Course : Optimization
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1. Show that (0, 0) is a saddle point of $f(x,y) = x^6 + (x-y)^3$.

   Ans. Let us observe the behavior of the function near (0, 0),

$$f(\delta^2, 0) - f(0,0) = \delta^6 + \delta^{12} > 0$$
$$f(0, \delta^2) - f(0,0) = -\delta^6 < 0$$

   Here, $f(x,y) - f(0,0) > 0$ and $f(x,y) - f(0,0) < 0$ in the neighborhood of (0, 0). (0,0) is a saddle point.

2. Solve the optimization problem $Minimize \sin x + \sin y + \sin(x+y) such\ that\ 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}$.

   Ans. Here $f(x,y) = \sin x + \sin y + \sin(x+y)$

$$\Rightarrow \begin{cases} f_x = \cos x + \cos(x+y) = 0 \\ f_y = \cos y + \cos(x+y) = 0 \end{cases} \Rightarrow \cos x - \cos y = 0 \Rightarrow x = (2k\pi \pm y)$$

$$\Rightarrow x = \pm y \Rightarrow x = y = \frac{\pi}{3}\ as\ 0 \le x \le \frac{\pi}{2}\ and\ 0 \le y \le \frac{\pi}{2}.$$

   Thus, $(\frac{\pi}{3}, \frac{\pi}{3})$ is a stationary point. Let us determine the Hessian matrix,

$$\nabla^2 f = \begin{pmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin y - \sin(x+y) \end{pmatrix}$$

$$\Rightarrow \nabla^2 f|_{(\frac{\pi}{3},\frac{\pi}{3})} = \begin{pmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

   This is a negative definite matrix. Hence $(\frac{\pi}{3}, \frac{\pi}{3})$ is a local maximum point.

3. Find the Lagrangian and KKT point for the following problem:

$$Minimize \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$$
$$subject\ to\ x_1 + x_2 + x_3 = b.$$

   The Lagrangian function may be defined as

$$L = (\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}) + \lambda(x_1 + x_2 + x_3 - b).$$

   Hence the optimality conditions are:

$$L_{x_1} = 0 \Rightarrow \left(\frac{1}{2\sqrt{x_1}}\right) + \lambda = 0 \ldots\ldots\ldots\ldots\ldots (1)$$
$$L_{x_2} = 0 \Rightarrow \left(\frac{1}{2\sqrt{x_2}}\right) + \lambda = 0 \ldots\ldots\ldots\ldots\ldots (2)$$
$$L_{x_3} = 0 \Rightarrow \left(\frac{1}{2\sqrt{x_3}}\right) + \lambda = 0 \ldots\ldots\ldots\ldots\ldots (3)$$

   From (1), (2) and (3) we get $x_1 = x_2 = x_3$.

   Feasible condition is $x_1 + x_2 + x_3 = b, \lambda \ge 0 \ldots\ldots\ldots\ldots\ldots (4)$

Thus we get $x_1 = x_2 = x_3 = \frac{b}{3}$ and $\lambda^2 = \frac{1}{4x_1} \Rightarrow \lambda = \sqrt{\frac{3}{4b}}$. Hence $\left(\frac{b}{3}\ , \frac{b}{3}\ , \frac{b}{3}, \sqrt{\frac{3}{4b}}\ \right)$ is a KKT point.

4. Test whether the KKT point in example 3 is optimal.

Ans. At KKT point $\left(\frac{b}{3}\ , \frac{b}{3}\ , \frac{b}{3}, \sqrt{\frac{3}{4b}}\ \right)$ we need to find the value of $\nabla^2 L$.

$$\nabla^2 L = \begin{pmatrix} -\frac{1}{4}x_1^{-\frac{3}{2}} & 0 & 0 & 1 \\ 0 & -\frac{1}{4}x_1^{-\frac{3}{2}} & 0 & 1 \\ 0 & 0 & -\frac{1}{4}x_1^{-\frac{3}{2}} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

At KKT point this is a negative definite matrix. Hence the above KKT point is a maximum point.

5. Find and classify the stationary points of $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - \frac{3}{2}x^2 - 4y$.

Ans. The stationary points are given by $f_x = 0, f_y = 0$. This condition gives *x*=0, 3 and *y* = 2, -2. Hence the stationary points are $\{(0,2), (0, -2), (3, 2), (3, -2)\}$.

Now $f_{xx} = 2x - 3, f_{xy} = 0\ and\ f_{yy} = 2y$. The corresponding Hessian matrix is $\begin{pmatrix} 2x-3 & 0 \\ 0 & 2y \end{pmatrix}$. At (0,2) the Hessian matrix is $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$, so (0,2) is a saddle point. At (0,-2) the Hessian matrix is $\begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$, which is a negative definite matrix. So (0,-2) is a local maximum. At (3,2) the Hessian matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, which is a positive definite matrix. So (3,2) is a local minimum. Again (3, -2) is a saddle point.

6. For the following problem establish $\nabla f = -\nabla g$:

$$Maximize\ f(x_1, x_2) = -\frac{1}{2}(x_1 + \left(x_2 + \frac{1}{2}\right)^2 + \frac{1}{8}$$

$$Subject\ to\ x_1 + x_2 - 2 = 0.$$

Ans. The Lagrangian function is

$$L(x_1, x_2, \lambda) = -\frac{1}{2}(x_1 + \left(x_2 + \frac{1}{2}\right)^2 + \frac{1}{8} + \lambda(x_1 + x_2 - 2)$$

The necessary conditions for optimality are: $L_{x_1} = 0, L_{x_2} = 0, L_\lambda = 0.$

From above three equations we get the stationary point as (2, 0). Now if we calculate ∇f and ∇g at the stationary point, the required result will be established.

7. Let $f(x) = 0.65 - \frac{0.75}{1+x^2} - 0.65x\, tan^{-1}\left(\frac{1}{x}\right)$. Find the minimum of *f(x)* using search with a fixed step 0.1 from the starting point 0.0.

Ans.

i	Value of step length	$x_i = x_{i-1} + s$	$f_i = f(x_i)$	Is $f_i > f_{i-1}$
1	-	0.0	-0.1	No
2	.1	0.1	-0.18819	No
3	.1	0.2	-0.249	No
4	.1	0.3	-0.2875	No
5	.1	0.4	-0.30602	No
6	.1	0.5	-0.30982	No
7	.1	0.6	-0.30331	Yes

Since $x_6 > x_7$, optimal must lie within [.4, .5].

8. Find minimum of $f(x) = 0.65 - \frac{0.75}{1+x^2} - 0.65x\, tan^{-1}\left(\frac{1}{x}\right)$ using Fibonacci method within [0, 3]. Achieve accuracy within 10% of exact value.

Ans. Here the initial interval of uncertainty is [0,3]. The ratio $\frac{L_n}{L_0}$ determines the required number of experiments in Fibonacci method. We know $\frac{L_n}{L_0} = \frac{1}{F_n} \leq \frac{1}{10}$, as we need to achieve accuracy 10%. Which gives the minimum value of *n* as 6.

Now the process starts, let all the iterations may be summarized as follows:

Ite. no.	Initial interval of uncertainty	Required calculations	Reason for decision	Final interval of uncertainty
1	[0,3]	$L_2^* = \frac{F_4}{F_6} \times 3 = 1.153846$ $x_1 = 0 + L_2^*$ $= 1.1513846$ $x_2 = 3 - L_2^*$ $= 1.846154$	$f(x_1) = -0.2072$ $f(x_2) = -0.115$ $f(x_2) > f(x_1)$	[0,1.846154]
2	[0,1.846154]	$L_3^* = \frac{F_3}{F_6} \times 3 = .692308$ $x_3 = .692308$	$f(x_1) = -0.2072$ $f(x_3) = -0.291$ $f(x_1) > f(x_3)$	[0, 1.1513846]
3	[0, 1.1513846]	$L_4^* = \frac{F_2}{F_6} \times 3 = .461538$ $x_4 = .461538$	$f(x_4) = -0.3098$ $f(x_3) = -0.291$ $f(x_3) > f(x_4)$	[0, .692308]
4	[0, .692308]	$L_5^* = \frac{F_1}{F_6} \times 3 = .23077$ $x_5 = .23077$	$f(x_4) = -0.3098$ $f(x_5)$ $= -0.2636$ $f(x_5) > f(x_4)$	[.23077, .692308]
5	[.23077, .692308]	$L_6^* = \frac{F_0}{F_6} \times 3 = .23076$ $x_6 = .461540$	$f(x_4) = -0.309811$ $f(x_6) = -.309810$ $f(x_6) > f(x_5)$	[.23077, 461540]

Hence the final interval of uncertainty is $[.23077, 461540]$. The middle value of the interval may be declared as optimal solution.

9. Apply geometric programming technique to solve the following:

$$Minimize\ f(x_1, x_2) = x_1^{-3}x_2 + x_1^{3/2}x_2^{-1} + x_1^2 x_2^{5/2}.$$

Ans. The degree of difficulty of this problem is zero. Hence we'll get unique solution. The dual of the given problem is

$$Maximize\ v(\delta) = \left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}$$

$$subject\ to \begin{pmatrix} -3 & \frac{3}{2} & 2 \\ 1 & -1 & \frac{5}{2} \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\\ \delta_3\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}, \delta_i > 0, i = 1,2,3$$

Solving the following equations we'll get the values of three unknowns

$$-3\delta_1 + \frac{3}{2}\delta_2 + 2\delta_3 = 0$$

$$\delta_1 - \delta_2 + \frac{5}{2}\delta_3 = 0$$

$$\delta_1 + \delta_2 + \delta_3 = 0$$

$$\Rightarrow \delta_1 = 0.3432, \delta_2 = 0.5671, \delta_3 = 0.0895$$

And corresponding objective functional value of dual problem is

$$v(\delta*) = \left(\frac{1}{.3432}\right)^{.3432}\left(\frac{1}{.5671}\right)^{.5671}\left(\frac{1}{.0895}\right)^{.0895} = 2.4702$$

Considering this as the optimal of primal objective function we get the optimal value for primal problem as $x_1 = .89175, x_2 = 0.6011$.

10. Suppose that we wish to minimize the posynomial

$f_0(x) = 40x_1^{-1}x_2^{-1/2}x_3^{-1} + 20x_1x_3 + 20x_1x_2x_3$ Subject to the constraint

$f_1(x) = \frac{1}{3}x_1^{-2}x_2^{-2} + \frac{4}{3}x_2^{1/2}x_3^{-1} \le 1$ and $x_1, x_2, x_3 > 0$.

This problem is having one degree of difficulty. The dual program associated with this problem consists of maximizing the dual function

$$v(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{20}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{1}{3\delta_4}\right)^{\delta_4}\left(\frac{4}{3\delta_5}\right)^{\delta_5}(\delta_4+\delta_5)^{(\delta_4+\delta_5)}$$

subject to the dual constraints

$$\delta_1 \ge 0, \delta_2 \ge 0, \delta_3 \ge 0, \delta_4 \ge 0, \delta_5 \ge 0$$
$$\delta_1 + \delta_2 + \delta_3 = 1$$
$$-\delta_1 + \delta_2 + \delta_3 - 2\delta_4 = 0$$
$$-\frac{1}{2}\delta_1 + \delta_3 - 2\delta_4 + \frac{1}{2}\delta_5 = 0$$
$$-\delta_1 + \delta_2 + \delta_3 - \delta_5 = 0$$

Finally solving dual constraints we get

$$\delta_1 = 1 - 2r, \delta_2 = r, \delta_3 = r, \delta_4 = -\frac{1}{2} + 2r, \delta_5 = -1 + 4r$$

It is clear from these equations that δ satisfies the positivity condition only when r is restricted so that $\frac{1}{4} \le r \le \frac{1}{2}$. Now we get dual objective function as

$$\max\ v(\delta) = v(r) = \left(\frac{40}{1-2r}\right)^{1-2r} \left(\frac{20}{r}\right)^{2r} \left(\frac{2}{3(4r-1)}\right)^{\frac{4r-1}{2}} \left(\frac{4}{3(4r-1)}\right)^{4r-1} \left(\frac{3(4r-1)}{2}\right)^{\frac{3(4r-1)}{2}}$$

where $\frac{1}{4} \le r \le \frac{1}{2}$

It is a function of single variable. By calculus we find maximum values of $v(r) = 99.9999$ when r=0.4. So $\delta_1 = 0.2, \delta_2 = 0.4, \delta_3 = 0.4, \delta_4 = 0.3, \delta_5 = 0.6$ and from primal-dual variable relationship we get $x_1 = 1, x_2 = 1, x_3 = 2$.

11. Solve the following problem using Dynamic programming technique

$$Minimize\ x_1 + x_2 + x_3$$
$$subject\ to\ x_1 x_2 x_3 = 15.$$

Ans. Let us define state variables as

$$s_1 = x_1 \qquad s_2 = x_1 x_2 \qquad s_3 = x_1 x_2 x_3$$

Considering the first stage we have, $f_1(s_1) = 15$. And in stage 2 $x_1 x_2 = 15 \Rightarrow x_1 = \frac{15}{x_2}$.

Then $f_2(s_2) = \underset{x_2}{Minimize}(x_1 + x_2) = \underset{x_2}{Minimize}\left(\frac{15}{x_2} + x_2\right)$

Using differential calculus technique we get $x_2 = 15^{\frac{1}{2}}$ and $x_1 = 15^{\frac{1}{2}}$. So $f_2(s_2) = 15$.

Now moving to the next stage 3, $(x_1 x_2 x_3 = 15) \Rightarrow s_2 = \frac{15}{x_3}$. Then

$$f_3(s_3) = \underset{x_3}{Minimize}\left(x_3 + f_2(s_2)\right) = \underset{x_3}{Minimize}\left(x_3 + 2\left(\frac{15}{x_3}\right)^{1/2}\right)$$

Differentiating with respect to x_3, we get $x_3 = 15^{\frac{1}{3}}$. And the corresponding optimal value of the objective function is $x_3 + 2\left(\frac{15}{x_3}\right)^{1/2} = 3 \times 15^{1/3}$.