

Module1: Numerical Solution of Ordinary Differential Equations

Lecture 3

Modified Euler Method

Keywords: Euler method, local truncation error, rounding error

Modified Euler Method: Better estimate for the solution than Euler method is expected if average slope over the interval (t_0, t_1) is used instead of slope at a point. This is being

used in modified Euler method. The solution is approximated as a straight line in the interval (t_0, t_1) with slope as arithmetic average at the beginning and end point of the interval.

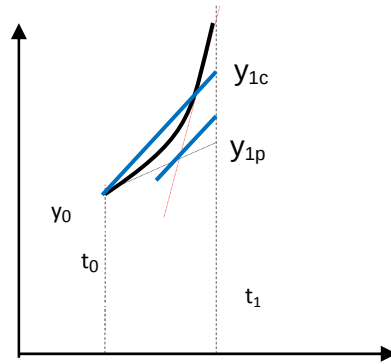


Fig1.3 Schematic Diagram for Modified Euler Method

Accordingly, y_1 is approximated as

$$y(t_1) \cong y_1 = y_0 + h \frac{(y_0' + y_1')}{2} = y_0 + h \frac{(f(t_0, y(t_0)) + f(t_1, y(t_1)))}{2}$$

However the value of $y(t_1)$ appearing on the RHS is not known. To handle this, the value of y_{1p} is first predicted by Euler method and then the predicted value is used in (1.6) to compute y_1' from which a better approximation y_{1c} to y_1 can be obtained:

$$y_{1p} = y_0 + hf(t_0, y_0) ; y_{1c} = y_0 + h \frac{f(t_0, y_0) + f(t_1, y_{1p})}{2}$$

The solution at t_{k+1} is computed as

$$y_{k+1p} = y_k + hf(t_k, y_k) ; y_{k+1} = y_k + h \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1p})}{2}$$

In the fig (1.3), observe that black dotted line indicates the slope $f(t_0, y(t_0))$ of the solution curve at t_0 , red line indicates the slope $f(t_1, y(t_1))$, at the end point t_1 . Since the solution at end point $y(t_1)$ is not known at the moment, its approximation y_{1p} as obtained from Euler

method is used. The blue line indicates the average slope. Accordingly, y_1 is a better estimate than y_{1p} . The method is also known as Heun's Method.

Algorithm 2 For numerical solution by Modified Euler method

Step 0 [initialization] $k=0$, $h=(b-t_0)/n$, $y(t_k)=y_k$,

Step 1 [predict solution] $y_{k+1p} = y_k + hf(t_k, y_k)$

Step 2 [correct solution] $y_{k+1} = y_k + h \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1p})}{2}$

Step 3 [increment] $t_{k+1}=t_k+h$, $k=k+1$

Step 4 [check for continuation] if $k < n$ go to step 1

Step 5 [termination] stop

Example 1.4: Solve the IVP in the interval (0.0, 2.0) using Modified Euler method with step size $h=0.2$

$$\frac{dy}{dt} = y - 2t^2 + 1; y(0) = 0.5$$

Solution: The computations are shown in the Table 1.2 .

To compute local truncation error consider

$$y(t_k + h) = y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + \frac{h^3}{6}y''(\xi), \quad \xi \in (t_k, t_k + h) \quad (1.6)$$

Replacing second derivative by finite difference gives

$$y(t_k + h) = y(t_k) + hy'(t_k) + \frac{h^2}{2} \left(\frac{y'(t_{k+1}) - y'(t_k)}{h} \right) + \frac{h^3}{6}y''(\xi), \quad \xi \in (t_k, t_k + h)$$

Further simplification gives local truncation error of modified Euler formula as $O(h^3)$:

$$y(t_k + h) = y(t_k) + \frac{h}{2}(y'(t_k) + y'(t_{k+1})) + \frac{h^3}{6}y''(\xi), \quad \xi \in (t_k, t_k + h)$$

The FGE in this method is of order h^2 . This means that halving the step size will reduce the error by 25%.

t	y0	f(t0,y0)	t1	y1p	f(t1,y1p)	y1c
0	0.5	1.5	0.2	0.8	1.72	0.822
0.2	0.822	1.742	0.4	1.1704	1.8504	1.18124
0.4	1.18124	1.86124	0.6	1.553488	1.833488	1.550713
0.6	1.550713	1.830713	0.8	1.916855	1.636855	1.89747
0.8	1.89747	1.61747	1	2.220964	1.220964	2.181313
1	2.181313	1.181313	1.2	2.417576	0.537576	2.353202
1.2	2.353202	0.473202	1.4	2.447842	-0.47216	2.353306
1.4	2.353306	-0.56669	1.6	2.239967	-1.88003	2.108634
1.6	2.108634	-2.01137	1.8	1.70636	-3.77364	1.530133
1.8	1.530133	-3.94987	2	0.740159	-6.25984	0.509162
2	0.509162	-6.49084	2.2	-0.78901	-9.46901	-1.08682

Table 1.2 Modified Euler Method Example 1.4

[Reference excel sheet [modified-euler.xlsx](#)]

The Euler method and Modified Euler methods are explicit single step methods as they need to know the solution at a single step. It may be observed that the Euler method is derived by replacing derivative by forward difference:

$$\left. \frac{dy}{dt} \right|_{t=t_k} = \frac{y_{k+1} - y_k}{h} + O(h)$$

The central and backward difference approximation can also be used to give single step methods

$$\left. \frac{dy}{dt} \right|_{t=t_k} = \frac{y_k - y_{k-1}}{h} + O(h) \quad \text{or} \quad \left. \frac{dy}{dt} \right|_{t=t_k} = \frac{y_{k+1} - y_{k-1}}{2h} + O(h^2)$$

