ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-41 NATURAL CONVECTION BLs

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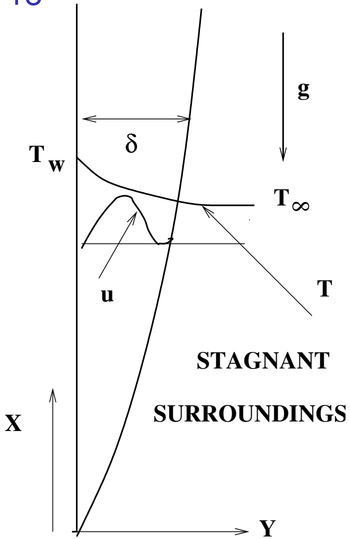
1. Problem Definition
2. Const Property LBL - Similarity Solns - Heat Transfer
    1. T_w = const and $v_w = 0$
    2. Effect of T_w variation
    3. Effect of Suction and Blowing
3. Integral Solns - Heat Transfer
    1. Laminar Bls
    2. Turbulent BLs
4. Simultaneous Heat and Mass Transfer

Problem Definition L41($\frac{1}{18}$)

Consider a vertical plate at T_w = const such that $T_w > T_\infty$. Then, fluid close to the wall heats up and, compared to the density of the stagnant surrounding, its density decreases. This density difference sets up an upward fluid motion due to buoyancy . If $T_w < T_\infty$, then a downward motion will be set up . The boundary layers thus developed are laminar to begin with (small x) but, turn turbulent at large x.



Governing Eqns - 1 L41($\frac{2}{18}$)

For constant properties, the governing eqns are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = - \frac{dp_\infty}{dx} + \mu \frac{\partial^2 u}{\partial y^2} - \rho g$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

where from hydrostatics, $dp_\infty / dx = - \rho_\infty g$. Now, using definition of volumetric coefficient of thermal expansion β, the vertical momentum eqn transforms to

$$\beta = - \frac{1}{\rho} \left(\frac{\rho - \rho_\infty}{T - T_\infty} \right)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g \beta (T - T_\infty)$$

Governing Eqns - 2 L41($\frac{3}{18}$)

Thus, the modified momentum eqn and the energy eqn must be solved simultaneously along with the continuity eqn. Solutions can be obtained by similarity, integral or finite-difference methods. The Integral forms of Eqns can be derived as

$$\frac{d}{dx}\left(\int_0^{\delta} u^2\, dy\right) = -\frac{\tau_w}{\rho} + g\,\beta \int_0^{\delta} (T - T_\infty)\, dy$$

$$\frac{d}{dx}\left\{\int_0^{\delta} u\,(T - T_\infty)\, dy\right\} = \frac{q_w}{\rho\, c_p}$$

The BCs are: at y = 0, u = 0 and T = T_w. At y = δ, u = 0 and T = T_∞ .

Similarity Soln - L41($\frac{4}{18}$)

Define $u \equiv \dfrac{\partial \Psi}{\partial y}, \quad v \equiv -\dfrac{\partial \Psi}{\partial x}, \quad \theta = \dfrac{T - T_\infty}{T_w - T_\infty}$ Then,

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial y\,\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \nu\,\frac{\partial^3 \Psi}{\partial y^3} + g\,\beta\,(T_w - T_\infty)\,\theta \quad (\text{ Mom })$$

$$\frac{\partial \Psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \theta}{\partial y} = \alpha\,\frac{\partial^2 \theta}{\partial y^2} - \theta\,\frac{\partial \Psi}{\partial y}\frac{dT_w}{dx} \quad (\text{ Energy })$$

Now, define similarity variables $\eta = y \times S(x)$ and $\Psi\,(x, \eta) = \nu \times f\,(\eta) \times G\,(x)$. Hence,

$$f''' + \left[\frac{g\,\beta\,(T_w - T_\infty)}{(G\,S^3)\,\nu^2}\right]\theta + (\frac{G'}{S})\,(f\,f'' - f'^2) - (\frac{G}{S^2})\,S'\,f'^2 = 0$$

$$\theta'' + Pr\left[(\frac{G'}{S})\,f\,\theta' - (\frac{G/S}{T_w - T_\infty}\frac{dT_w}{dx})\,f'\,\theta\right] = 0$$

Similarity Soln - T_w = const - L41($\frac{5}{18}$)

1. For $dT_w/dx = 0$, let
$$(\frac{G'}{S}) = C_1, \quad \frac{g\,\beta\,(T_w - T_\infty)}{(G\,S^3\,\nu^2)} = C_2 \;\; \text{and} \;\; \frac{G\,S'}{S^2} = C_3$$
2. Combining expressions for C_1 and C_2 gives
G (x) $\propto\; x^{3/4}$ and S (x) $\propto\; x^{-1/4}$
3. If we take C_3 = -1 then, C_1 = 3 and C_2 = 1. Hence,
$G = 4 \times (Gr_x/4)^{1/4}$ and $S = (Gr_x/4)^{1/4}/x$, where
$$Gr_x = \frac{g\,\beta\,(T_w - T_\infty)\,x^3}{\nu^2} \quad = \quad \text{(Grashof Number) and}$$
$$f''' + \theta + 3\,f\,f'' - 2\,f'^2 = 0 \qquad \text{BCs } f(0) = f'(0) = f(\infty) = 0$$
$$\theta'' + 3\,Pr\,f\,\theta' = 0 \qquad \text{BCs } \theta(0) = 1, \theta(\infty) = 0$$
$$\eta = \frac{y}{x}\,(\frac{Gr_x}{4})^{1/4} \qquad f' = \frac{u\,x/\nu}{2\,\sqrt{Gr_x}}$$

Soln (Contd) - 1 - L41($\frac{6}{18}$)

Unlike in Forced convection, the equations are best solved by FD method using Tri-Diagonal Matrix Algorithm.

$$\frac{f_i - f_{i-1}}{\Delta\eta} = f_i'$$

$$(AE + AW)\, f_i' = AE\, f_{i+1}' + AW\, f_{i-1}' + \theta_i \text{ where}$$

$$AE = (\frac{1}{\Delta\eta^2} + 1.5\,\frac{f_i}{\Delta\eta}) \text{ and } AW = (\frac{1}{\Delta\eta^2} - 1.5\,\frac{f_i}{\Delta\eta})$$

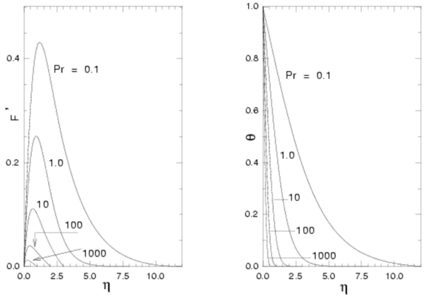
$$(AE + AW)\,\theta_i = AE\,\theta_{i+1} + AW\,\theta_{i-1} \text{ where}$$

$$AE = (\frac{1}{\Delta\eta^2} + 1.5\,Pr\,\frac{f_i}{\Delta\eta}) \text{ and } AW = (\frac{1}{\Delta\eta^2} - 1.5\,Pr\,\frac{f_i}{\Delta\eta})$$

$$\theta'(0) = 2 \times (\frac{\theta(2) - \theta(1)}{\Delta\eta}) \text{ and } f''(0) = 2 \times (\frac{f'(2) - f'(1)}{\Delta\eta})$$

$$\text{Soln: } Nu_x = \frac{q_w}{T_w - T_\infty}(\frac{x}{k}) = -\theta'(0) \times (Gr_x/4)^{1/4}$$

Soln (Contd) - 2 - L41($\frac{7}{10}$)



Pr	0.01	0.1	1.0	10	100	1000
η_{max}	22	12	7.5	3	2	1
$\frac{Nu_x}{Gr_x^{0.25}}$	0.059	0.164	0.402	0.821	1.54	2.72
$f''(0)$	0.9855	0.859	0.6419	0.4145	0.248	0.137

Soln for $T_w - T_\infty = A\,x^n$ - L41($\frac{8}{18}$)

Governing eqns for $T_w - T_\infty = A\,x^n$ are

$$f''' + (n+3)\,f\,f'' - (2n+2)\,f'^2 + \theta = 0$$
$$\theta'' + Pr\,\left\{(n+3)\,f\,\theta' - 4\,n\,f'\,\theta\right\} = 0$$

Solns for n = 0.2 and n = 1.0

n	Pr	0.01	0.1	1.0	10	100	1000
0.2	$\frac{Nu_x}{Gr_x^{0.25}}$	0.068	0.189	0.457	0.924	1.705	3.03
	$f''(0)$	0.934	0.813	0.607	0.391	0.230	0.13
1.0	$\frac{Nu_x}{Gr_x^{0.25}}$	0.093	0.354	0.597	1.184	2.178	3.87
	$f''(0)$	0.807	0.702	0.523	0.336	0.197	0.11

For n = 0.2,

$q_w = -k\,(T_w - T_\infty)\,\theta'\,(0)\,S \propto (T_w - T_\infty)^{1.25} \times x^{-0.25} = \text{const}$.

The Nu_x values are greater than those for T_w = const.

Soln for $T_w - T_\infty = A\,x^n$ & v_w - L41($\frac{9}{18}$)

For finite v_w, boundary condition is changed

$$v = -\frac{\partial \Psi}{\partial x} = -\nu\,(f\,G' + G\,f'\,y\,S') \quad \text{hence}$$

$$f(0) = -\frac{(v_w\,x/\nu)}{(Gr_x/4)^{0.25}} \times (\frac{1}{n+3}) = \frac{-\,v_w^*}{n+3} = \text{const}$$

Solns for n = 0 (T_w = const) and Pr = 0.7

	Suction			Blowing		
v_w^*	-3	-2	-1	+1	+2	+3
$Nu_x\,Gr_x^{-1/4}$	1.513	1.06	0.664	0.147	0.0504	0.0055
$f''(0)$	0.446	0.574	0.678	0.576	0.434	0.326

For v_w^* = 0, $Nu_x\,Gr_x^{-1/4}$ = 0.353.

Curve-fit Correlations - L41($\frac{10}{18}$)

1. For T_w = const and v_w = 0

$$Nu_x = \frac{3}{4}\left[\frac{2\,Pr}{5\,(1+2\,(\sqrt{Pr}+Pr)}\right]^{0.25}(Gr_x\,Pr)^{0.25} \quad \text{(Ede)}$$

$$Nu_x = (\frac{Gr_x}{4})^{0.25}\left[\frac{0.676\,\sqrt{Pr}}{(0.876+Pr)^{0.25}}\right] \quad \text{(Ostrach)}$$

2. For $(T_w - T_\infty) = A\,x^{0.2}$ (q_w = const) and v_w = 0

$$Nu_x = \left[\frac{Pr}{4+9\,\sqrt{pr}+10\,Pr}\right]^{0.2}(Gr_x^*\,Pr)^{0.2} \quad \text{(Fujii \& Fujii)}$$

$$Gr_x^* = Gr_x\,Nu_x = \frac{g\,\beta\,q_w\,x^4}{k\,\nu^2}$$

Integral Solns T_w = const, v_w = 0 - L41($\frac{11}{18}$)

Here integral eqns of slide 3 are evaluated by assuming

$$\frac{u}{U_{ref}} = \frac{y}{\delta}\,(1-\frac{y}{\delta})^2, \quad \text{and} \quad \frac{T-T_\infty}{T_w-T_\infty} = (1-\frac{y}{\delta})^2$$

$$\frac{1}{105}\,\frac{d\,U_{ref}^2\,\delta}{dx} = \frac{g\,\beta}{3}\,(T_w-T_\infty)\,\delta - \frac{\nu\,U_{ref}}{\delta} \quad \text{(Mom)}$$

$$\frac{1}{30}\,\frac{d\,U_{ref}\,\delta}{dx} = \frac{2\,\alpha}{\delta} \quad \text{(Energy)}$$

Soln obtained assuming $U_{ref} = C_1\,x^m$ and $\delta = C_2\,x^n$ to give m = 0.5 and n = 0.25. The result

$$\frac{\delta}{x} = 3.93\,(\frac{0.952+Pr}{Pr^2})^{0.25}\,Gr_x^{-0.25} \quad \rightarrow \; \delta \propto x^{0.25}$$

$$Nu_x = 0.508\,(\frac{0.952+Pr}{Pr^2})^{-0.25}\,Gr_x^{0.25} \quad \rightarrow \; h_x \propto x^{-0.25}$$

Transition to Turbulence - L41($\frac{12}{18}$)

1. If we write the differential mom eqn for y = 0 and $y \geq \delta$ then,

$$U_{ref} \frac{dU_{ref}}{dx} = \frac{(\rho_\infty - \rho_w)\, g}{\rho} = g\,\beta\,(T_w - T_\infty)$$

Hence, $U_{ref} = \sqrt{g\,\beta\,(T_w - T_\infty)\,x}$ and
$Re_x = U_{ref}\, x/\nu = Gr_x^{1/2}$

2. Generally, transition occurs at $10^9 \leq Gr_{x,tr} \leq 10^{10}$. For gases and organic liquids, $Gr_{x,tr} \rightarrow 10^9$.
For $Pr > 100$, $Gr_{x,tr} \rightarrow 10^{10}$.
3. In practical work, Rayleigh number $Ra_{tr} = Gr_x\, Pr = 10^9$ is taken as transition criterion.

Turbulent BL - L41($\frac{13}{18}$)

Integral eqns are solved with

$$\frac{u}{U_{ref}} = (\frac{y}{\delta})^{1/7}(1-\frac{y}{\delta})^4, \quad \text{and} \quad \frac{T-T_\infty}{T_w-T_\infty} = 1-(\frac{y}{\delta})^{1/7}$$

$$\tau_w = 0.0225\,\rho\,U_{ref}^2\,(\frac{U_{ref}\,\delta}{\nu})^{-0.25}$$

$$St_x = \frac{q_w/(T_w-T_\infty)}{\rho\,c_p\,U_{ref}} = 0.0225\,(\frac{U_{ref}\,\delta}{\nu})^{-0.25}\,Pr^{-2/3}$$

Soln obtained assuming $U_{ref} = C_1\,x^m$ and $\delta = C_2\,x^n$ to give m = 0.5 and n = 0.7. The result

$$Nu_x = 0.0295\,Pr^{7/15}\,(\frac{Gr_x}{1+0.494\,Pr^{2/3}})^{0.4} \quad \rightarrow \quad h_x \propto x^{+0.2}$$

This is unlike Laminar boundary layer.

Overall Correlation - L41($\frac{14}{18}$)

For the entire range of Rayleigh Numbers and for T_w = const and $v_w = 0$, the currently accepted correlation[1] is

$$\overline{Nu}_L = 0.68 + \frac{0.67\ Ra_L^{0.25}}{[1 + (0.492/Pr)^{9/16}]^{4/9}} \quad 10^{-1} < Ra_L < 10^9$$

$$\overline{Nu}_L = \left\{ 0.825 + \frac{0.387\ Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}} \right\}^2 \quad 10^9 < Ra_L < 10^{12}$$

where $\overline{h}_L = \frac{1}{L} \int_0^L h_x\, dx$ and $\overline{Nu}_L = \overline{h}_L\, L/k$.

Correlations for natural convection from other geometries such as inclined/horizontal plates, cylinders, cavities etc (see, Incropera F P and Dewitt D P, Fundamentals of Heat and Mass Transfer, 4th Edition, John Wiley and Sons, New York, 1996)

[1] Churchill and Chu, IJHMT, vol 18, p 1323, (1975)

Simultaneous HMT - L41($\frac{15}{18}$)

We consider inert mass transfer with heat transfer at small mass transfer rates and assume constant properties. Then for $\Delta T = T_w - T_\infty = A\, x^n$ and $\Delta\omega_v = \omega_{v,w} - \omega_{v\infty} = A\, x^n$, it can be shown that

$$
\begin{aligned}
f^{'''} + (n+3)\, f\, f^{''} - (2n+2)\, f^{'2} + \theta + F_\beta\, \Phi &= 0 \\
\theta^{''} + Pr\, \left\{(n+3)\, f\, \theta^{'} - 4\, n\, f^{'}\, \theta\right\} &= 0 \\
\Phi^{''} + Sc\, \left\{(n+3)\, f\, \Phi^{'} - 4\, n\, f^{'}\, \Phi\right\} &= 0 \\
\text{and, as before } \eta = \frac{y}{x}\, \left(\frac{Gr_{x,\Delta T}}{4}\right)^{1/4} &
\end{aligned}
$$

where $\Phi = (\omega_v - \omega_{v,\infty})/\Delta\omega_v$ and $F_\beta = (\beta^*\, \Delta\omega_v)/(\beta\, \Delta T)$. The BCs are $f(0) = f^{'}(0) = 0$, $\theta\,(0) = \Phi\,(0) = 1$ and $f^{'}\,(\infty) = \theta\,(\infty) = \Phi\,(\infty) = 0$. Notice from slide 9 that as $v_w \to 0$, f(0) = 0.

Solutions for (n = 0, Pr = 0.7, F_β = 1) - L41($\frac{16}{18}$)

F_β = 1 implies *aiding buoyancies*.

$$Nu_x = \frac{h_x\, x}{k} = -\,\theta^{'}(0)\,(\frac{Gr_{x,\Delta T}}{4})^{1/4}$$
$$Sh_x = \frac{g^*\, x}{\rho\, D} == -\,\Phi^{'}(0)\,(\frac{Gr_{x,\Delta T}}{4})^{1/4}$$

Sc	$-\theta^{'}(0)$	$-\Phi^{'}(0)$	$f^{''}(0)$	$\frac{\delta_\omega}{\delta_T}$	Le = Pr / Sc
0.5	0.431	0.362	1.17	1.01	1.4
0.7	0.421	0.421	1.14	1.00	1.0
1.0	0.410	0.490	1.111	0.97	0.7
5.0	0.379	0.917	0.985	0.329	0.14
10.0	0.371	1.176	0.937	0.233	0.07

Solutions for (n = 0, Pr = 0.7, F_β = -0.5) - L41($\frac{17}{18}$)

F_β = -0.5 implies *opposing buoyancies*

Sc	$-\theta^{'}(0)$	$-\Phi^{'}(0)$	$f^{''}(0)$	$\frac{\delta_\omega}{\delta_T}$	Le = Pr / Sc
0.5	0.278	0.231	0.383	1.004	1.4
0.7	0.297	0.297	0.404	1.00	1.0
1.0	0.309	0.367	0.425	0.984	0.7
5.0	0.337	0.776	0.507	0.350	0.14
10.0	0.342	1.027	0.536	0.256	0.07

Compared to F_β = 1, $f^{''}(0)$, Nu and Sh have now reduced .
For other problems, see (Y. Jaluria Natural Convection Heat and Mass Transfer, Pergamon Press, NY, 1980)

Large Mass Transfer Rates - L41($\frac{18}{18}$)

At large mass tranfer rates, it can be shown that

$$f(0) = \frac{-v_w}{\nu G'} = \frac{-\Phi'(0)}{Sc\,(n+3)} \times B_m = \frac{-\theta'(0)}{Pr\,(n+3)} \times B_h$$

Solutions for n = 0, Pr = Sc = 0.7 and F_β = 1.

B_m	$Sh_x\,(Gr_{x,\Delta T}/4)^{-0.25}$	$f''(0)$	Sh/Sh^*	$\frac{\ln(1+B_m)}{B_m}$
0.0	0.421	1.142	1.0	1.0
0.1	0.400	1.135	0.951	0.953
0.2	0.381	1.128	0.907	0.912
0.3	0.364	1.121	0.866	0.875
0.4	0.350	1.114	0.833	0.841
0.5	0.336	1.107	0.800	0.811

The data show that $Sh/Sh^* \simeq$ ln (1 + B) / B. So,
for const properties, Reynolds flow model is verified.