m}} = \text{natural frequency} \\[2mm] &c_c = 2m\omega_n = \text{Critical damping} \\[2mm] &\zeta = \dfrac{c}{c_c} = \text{damping ratio} \\[2mm] &\dfrac{c\omega}{k} = \dfrac{c}{c_c}\dfrac{c_c}{k} = 2\zeta\dfrac{\omega}{\omega_n} \\[2mm] &\dfrac{c}{m} = 2\zeta\omega_n \end{aligned}}$$

One can write

$$X = \dfrac{X_0}{\sqrt{\left(1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\dfrac{\omega}{\omega_n}\right)\right)^2}} \quad \text{and} \quad \phi = \tan^{-1}\dfrac{2\zeta\left(\dfrac{\omega}{\omega_n}\right)}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}$$ (1.2.5)

The total response of the system is the summation of transient and steady state response which is given below.

$$X(t) = x_1 e^{-\zeta\omega_n t}\sin\left(\sqrt{1 - \zeta^2}\,\omega_n t + \phi_1\right) + \dfrac{F_0}{k}\dfrac{\sin\left(\omega t - \phi\right)}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}$$ (1.2.6)

As the ratio F/k is the static deflection (Xo) of the spring, $Xk/F = X/X_0$ is known as the **magnification factor or amplitude ratio** of the system. Figure 1.2.2 shows the magnification factor ~ frequency ratio and phase angle (ϕ)~ frequency ratio plot.

 Following observations can be made from these plots.

- For undamped system (i.e., $\zeta = 0$) the magnification factor tends to infinity when the frequency of external excitation equals natural frequency of the system ($\frac{\omega}{\omega_n} = 1.$).

- But for underdamped systems the maximum amplitude of excitation has a definite value and it occurs at a frequency $\frac{\omega}{\omega_n} < 1$.

- For frequency of external excitation very less than the natural frequency of the system, with increase in frequency ratio, the dynamic deflection (*X*) dominates the static deflection (*X₀*), the magnification factor increases till it reaches a maximum value at resonant frequency (ω_r).

- For $\omega > \omega_r$, the magnification factor decreases and for very high value of frequency ratio (say $\frac{\omega}{\omega_n} > 2$), the vibration is very much attenuated.

- One may observe that with increase in damping ratio, the resonant response amplitude decreases.

- Irrespective of value of ζ, at $\frac{\omega}{\omega_n} = 1$, the phase angle $\phi = 90^0$.

- For $\frac{\omega}{\omega_n} < 1$, phase angle $\phi < 90^0$.

- For, $\frac{\omega}{\omega_n} > 1$ phase angle ϕ approaches 180^0 for very low value of ζ.

- From phase angle ~frequency ratio plot it is clear that, for very low value of frequency ratio, phase angle tends to zero and at resonant frequency it is 90^0 and for very high value of frequency ratio it is 180^0.

- The resonant frequency $\omega_r = \sqrt{1 - 2\zeta^2}\,\omega_n$ and

- The resonant amplitude of vibration $X = \dfrac{X_0}{2\zeta\sqrt{1 - \zeta^2}}$
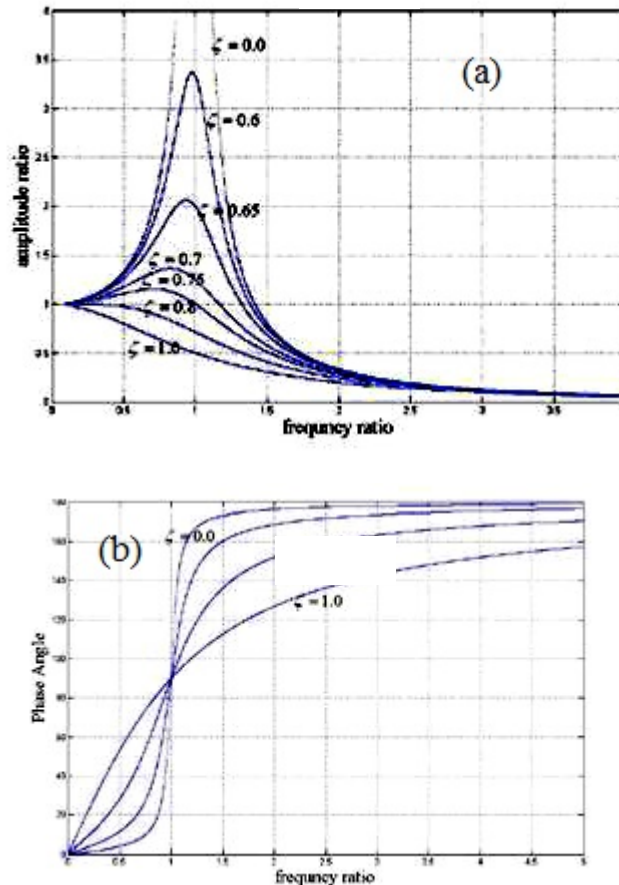
Figure 1.2.2: (a) Magnification factor ~ frequency ratio and (b) phase angle ~frequency ratio for different values of damping ratio.

Here, it may be noted that one can convert this linear spring-mass-damper system into a nonlinear system by introducing nonlinearity in mass, stiffness and damping terms.
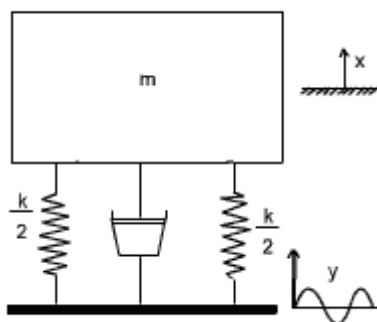
Support Motion:

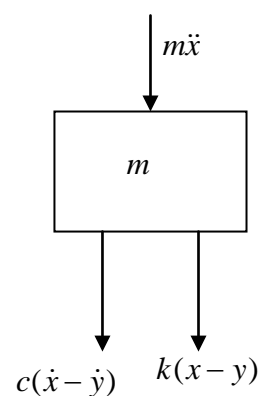Figure 1.2.3: A system subjected to support motion   Figure 1.2.4: Freebody diagram

Consider a system as shown in Fig. 1.2.3 where the support is moving with $y = y \sin \omega t$. It is required to find the motion of mass m which is supported by spring and damper with spring constant k and damping factor c. The equation of motion of the system is given by

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y}) \tag{1.2.7}$$

Substituting $z=x-y$ (1.2.8)

In Eq. (1.2.8) one obtains

$$m\ddot{z} + kz + c\dot{z} = -m\ddot{y} = m\omega^2 y \sin\omega t \tag{1.2.9}$$

The solution of this equation can be given by
$$z = Z\sin(\omega t - \phi) \tag{1.2.10}$$

where, $Z = \dfrac{m\omega^2 y}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$ and $\tan\phi = \dfrac{c\omega}{k-m\omega^2}$ (1.2.11)

Taking $x = X\sin(\omega t - \varphi)$ (1.2.12)

One can obtain

$$x = \operatorname{Im}\left[\left(\frac{k-m\omega^2 + ic\omega + m\omega^2}{k-m\omega^2 + ic\omega}\right)Ye^{i\omega t}\right] \tag{1.2.13}$$

From which one obtains

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{1+(2\zeta r)^2}{\left[1-(r)^2\right]^2 + (2\zeta r)^2}} \tag{1.2.14}$$
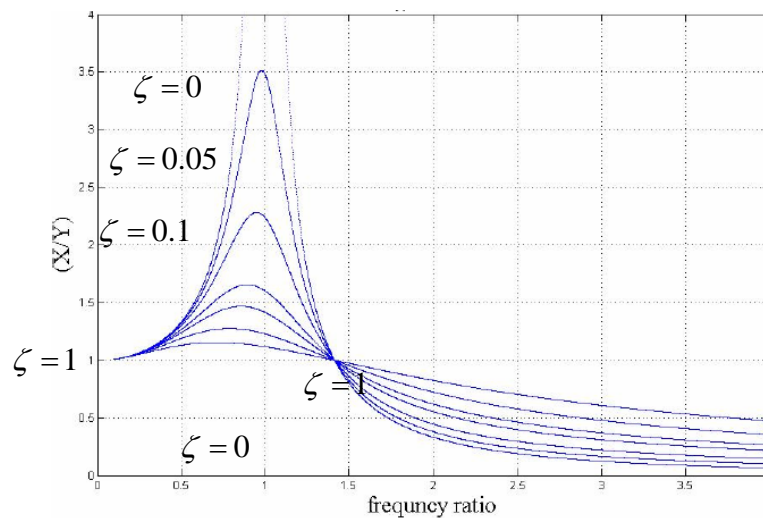
where $r = \omega/\omega_n$

Figure 1.2.5: Amplitude ratio ~ frequency ratio plot for system with support motion

From figure 1.2.5, it is clear that when the frequency of support motion nearly equals to the natural frequency of the system, resonance occurs in the system. This resonant amplitude decreases with increase in damping ratio for $r < \sqrt{2}$. At $r = \sqrt{2}$, irrespective of damping

ratio, the mass vibrate with an amplitude equal to that of the support and for $r > \sqrt{2}$, amplitude ratio becomes less than 1, indicating that the mass will vibrate with an amplitude less than the support motion. But with increase in damping, in this case, the amplitude of vibration of the mass will increase. So in order to reduce the vibration of the mass, one should operate the system at a frequency very much greater than $\sqrt{2}$ times the natural frequency of the system. This is the principle of vibration isolation.

One may consider a number of problems where the system can be reduced to that of a single-degree of freedom system. Next we will review about two degree of freedom system and continuous systems.

TWO DEGREE OF FREEDOM SYSTEMS

Tuned Vibration Absorber: Figure 1.2.6 (a) shows a spring mass system which can be thought of the model of a harmonically excited system. To absorb the vibration, generally another spring-mass is added to the primary system as shown in Fig. 1.2.6 (b). This system is a two degree of freedom system and the equation of motion of this system can be given by the following equation.
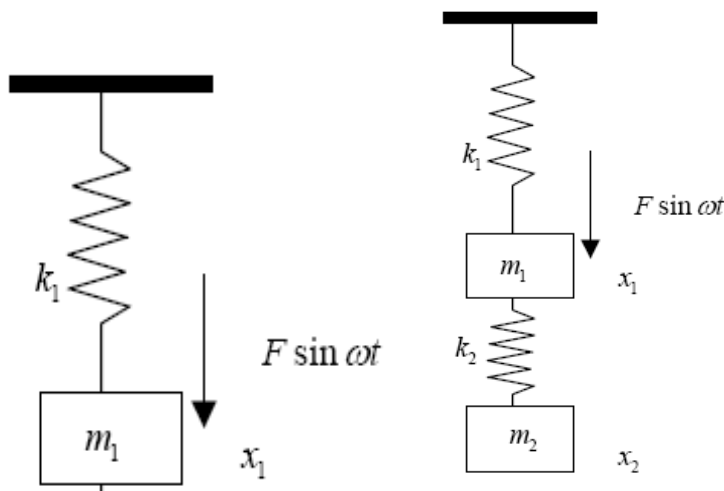

Fig. 1.2.6 (a) Single spring-mass system subjected to harmonic forcing, (b) secondary spring-mass added to the system shown in (a).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \sin \omega t \\ 0 \end{Bmatrix} \qquad (1.2.15)$$

In the absence of damping the steady state response of the primary system can be given by

$$X = \frac{F \sin(\omega t - \phi)}{m\left(\omega_n^2 - \omega^2\right)} \qquad (1.2.16)$$

Or, $\left| \dfrac{kX}{F} \right| = \left| \dfrac{1}{\left(1 - r^2\right)} \right|$ (1.2.17)

where, $r = \omega / \omega_n$ and $\omega_n = \sqrt{k_1 / m_1}$

Hence, when $r = 1$ or, $\omega = \omega_n$ the response of the system tends to infinity. Now one may find the steady state response of the system by substituting $x_1 = X_1 \sin \omega t$ and $x_2 = X_2 \sin \omega t$ in Eq.(1.2.15) which yield

$$\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t = \begin{Bmatrix} F \sin \omega t \\ 0 \end{Bmatrix}$$ (1.2.18)

Or, $\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$ (1.2.19)

If we want to make the primary system stationary i.e., $X_1 = 0$, from Eq. (1.2.19) one can write $\omega^2 = k_2 / m_2$. This is the condition for tuned vibration absorber. But in general the amplitude of response of the primary and secondary system can be written as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix}^{-1} \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$ (1.2.20)
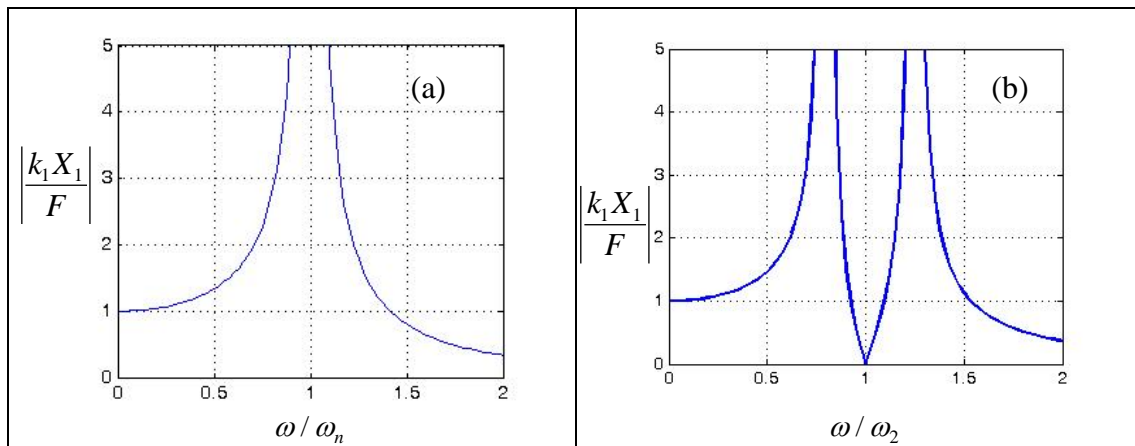

Figure 1.2.7 (a) Response of the primary system (b) response of the primary system in the presence of secondary mass and damper system.

Figure 1.2.7 (a) shows the steady state response of the primary system and (b) shows the response in the presence of secondary spring-mass system. It is clearly observed that when $\omega / \omega_2 = 1$, there is no vibration of the primary system. Hence, at this frequency the secondary system absorbs the vibration of the primary system and so the system is known as tuned vibration absorber.

For multi degree of freedom system one may revise the following points

- Normal mode of vibration: In this mode of vibration all the masses of the multi-degree of freedom system vibrating with same frequencies and passes the

equilibrium position at the same time. For example, in case of a double pendulum the two modes of vibration are shown in Fig 1.2.8.
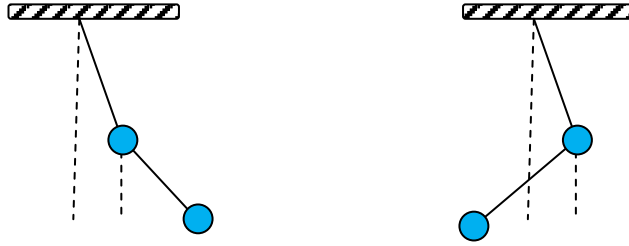

Fig. 1.2.8 (a) First mode (b) second mode of vibration of a double pendulum

If the first and second links make an angle of θ_1 and θ_2 with respect to the vertical axis, then the frequencies and first and second normal modes can be found as given below.

$$\omega_1 = \sqrt{\frac{g}{l}\left(2 - \sqrt{2}\right)} = 0.764\sqrt{\frac{g}{l}}, \omega_2 = \sqrt{\frac{g}{l}\left(2 + \sqrt{2}\right)} = 1.850\sqrt{\frac{g}{l}} \tag{1.2.21}$$

$$\phi_1 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{\lambda = \lambda_1} = \begin{Bmatrix} \theta_1 / \theta_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix}, \tag{1.2.22}$$

$$\phi_2 = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{\lambda = \lambda_2} = \begin{Bmatrix} \theta_1 / \theta_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix} \tag{1.2.23}$$

The resulting free vibration of the system is a combination of normal modes having different modal participation.

Hence
$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = c_1 \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_{\lambda = \lambda_1} + c_2 \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_{\lambda = \lambda_2} + \cdots + c_n \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_{\lambda = \lambda_n} \tag{1.2.24}$$

Or, $x = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \cdots + c_n\phi_n$ \qquad (1.2.25)

where c_n is the participation factor of the nth mode and ϕ_n is the nth normal mode obtained by finding the eigenvalues corresponding to the nth eigenvector of the dynamic matrix $A = M^{-1}K$. It may be noted that the eigenvector is equal to the square of the modal frequency of the system.

- Orthogonality principle of the normal modes: Normal modes are orthogonal. Hence, one may obtain diagonal / uncoupled mass and stiffness matrices by using this principle.
- Modal matrix: (P)The nth column vector of this matrix is the eigenvector corresponding to the nth eigenvalue of the dynamic matrix.
  So, $P = \begin{bmatrix} \phi_1 & \phi_2 & & \phi_n \end{bmatrix}$ \qquad (1.2.26)
- Generalized mass matrix: It is a diagonal matrix which is given by $M_g = P'MP$
- Generalized stiffness matrix : It is a diagonal matrix which is given by $K_g = P'KP$
- Weighted modal matrix ($\tilde{P}$): It is obtained when each column of the modal matrix is divided by the square root of the corresponding generalized mass (i.e., n^{th} column of P matrix is divided by square root of the nth generalized mass). It may be noted that $\tilde{P}'M\tilde{P} = I$ and $\tilde{P}'K\tilde{P} = \lambda$.

- Modal analysis: It is used to uncouple the equation of motion of the coupled multi-degree of freedom system. For example consider the coupled equation of motion
  $M\ddot{x} + Kx + c\dot{x} = F\sin\omega t$

  Where the mass matrix M, stiffness matrix K and damping matrix C are coupled matrices (i.e., they have off-diagonal terms). Now one may use any of the following procedure.

  Procedure 1:
  - Find the modal matrix P
  - Assuming Rayleigh damping one my write $C = \alpha M + \beta K$ (1.2.28)
  - Substitute $x = Py$ in Eq. $M\ddot{x} + Kx + C\dot{x} = F\sin\omega t$ and premultiply P' in both sides of the resulting equation.
  - So one obtains $P'MP\ddot{y} + P'KPy + \alpha P'MP\dot{y} + \beta P'KP\dot{y} = P'F\sin\omega t$ (1.2.28)
  - Here, all the matrices are diagonal matrices and one may solve the resulting equations as that of single degree of freedom system.

  Procedure 2

  Find the weighted modal matrix $\tilde{P}$

  Assuming Rayleigh damping one my write $C = \alpha M + \beta K$

  Substitute $x = \tilde{P}y$ in Eq. $M\ddot{x} + Kx + C\dot{x} = F\sin\omega t$ and pre multiply $\tilde{P}'$ in both sides of the resulting equation.

  So one obtains $\tilde{P}'M\tilde{P}\ddot{y} + P'K\tilde{P}y + \alpha\tilde{P}'M\tilde{P}\dot{y} + \beta\tilde{P}'K\tilde{P}\dot{y} = \tilde{P}'F\sin\omega t$ (1.2.29)

  Or, $I\ddot{y} + \lambda y + \alpha I\dot{y} + \beta\lambda\dot{y} = \tilde{P}'F\sin\omega t$ (1.2.30)

  Unlike the previous procedure here one has to calculate only the $\tilde{P}'F$ vector and then solve the resulting equations as that of single degree of freedom system.

  Normal mode summation method: Most of the time it is not required to consider all the modes of the system as only the first few modes play dominant role in the resulting vibration. Hence, instead of taking an $n \times n$ P or $\tilde{P}$ matrix, one may consider $m \times n$ matrix corresponding to the first m modes only. Now one may follow the above mentioned procedure where the number of resulting equations will be m only. Hence, this reduces the computational time and memory.

Continuous or Distributed Mass System:

Figure 1.2.10 shows few examples of continuous system. The first column shows the beams with fixed-fixed, simply supported, fixed-free (cantilevered) and free-free end conditions.
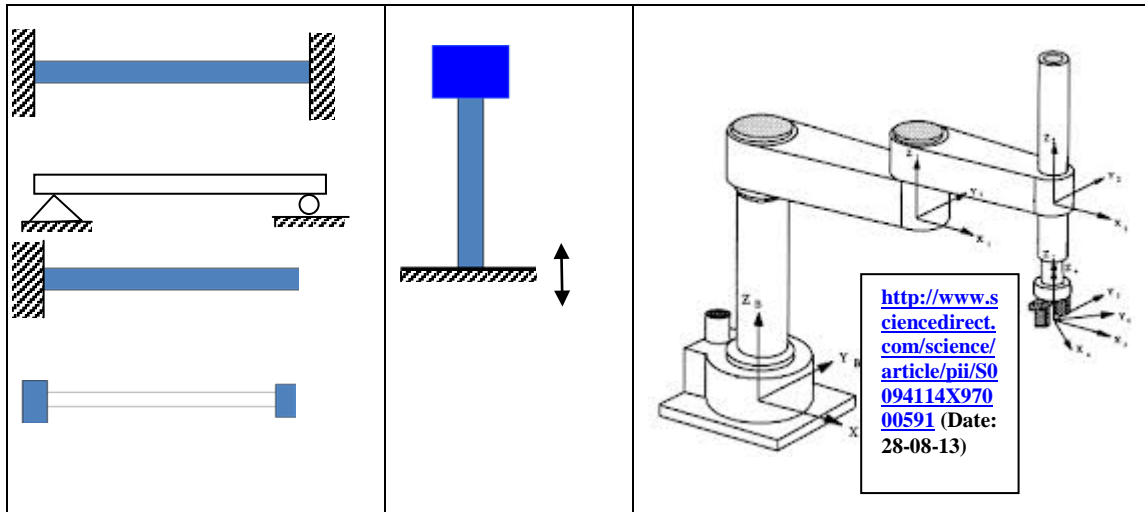
Fig.: 1.2.10: Example of continuous system (a)

The second column shows a base excited beam with a tip mass and last column shows a SCARA robot. In all these cases the system can be modelled as a continuous or distributed mass system. Unlike the case of multi-degree of freedom system or discrete mass system where the governing equations are written as ordinary differential equation, here, partial differential equations are used represent the motion of the system. Few typical cases are discussed below.
For the following systems the governing equations are represented by wave equations.
- Lateral vibration of taut string
- Longitudinal vibration of rod
- Torsional Vibration of Shaft

The wave equation is given by
$$\frac{\partial^2 w}{\partial t^2} = C^2 \frac{\partial^2 w}{\partial x^2} \qquad (1.2.31)$$
Here w is the displacement which is a function of both space variable x and time t.

The general solution of the system is $w = \phi(x)\sin \omega t$ where the mode shape $\phi(x)$ is given by the following equation.
$$\phi(x) = a\cos \beta x + b\sin \beta x \qquad (1.2.32)$$

These constants can be obtained by applying the boundary conditions.
For transverse vibration of the beam with young's modules E, Moment of inertia I and mass per unit length ρ, length L due to pure bending one may use Euler Bernoulli Beam which is given by the following equation.
$$EI \frac{d^4 y}{dx^4} + \rho \frac{d^2 y}{dt^2} = 0 \qquad (1.2.33)$$
The general solution of the system is $y = \phi(x)\sin \omega t$ where the mode shape $\phi(x)$ is given by the following equation.
$$\phi(x) = a\cosh \beta x + b\sinh \beta x + c\cos \beta x + d\sin \beta x \qquad (1.2.34)$$
The frequency of the system can be obtained from the following equation.

$$\omega = \beta^2 l^2 \sqrt{\frac{EI}{\rho l^4}} \qquad (1.2.35)$$

Table 1.2.1 gives the values of $\beta^2 l^2$ for different end conditions of the beams for the first three mode.

Table : 1.2.1: Values of $\beta^2 l^2$ for different end conditions of the beams

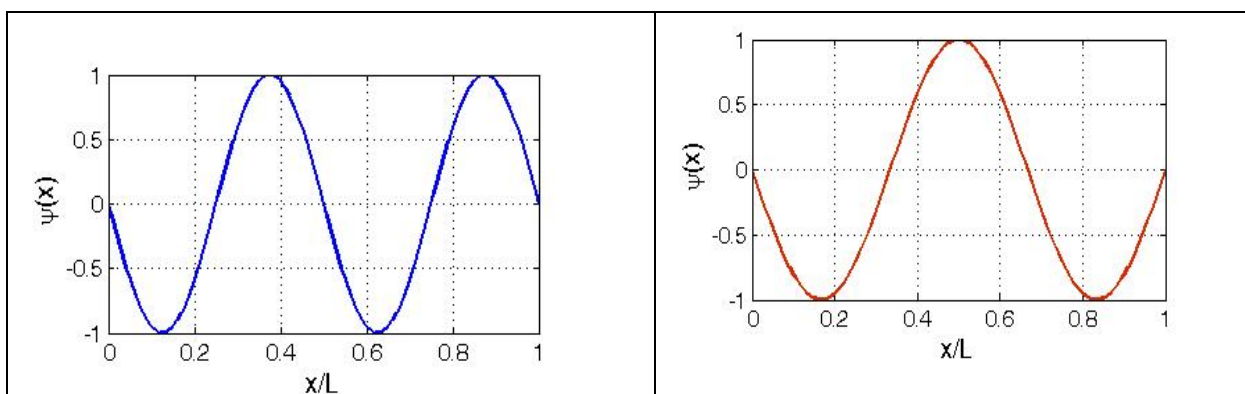| Beam Configuration	First mode	Second mode	Third mode
Simply supported	9.87	39.5	88.9
Cantilever	3.52	22.0	61.7
Free-free	22.4	61.7	121.0
Clamped-clamped	22.4	61.7	121.0
Clamped-hinged	15.4	50.0	104.0
Hinged-free	0	15.4	50.0

Mode shapes
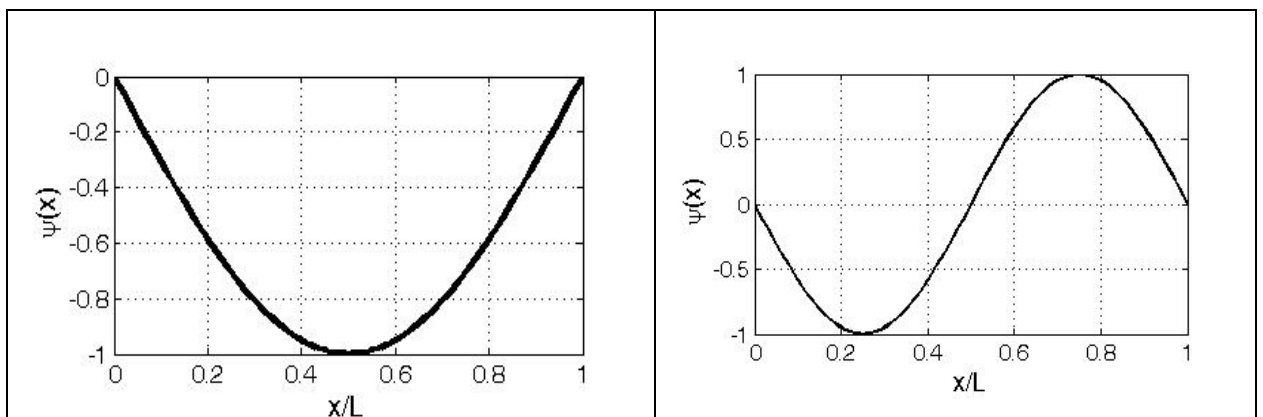
Fig. 1.2.11: First four mode shapes for a simply supported beam.

Exercise problems

1. Considering the springs and damper to be nonlinear write the equation of motion of a single, two and multi-degree of freedom system.

2. Write the equation of motion of the tuned vibration absorber. Replace the secondary spring by a spring with cubic nonlinearity and write the resulting equation of motion.

3. Consider a pendulum vibration absorber. Considering the link to be flexible, derive the equation of motion on the system.

4. Plots the mode shapes of a (a) cantilever beam (b) cantilever beam with tip mass (c) beam with fixed and roller supported end conditions.

Module 1 Lecture 3

In this lecture the qualitative analysis of nonlinear conservative system is introduced and commonly observed nonlinear phenomena are briefly described.

Qualitative analysis of nonlinear conservative systems

Consider a nonlinear conservative system which is given by the equation

$$\ddot{u} + f(u) = 0 \tag{1.3.1}$$

Multiplying $\dot{u}$ and the resulting equation

Upon integrating one obtains

$$\int (\dot{u}\ddot{u} + \dot{u}f(u))dt = h \tag{1.3.2}$$

or, $\dfrac{1}{2}\dot{u}^2 + F(u) = h, \quad F(u) = \int f(u)du$ (1.3.3)

This represents that the sum of the kinetic energy and potential energy of the system is constant. Hence, for a particular energy level h, the system will be under oscillation, if the potential energy $F(u)$ is less than the total energy h. From the above equation, one may plot the phase portrait or the trajectories for different energy level and study qualitatively about the response of the system.

Example 1.3.1: Perform qualitative analysis to study the response of the dynamic system

$$\ddot{x} + x - 0.1x^3 = 0 \tag{1.3.4}$$

Solution: For this system $F(x) = \int f(x)dx = \int (x - 0.1x^3)dx = \dfrac{1}{2}x^2 - \dfrac{1}{40}x^4$ (1.3.5)

Figure 1.3.1 shows the variation of potential energy $F(x)$ with x. It has optimum values corresponding to $x = 0$ or $\pm\sqrt{20}$ While x equal to zero represents the system with minimum potential energy, the other two points represent the points with maximum potential energy
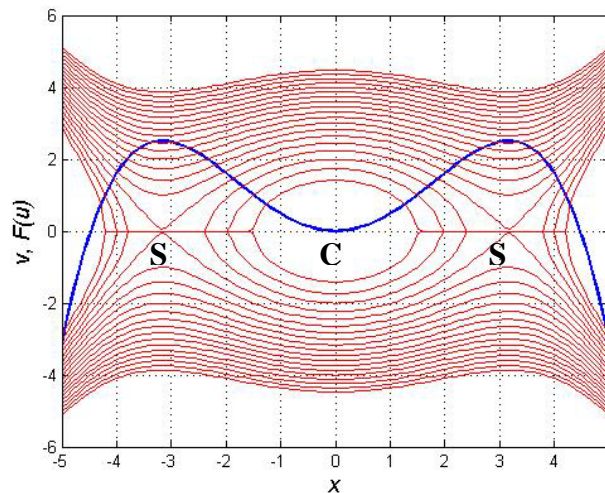

Fig. 1.3.1 Potential well and phase portrait showing saddle point and center corresponding to maximum and minimum potential energy.

Now by taking different energy level h, one may find the relation between the velocity v and displacement x as $v = \dot{x} = \sqrt{2(h - F(x))} = 2\sqrt{2(h - (0.5x^2 - 0.025x^4))}$ (1.3.6)

Now by plotting the phase portrait one may find the trajectory which clearly depicts that the motion corresponding to maximum potential energy is unstable and the bifurcation point is of saddle-node type (marked by point S) and the motion corresponding to the minimum potential energy is stable center type (marked by point C).

There are several approximate solution method based on the perturbation techniques to solve the nonlinear equation of motions of the system.

Types of Nonlinear response

➢ Fixed point response
➢ Periodic response
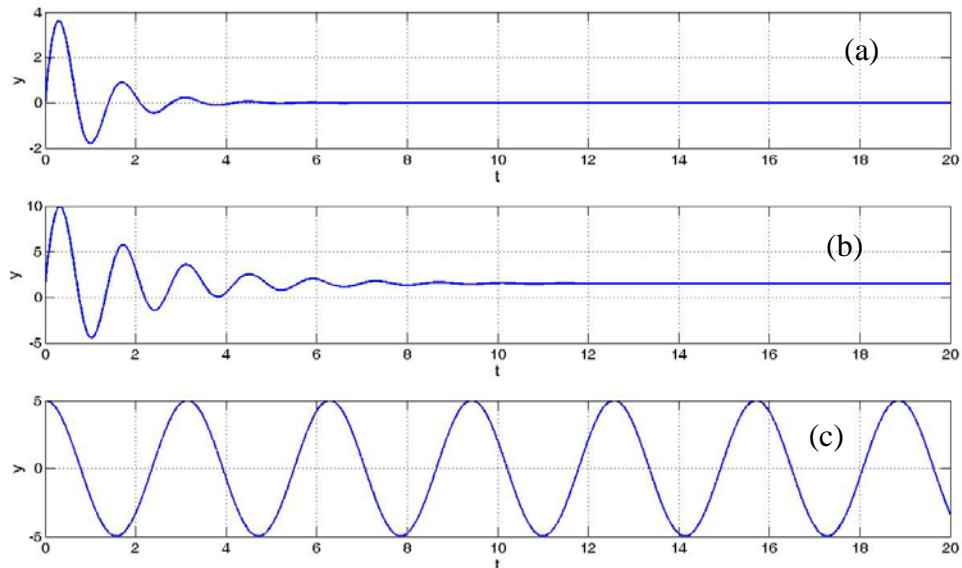➢ Quasiperiodic response
➢ Chaotic response

Fig. 1.3.2: (a) fixed-point trivial response (b) fixed-point non-trivial response, (c) periodic response
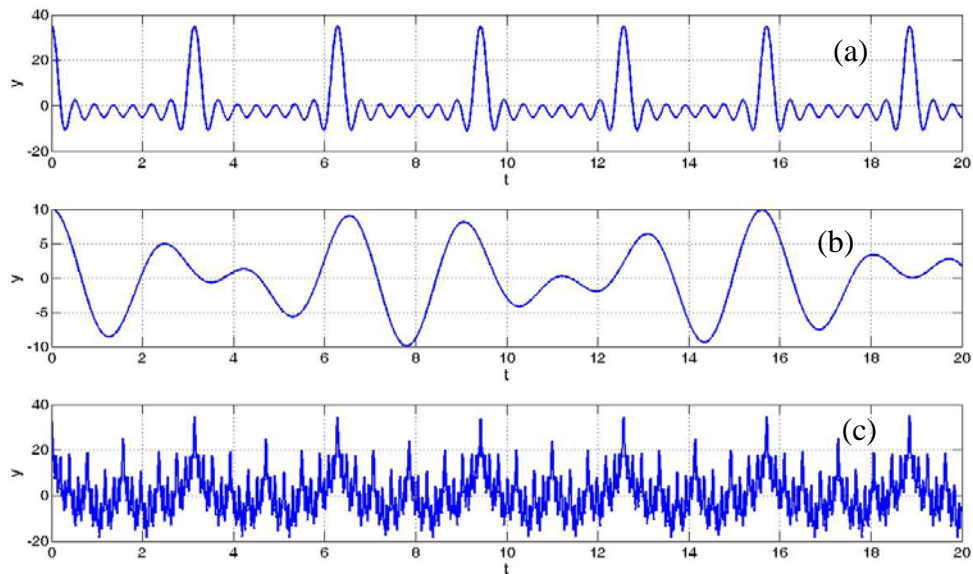

Fig. 1.3.3: (a) periodic response with multi-frequency (b) quasi-periodic response (c) chaotic response.

Figures 1.3.2(a, b) show the time response where the steady state response is a fixed point response. While in the first case the steady state response leads to a trivial state, in the second case it is a non-trivial state. In Fig. (1.3.2 c) a periodic response with single frequency is shown. A periodic response with multi-frequency is shown in Fig. (1.3.3(a)). Figure 1.3.3(b) shows the time response when the ratio of the considered two frequencies is an irrational number. Such responses are known as aperiodic or quasi-periodic response.

The plot shown in Fig. 1.3.3(c) is a chaotic response which is a deterministic bounded response but without following any specific pattern.

A Matlab code is given below to plot the time responses of fixed-point, periodic, quasi-periodic and chaotic responses. One may change the parameters to obtain a wide range of responses.

Matlab code to plot Fig. 1.3.2 and Fig. 1.3.3
```
% To plot fixed-point, eriodic, quasi-periodic and chaotic responses
clc
clear all
a0=5;

t=0:0.01:20;
omega=2;
beta=0;
r=sqrt(2);
u0=5*exp(-0.2*5*t).*sin(4.5*t);
u1=1.5+10*exp(-0.1*5*t).*sin(4.5*t);
u2=a0*cos(omega*t+beta);

u3=a0*(cos(omega*t)+cos(omega*r*t));
u4=0;
u5=0;
ii=1;
for ip=1:1:7;
u4=u4+a0*cos(ii*omega*t);
u5=u5+a0*cos(ip*omega*t);
ii=2^ip
end

figure(1)
subplot(3,1,1)
plot(t,u0)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on

subplot(3,1,2)
plot(t,u1)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on

subplot(3,1,3)
plot(t,u2)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on
```

```
figure(2)

subplot(3,1,1)
plot(t,u5)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on

subplot(3,1,2)
plot(t,u3)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on

subplot(3,1,3)
plot(t,u4)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('y','fontsize',14,'fontweight','b');
grid on
```

To characterize these responses one may use

- ✓ Time response
- ✓ Phase portrait
- ✓ Poincare' section
- ✓ Lyapunov exponent

A detailed discussion to numerically obtain time response, phase portrait, Poincare' section and Lyapunov exponent have been carried out in module 5.

Classification of fixed point response

For a dynamical system $\dot{x}=F(x; m)$ substituting $\dot{x}=0$ one can obtain the steady state solution or equilibrium Point x_o by substituting $\dot{x}=0$ and solving F($x;m$) =0. This solution is stable or unstable can be studied by performing the solution by substituting $x = x_o + \Delta x$ in the equation $x=F(x,m)$ which yields the equation $\Delta \dot{x} = A \Delta x$.

- ➢ Hyperbolic fixed point: when all of the eigenvalues of A have nonzero real parts it is known as hyperbolic fixed point.
- ➢ Sink: If all of the eigenvalues of A have negative real part. The sink may be of stable focus if it has nonzero imaginary parts and it is of stable node if it contains only real eigenvalues which are negative.
- ➢ Source: If one or more eigenvalues of A have positive real part. Here, the system is unstable and it may be of unstable focus or unstable node.

> ➢ Saddle point: when some of the eigenvalues have positive real parts while the rest of the eigenvalues have negative.
> ➢ Marginally stable: If some of the eigenvalues have negative real parts while the rest of the eigenvalues have zero real parts

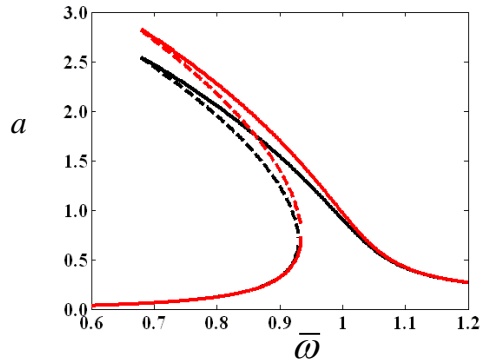**Typical Frequency response curves**

Fig. 1.3.4: A typical frequency response curves showing the stable and unstable branch (solid line stable, dotted line unstable branch)

In nonlinear systems, while plotting the frequency response curves of the system by changing the control parameters, one may encounter the change of stability or change in the number of equilibrium points. These points corresponding to which the number or nature of the equilibrium point changes, are known as bifurcation points. For fixed point response, they may be divided into static or dynamic bifurcation points depending on the nature of the eigenvalues of the system. If the eigen values are plotted in a complex plane with their real and imaginary parts along *X* and *Y* directions, a static bifurcation occurs, if with change in the control parameter, an eigenvalue of the Jacobian matrix crosses the origin of the complex plane. In case of dynamic bifurcation, a pair of complex conjugate eigenvalues crosses the imaginary axis with change in control parameter of the system. Hence, in this case the resulting solution is stable or unstable periodic type. A detailed discussion on the stability and bifurcation of fixed point and periodic responses are given in module 4.
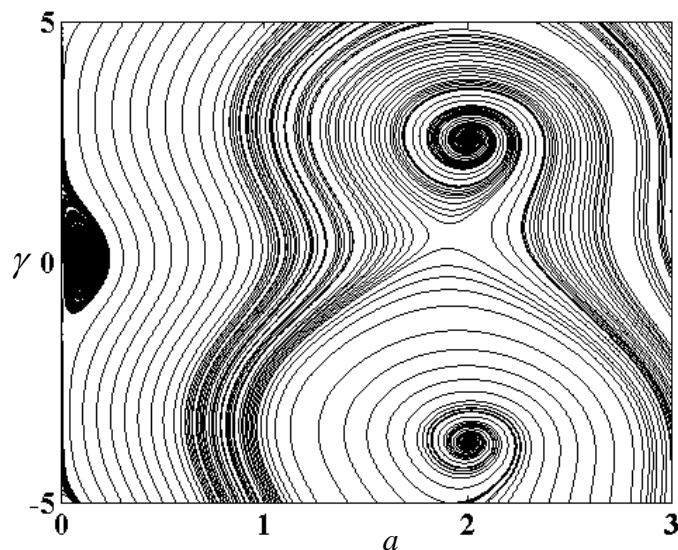
Fig. 1.3.5: Basin of attraction

Commonly used nonlinear equation of motion

Duffing equation (Free vibration with quadratic and cubic nonlinear term)

$$\frac{d^2u}{dt^2} + \omega_0^2 u + \varepsilon\alpha_2 u^2 + \varepsilon\alpha_3 u^3 = 0 \tag{1.3.7}$$

Duffing equation with damping and weak forcing terms

$$\ddot{x} + \omega_n^2 x + 2\varepsilon\zeta\omega_n\dot{x} + \varepsilon\alpha x^3 = \varepsilon f \cos\Omega t \tag{1.3.8}$$

Duffing equation with damping and strong forcing terms

$$\ddot{x} + \omega_n^2 x + 2\varepsilon\zeta\omega_n\dot{x} + \varepsilon\alpha x^3 = f \cos\Omega t \tag{1.3.9}$$

Duffing equation with multi-frequency excitation

$$\ddot{x} + \omega_n^2 x + 2\varepsilon\zeta\omega_n\dot{x} + \varepsilon\alpha x^3 = f_1 \cos\Omega_1 t + f_2 \cos\Omega_2 t + f_3 \cos\Omega_3 t + \cdots \tag{1.3.10}$$

Rayleigh's equation

$$\frac{d^2u}{dt^2} + \omega_0^2 u - \varepsilon(\dot{u} - \dot{u}^3) = 0 \tag{1.3.11}$$

van der Pol's equation

$$\frac{d^2v}{dt^2} + \omega_0^2 v = \varepsilon(1-v^2)\frac{dv}{dt} \tag{1.3.12}$$

Hill's equation

$$\ddot{x} + p(t)x = 0 \tag{1.3.13}$$

Mathieu's equation

$$\ddot{x} + \left(\omega_n^2 + 2\varepsilon f \cos\Omega t\right)x = 0 \tag{1.3.14}$$

Mathieu's equation with cubic nonlinearies and forcing terms

$$\ddot{x} + \left(\omega_n^2 + 2\varepsilon f_1 \cos\Omega_1 t\right)x + \varepsilon\alpha x^3 = \varepsilon f_2 \cos\Omega_2 t \tag{1.3.15}$$

Method of solutions of these equations will be discussed in module 3.

Commonly observed phenomena

- ✓ Jump up phenomena
- ✓ Jump down phenomena
- ✓ Multi stable region
- ✓ Butterfly effect
- ✓ Primary resonance
- ✓ Secondary resonance
- ✓ Super harmonic and sub harmonic resonance
- ✓ Principal parametric resonance
- ✓ Combination resonance of sum type
- ✓ Combination resonance of difference type
- ✓ Simultaneous resonance conditions
- ✓ Relaxation oscillation
- ✓ Internal resonance condition

A detailed discussion on these points will be made in module 6.

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