ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-23 SUSTAINING MECHANISM OF TURBULENCE-2

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1. Spectral Analysis
2. Vorticity Dynamics

Spectral Analysis L23($\frac{1}{18}$)

1. In the previous lecture, we discovered the largest (l_{int}), intermediate (Taylor micro-scales $l_{f,g}$) and the smallest (Komogorov l_ϵ) scales or eddies.
2. Spectral analysis explains how turbulence energy is *distributed* among the range of scales and how the energy exchange between eddies of different scales takes place.
3. Spectra are decompositions of a non-linear function into waves of different wavelengths (or periods).
4. The value of the spectrum at a given wavelength (or frequency) is the mean energy in that wave.
5. Spectral analysis leads to the understanding that turbulence receives its energy at the large scales, and while its energy dissipates at very small scales; there also exist waves within a range of wavelengths (called the *inertial range*) which are not directly affected by the sustenance mechanism of turbulence.

Main Postulate - 1 - L23($\frac{2}{18}$)

1. The spatial correlation tensor $B_{ij}\,(\vec{r} = \Delta\, x_1, \Delta\, x_2, \Delta\, x_3)$ is related to the spectral tensor $\Phi_{ij}\,(\vec{k} = k_1, k_2, k_3)$ via the 3D *Fourier transform* as:
$B_{ij}\,(\vec{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}\,(\vec{k})\exp\left(i\,\vec{k}.\vec{r}\right) d\vec{k}$
and its Inverese transform
$\Phi_{ij}\,(\vec{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_{ij}\,(\vec{r})\exp\left(-i\,\vec{k}.\vec{r}\right) d\vec{r}$
2. Therefore, spectral interpretation of the Reynolds stress (one-point correlation) tensor $-\rho\,\overline{u_i'\,u_j'}$ is
$-\rho\,\overline{u_i'\,u_j'} = -\rho\,B_{ij}\,(\vec{r} = 0) = -\rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ij}\,(\vec{k})\,d\vec{k}$
3. Now, since $\overline{u_i'\,u_j'}$ determine the energy in the various velocity components , the value of $\Phi_{ij}\,(\vec{k})$ gives the *division* of this energy in different eddy sizes or wave numbers[1] .
Consequently, $\Phi_{ij}\,(\vec{k})$ is called the *energy spectrum* tensor.

[1] Small values of wave numbers correspond to large eddies or wavelengths and vice versa.

Main Postulate - 2 - L23($\frac{3}{18}$)

1. Further, the sum of the diagonal components of the tensor gives the turbulent kinetic energy at a given wavenumber. $B_{ii}\left(\vec{r}=0\right)=\overline{u_i^{\prime}\,u_i^{\prime}}=2\,e=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{ii}\left(\vec{k}\right)d\vec{k}$
2. The spectral tensor $\Phi_{ii}\left(\vec{k}\right)$ is a function of 3 wavenumber components. In order that physical interpretation becomes easier, it is customary to remove directional dependence by integrating $\Phi_{ii}\left(\vec{k}\right)$ over a spherical shell of radius k (scalar) where $k=+\sqrt{k_1^2+k_2^2+k_3^2}$.
3. If dA is the area of an element on the surface of the spherical shell of radius k then

$$e(k)=\frac{1}{2}\int\int\Phi_{ii}\left(\vec{k}\right)dA \quad\rightarrow\quad e=\int_0^{\infty}e(k)\,dk$$

4. The function e(k) is called the *scalar kinetic energy spectrum*.

TKE Eqn in k-Space - 1 - L23($\frac{4}{18}$)

To derive the transport eqn for e(k), an equation for B_{ij} is first derived for a non-homogeneous, anisotropic and *steady* turbulent flow.Thus, the instantaneous ($\hat{u} = u + u^{'}$) form of the NS equations (with $\partial u_k^{'}/\partial x_k = \partial u_k/\partial x_k = 0$) is

$$\begin{aligned}
\frac{\partial u_i^{'}}{\partial t} + \hat{u}_k \frac{\partial \hat{u}_i}{\partial x_k} &= -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial x_i} + \nu \frac{\partial^2 \hat{u}_i}{\partial x_l \, \partial x_l} \\
\frac{\partial u_i^{'}}{\partial t} + (u_k + u_k^{'})\frac{\partial u_i}{\partial x_k} &+ u_k^{'}\frac{\partial u_i^{'}}{\partial x_k} + \frac{\partial u_k^{'} u_i^{'}}{\partial x_k} \\
&= -\frac{1}{\rho}\frac{\partial (p + p^{'})}{\partial x_i} + \nu \frac{\partial^2 (u_i + u_i^{'})}{\partial x_l \, \partial x_l}
\end{aligned}$$

We now subtract the RANS momentum eqn form this eqn (next slide)

TKE Eqn in k-Space - 2 - L23($\frac{5}{18}$)

Subtraction results in eqn for position $\vec{r}_1$

$$\frac{\partial u_i^{'}}{\partial t} + u_k^{'} \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial u_i^{'}}{\partial x_k} + \frac{\partial}{\partial x_k}(u_k^{'} \, u_i^{'} - \overline{u_k^{'} \, u_i^{'}}) = -\frac{1}{\rho} \frac{\partial p^{'}}{\partial x_i} + \nu \frac{\partial^2 u_i^{'}}{\partial x_l \, \partial x_l}$$

and a similar equation for $u_j^{'}$ at position $\vec{r}_2$

$$\frac{\partial u_j^{'}}{\partial t} + u_k^{'} \frac{\partial u_j}{\partial x_k} + u_k \frac{\partial u_j^{'}}{\partial x_k} + \frac{\partial}{\partial x_k}(u_k^{'} \, u_j^{'} - \overline{u_k^{'} \, u_j^{'}}) = -\frac{1}{\rho} \frac{\partial p^{'}}{\partial x_j} + \nu \frac{\partial^2 u_j^{'}}{\partial x_l \, \partial x_l}$$

Now, multiplying first equation by $u_j^{'}$ at $\vec{r}_2$ and second equation by $u_i^{'}$ at $\vec{r}_1$ and, adding and time-averaging, yields the required equation for B_{ij} (next slide) in terms of two (in fact, six) independent variables namely: $\xi_k = x_k|_{r_2} - x_k|_{r_1}$ (separation) and $x_k|_m = \frac{1}{2}\,(x_k|_{r_1} + x_k|_{r_2})$ (mid-point)

TKE Eqn in k-Space - 3 - L23($\frac{6}{18}$)

$$
\begin{aligned}
\frac{\partial B_{ij}}{\partial t} &+ \left[B_{kj} \left(\frac{\partial u_i}{\partial x_k}\right)_{r_1} + B_{ik} \left(\frac{\partial u_j}{\partial x_k}\right)_{r_2} \right] \\
&+ \frac{1}{2} \left(u_{k,r_1} + u_{k,r_2}\right) \left.\frac{\partial B_{ij}}{\partial x_k}\right|_m + \left(u_{k,r_2} - u_{k,r_1}\right) \frac{\partial B_{ij}}{\partial \xi_k} \\
&= -\frac{1}{2} \frac{\partial}{\partial x_k} \left.\left(T_{i,kj} + T_{ik,j}\right)\right|_m - \frac{\partial}{\partial \xi_k} \left(T_{i,kj} - T_{ik,j}\right) \\
&- \frac{1}{2\,\rho} \left[\left.\frac{\partial C_{p,j}}{\partial x_i}\right|_m + \left.\frac{\partial C_{p,i}}{\partial x_j}\right|_m \right] + \frac{1}{\rho} \left[\frac{\partial C_{p,j}}{\partial \xi_i} - \frac{\partial C_{p,i}}{\partial \xi_j} \right] \\
&\nu \left[\frac{1}{2} \left.\frac{\partial^2 B_{ij}}{\partial x_l\, \partial x_l}\right|_m + 2\, \frac{\partial^2 B_{ij}}{\partial \xi_l\, \partial \xi_l} \right]
\end{aligned}
$$

$$
T_{i,kj} \equiv \overline{(u_i')_{r_1} \, (u_j')_{r_2} \, (u_k')_{r_2}} \qquad T_{ik,j} \equiv \overline{(u_i')_{r_1} \, (u_k')_{r_1} \, (u_j')_{r_2}}
$$

$$
C_{p,j} \equiv \overline{(p')_{r_1} \, (u_j')_{r_2}} \qquad C_{p,i} \equiv \overline{(p')_{r_2} \, (u_i')_{r_1}} \qquad B_{ij} = \overline{(u_i') r_1 \, (u_j')_{r_2}}
$$

TKE Eqn in k-Space - 4 - L23($\frac{7}{18}$)

1. The eqn of the previous slide represents complete eqn for non-homogeneous non-isotropic turbulent flow. The eqn is not tractable.
2. For *homogeneous* turbulence , however, all derivatives of the correlations with x_k vanish but are finite w.r.t. ξ_k.

$$
\begin{aligned}
\frac{\partial B_{ij}}{\partial t} &= \xi_l \left.\frac{\partial u_k}{\partial x_l}\right|_m \frac{\partial B_{ij}}{\partial \xi_k} \quad \text{(mean convection)} \\
&- \left[B_{kj} \left(\frac{\partial u_i}{\partial x_k}\right)_{r_1} + B_{ik} \left(\frac{\partial u_j}{\partial x_k}\right)_{r_2} \right] \quad \text{(production)} \\
&- \frac{\partial}{\partial \xi_k} \left(T_{i,kj} - T_{ik,j} \right) \ \text{(v-diffu)} \\
&- \frac{1}{\rho} \left[\frac{\partial C_{p,i}}{\partial \xi_j} - \frac{\partial C_{p,j}}{\partial \xi_i} \right] \ \text{(p-diffu)} + 2\nu \frac{\partial^2 B_{ij}}{\partial \xi_l \, \partial \xi_l} \quad \text{(diss)}
\end{aligned}
$$

TKE Eqn in k-Space - 5 - L23($\frac{8}{18}$)

1. The mean convection term is really $(u_{k,r_2} - u_{k,r_1})\,\partial B_{ij}/\partial \xi_k$. However, $(u_{k,r_2} - u_{k,r_1}) = \xi_l\,\partial u_k/\partial x_l|_m$. The v-diffu and p-diffu terms represent diffusion of energy due to velocity and pressure fluctuations respectively.
2. In order to study the transfer process , each term is Fourier transformed so as to yield an equation for $\Phi_{ij}(\vec{k})$.
3. Then, setting i = j, the equation for $\Phi_{ii}(\vec{k})$ results.
4. Further, to achieve directional independence, each term is integrated over a spherical shell of radius k to yield

$$\frac{\partial e(k)}{\partial t} = P(k) - \frac{\partial T_t(k)}{\partial k} - D(k)$$

where, P(k) is production, D(k) is dissipation and $T_t(k) = T_{conv}(k) + T_{v-diffu}(k) + T_{p-diffu}(k)$

TKE Eqn in k-Space - 6 - L23($\frac{9}{18}$)

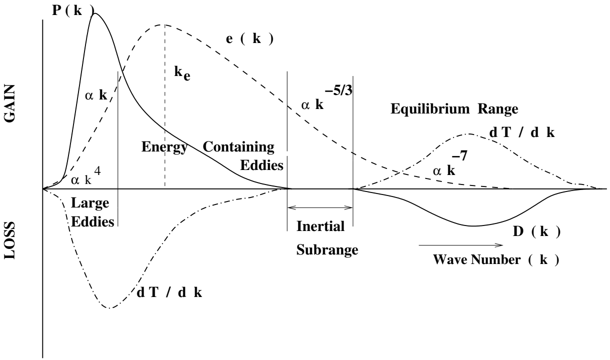
1. This is spectral form of the TKE eqn for homogeneous turbulence . It can be regarded as a 1D eqn representing energy balance over CV (dk) in the wavenumber space.
2. The P(k) term comprises of spectral functions (arising out of B_{kj} and B_{ik}) and mean velocity gradients, is not expected to be large at high wave numbers (small scale motions) but significant at small wave numbers.
3. The gradient transport of $T_t\,(k)$ vanishes when integrated from $k = 0$ to $k = \infty$ giving

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial t} \int_0^\infty e(k)\,dk = \int_0^\infty (P(k) - D(k))\,dk$$

   This transfer term simply redistributes energy both directionally and among the different wave numbers.
4. Dissipation $D(k) = 2\,\nu\,k^2\,e(k)$. The presence of k^2 confirms that it is significant only at high wave numbers.

Typical Solution - L23($\frac{10}{18}$)

We consider pure homogeneous shear flow with v = w = 0 and $\partial u/\partial y$ = const at $Re_{l_g} \simeq 100$.



Discussion -1 - L23($\frac{11}{18}$)

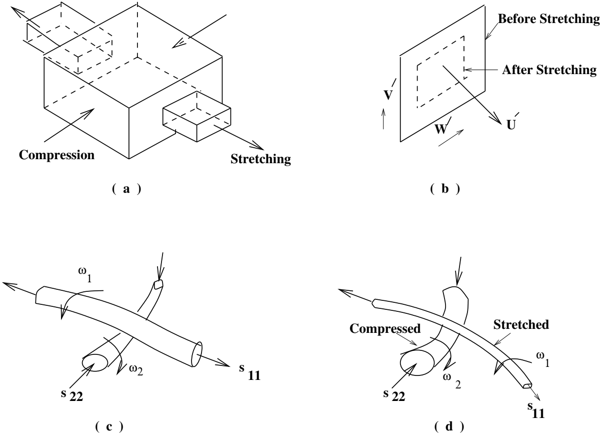
1. D (k) term dominates at high k (Kolmogorov small eddies, say $(0.1/l_\epsilon) < k < (1/l_\epsilon)$). Energy is mainly supplied by transfer term $\partial T_t(k)/\partial k$ and energy extraction from mean motion is minimal $P(k) \simeq 0$.
2. $\partial T_t(k)/\partial k$ is negative at small k and positive at higher k indicating that the energy is indeed transferred out of low-k region and into high-k region. $\int_0^\infty (\partial T_t(k)/\partial k)dk = 0$
3. The dominance of P(k) in low-k region indicates that most of the production is brought about by large eddies.
4. As $k \rightarrow 0$, very large eddies dominate and the e(k) spectrum is not expected to be universal being influenced by mean velocity gradients. Also $\partial e(k)/\partial t$ is small and $\partial T_t(k)/\partial k = P(k)$. This is region of *rapid distortion* .
5. The e(k) is maximum near k_e which characterises the *most energetic eddies* . l_{int} is largely determined by these eddies.

Discussion - 2 - L23($\frac{12}{18}$)

1. If $k_{diss} >> k_e$, then an *inertial equilibrium range* ($k < (0.1/l_\epsilon)$) identified with Taylor micro-scale exists in which the conditions for isotropy of the small scale eddies and of independence of the turbulence structure from energy containing eddies are simultaneously satisfied. For this region $e(k) \propto k^{-\frac{5}{3}}$.
2. The existence, or otherwise, of the inertial range has considerable significance for the near-wall turbulence .
3. Finally, at very high wave numbers, where $k > (1/l_\epsilon)$, the energy spectrum varies as $e(k) \propto k^{-7}$. At this point, D(k) is maximum.

Vorticity Dynamics-1 - L23($\frac{13}{18}$)

Spectral analysis shows how energy is transferred from large eddies to small eddies. The process of breakdown of eddies, can be understood from vorticity dynamics of the fluctuations.



Vorticity Dynamics-2 - L23($\frac{14}{18}$)

1. Fig (a) shows 3D cubic fluid elementwhich is stretched in the direction of the linear strain s_{11}.
2. Then, the cross-section in a plane perpendicular to the strain will become smaller as shown in Fig (b)
3. Similarly, in Fig (c) a vortex element is considered. The vortex in the direction of strain s_{11} becomes smaller in cross-section while the cross-section normal to the strain becomes larger as shown in Fig (d)
4. Intuitively, this is understandable. But, it is useful to consider Eqn for vorticity of fluctuations . (next slide)

Vorticity Dynamics-3 - L23($\frac{15}{18}$)

1. Consider large eddy structure where effects of viscosity are small. Then vorticity eqn is $\partial \omega_i' / \partial t = \omega_i' \, s_{ij}$
2. Now, for simplicity, consider a 2D strain field with $s_{11} = - s_{22} = s$ (a constant) for all times $t > 0$ and, $s_{12} = 0$. Then, if ω_0' is the vorticity at $t = 0$,

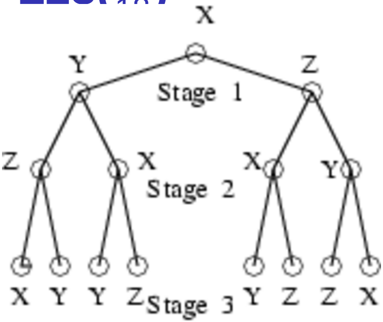
$$\frac{\partial \omega_1'}{\partial t} = \omega_1' s \rightarrow \frac{\omega_1'}{\omega_0'} = \exp(st), \quad \frac{\partial \omega_2'}{\partial t} = -\omega_2' s \rightarrow \frac{\omega_2'}{\omega_0'} = \exp(-st)$$

Hence, $(\omega_1')^2 + (\omega_2')^2 = (\omega_0')^2 \, (\exp(2\,st) + \exp(-2\,st))$

3. The total vorticity thus increases with $s \times t$. At large values of $s \times t$, ω_1' in the direction of stretching increases rapidly and ω_2' in the direction of compression decreases slowly. Eddies are thus stretched at a rapid rate into smaller eddies. Their growth to larger sizes occurs at a much slower rate resulting in net reduction in their size.

Vorticity Dynamics-4 - L23($\frac{16}{18}$)

1. When effect of μ is small, angular momentum is conserved . $(\omega')^2\, r$ = const.
2. In Fig (b), element is stretched in x dirn. Then, KE of rotation in the y-z plane increases at the expense of the KE of velocity component u' which does the stretching.
3. Length scales of motion in y-z plane decrease and hence, v' and w' increase.



v' and w' bring about further stretching in y and z directions and, so on. At each stretching, however, the length scale of the element decreases. This is called the breaking down of the eddies.

Summary -1 - L23($\frac{17}{18}$)

The tree demonstrates that stretching in x direction (say) intensifies motions in y and z directions; producing smaller scale stretching in these directions and intensifying motions in x, y and, z directions at the end of the second stage and, so on to further stages. As the length scales are progressively reduced, the effects of mean motion are weakened and the small eddies tend towards a universal structure that is homogeneous and isotropic despite the fact that the mean flow and the large scale structure are anisotropic and inhomogeneous.
The breaking down of the eddies would continue indefinitely if it were not for the action of viscosity which kills all fluctuations and maintains the fluid continuum[2] .

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[2] P. Bardshaw, *An Introduction to Turbulence and Its Measurement*, Pergamon Press, Oxford, (1971)

Summary -2 - L23($\frac{18}{18}$)

Big whirls have little whirls,
which feed on their velocity;
And little whirls have lesser whirls
And so on to viscosity .

Richardson (1922)