

Course: Optimization

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Problem 1. Solve the following system of Linear equation:

$$x_1 + x_2 + x_3 = 2$$

$$3x_1 + 5x_2 + x_3 = 4$$

$$x_1 + 2x_2 + 3x_3 = 4$$

Solution:

Since the given matrix is non-homogeneous and the coefficient matrix of the given equation is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 2 & 3 \end{pmatrix} \text{ which is non-singular; the system has unique solution.}$$

$$\text{Also the determinant of } A = \det. \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 2 & 3 \end{pmatrix} = 6$$

Now, using Cramer's rule

$$x_1 = \frac{1}{6} \det. \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 1 \\ 4 & 2 & 3 \end{pmatrix} = \frac{6}{6} = 1$$

$$x_2 = \frac{1}{6} \det. \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 1 & 4 & 3 \end{pmatrix} = \frac{0}{6} = 0$$

$$x_3 = \frac{1}{6} \det. \begin{pmatrix} 1 & 1 & 2 \\ 3 & 5 & 4 \\ 1 & 2 & 4 \end{pmatrix} = \frac{6}{6} = 1$$

We obtain following solution: $x_1 = 1, x_2 = 0, x_3 = 1$

Problem 2. Show that following set vectors are linearly independent or not?

$$\{(2,11,13),(5,7,3),(8,17,19)\}$$

Simple and best way to check this using determinant. If the determinant vanishes vectors are independent ; otherwise dependent.

Here

$$\det. \begin{pmatrix} 2 & 11 & 13 \\ 5 & 7 & 3 \\ 8 & 17 & 19 \end{pmatrix} = -240$$

Which means that determinant is nonzero; hence vectors are independent.

Problem 3 (Production Planning Problem)

Avatar & Co. plans to produce two types of electric bulbs: type A and type B. Each type-A bulb will result in a profit of \$1, and each type-B bulb will result in a profit of \$1.20. To manufacture a type-A bulb requires 2 minutes on machine I and 1 minute on machine II. A type-B bulb requires 1 minute on machine I and 3 minutes on machine II. There are 3 hours available on machine I and 5 hours available on machine II. How many bulbs of each type should Ace make in order to maximize its profit?

Solution: As a first step toward the mathematical formulation of this problem, we arrange given information in the following table:

TABLE 1

	Type A	Type B	Time Available
Machine I	2 min	1 min	180 min
Machine II	1 min	3 min	300 min
Profit/Unit	\$1	\$1.20	

Let x be the number of type - A bulbs and y the number of type -B bulbs to be made . Then the total profit P (in dollars) is given by

$$P=x+1.2y$$

which is the objective function to be maximized.

The total amount of time that machine I is used is given by $2x+y$ minutes and must not exceed 180 minutes. Thus, we have the inequality $2x + y \leq 180$

Similarly, the total amount of time that machine II is used is $x + 3y$ minutes and cannot exceed 300 minutes, so we are led to the inequality $x + 3y \leq 300$.

Finally, neither x nor y can be negative, so $x \geq 0$ $y \geq 0$

To summarize, the problem at hand is one of maximizing the objective function

$$P = x + 1.2y$$

subject to the system of inequalities

$$2x + y \leq 180$$

$$x + 3y \leq 300$$

$$x \geq 0, y \geq 0$$

Problem 4: Solve the following Linear Programming Problem using Graphical Method:

Maximize $z = 2x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 1$

$$3x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

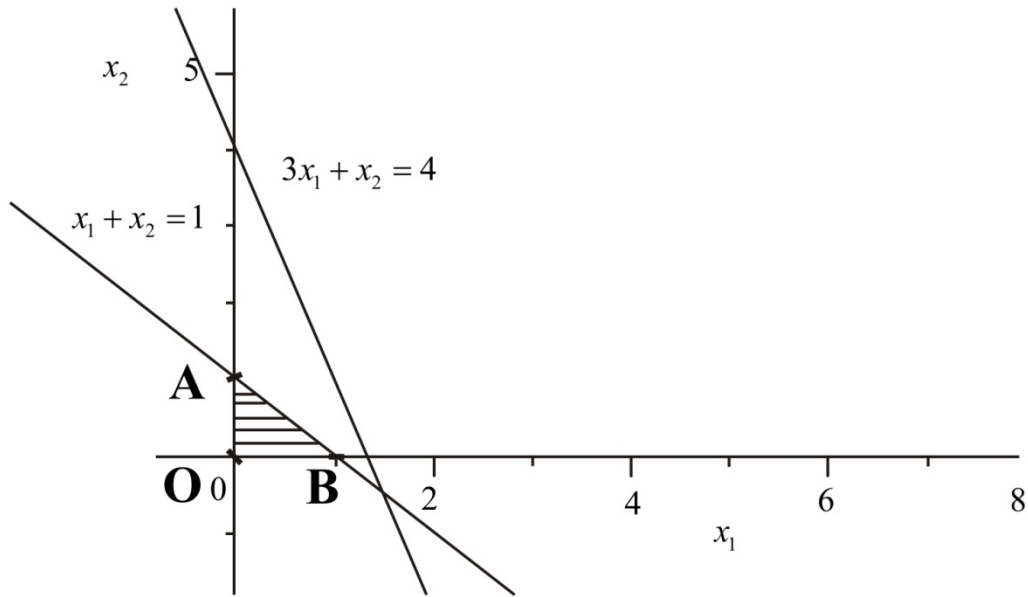
Solution:

Let us convert both of the constraints from ' \leq ' type to '=' type.

$$x_1 + x_2 = 1$$

$$3x_1 + x_2 = 4$$

We now draw corresponding lines for each equation as follows and shade the feasible region OAB.



We know that optimal solution occur at the corner point of the feasible region, which is either at $O(0,0)$ or $A(0,1)$ or $B(1,0)$.

Extreme point	$O(0,0)$	$A(0,1)$	$B(1,0)$.
Value of z	0	3	2

Since, value of z is maximum at $A(0,1)$. The optimal solution occurs at A .

Explicitly,

$$x_1 = 0, x_2 = 1 \text{ and } z = 3$$

[NB:-This problem is solved by simplex method in the next chapter's assignment.]

Problem 5. Solve the following Linear Programming Problem by Graphical Approach:

Maximize $z = 7x_1 + 2x_2$

Subject to

$$x_1 - 2x_2 \geq 1$$

$$x_1 + x_2 = 2$$

$$2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution:

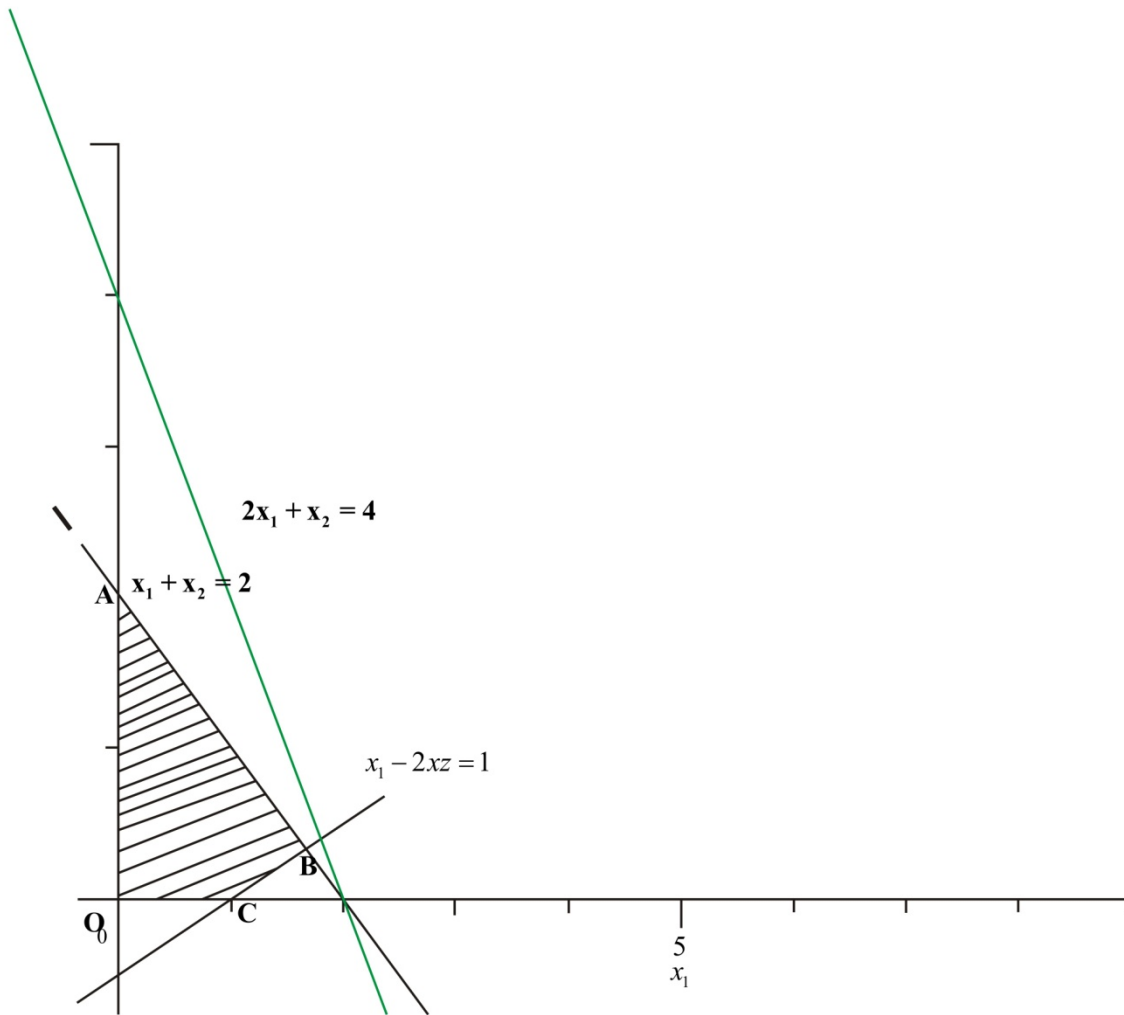
Let us convert each both of the constraints from ' \leq ' type to ' $=$ ' type.

$$x_1 - 2x_2 = 1$$

$$x_1 + x_2 = 2$$

$$2x_1 + x_2 = 4$$

We now draw corresponding lines for each equation as follows and shade the feasible region OABC



We know that optimal solution occur at the corner point of the feasible region, which is either at $O(0,0)$ or $A(0,2)$ or $B(5/3,1/3)$ or $C(1,0)$.

We now evaluate the value of z at each extreme pont(corner point).

Extreme point	O(0,0)	A(0,2)	B(5/3,1/3)	C(1,0)
Value of z	0	4	37/3	7

Thus we see that extreme point occurs at B(5/3,1/3).

Explicitly, we can write:

$$x_1 = \frac{5}{3}, x_2 = \frac{1}{3} \text{ and optimal value of objective function } z = \frac{37}{3}$$

Problem 6: Solve the following Linear Programming Problem:

$$\text{Maximize } z = 2x_1 + 3x_2$$

Subject to

$$x_1 + x_2 \leq 1$$

$$3x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution:

Let us introduce slack variables $x_3 \geq 0, x_4 \geq 0$ since both of the constraints are \leq type.

Standard form of given LPP becomes:

$$\text{Maximize } z = 2x_1 + 3x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

Subject to

$$x_1 + x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 1$$

$$3x_1 + x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 4$$

$$\text{where } x_1, x_2, x_3, x_4 \geq 0$$

We obtain follc

c_j	2	3	0	0
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c_B	B	x_B	b	a_1	a_2	a_3	a_4	$\frac{x_{Br}}{y_{rj}}$
0	a_3	x_3	1	1	1	1	0	$1 \rightarrow$
0	a_4	x_4	4	3	1	0	1	4
$z_j - c_j$				-2	-3 \uparrow	0	0	

Since $z_j - c_j < 0 \forall j = 1, 2$

This means the solution is not optimal. We proceed to the next tableau.

			c_j	2	3	0	0
c_B	B	x_B	b	a_1	a_2	a_3	a_4
3	a_2	x_2	1	1	1	1	0
0	a_4	x_4	3	2	0	-1	1
$z_j - c_j$				1	0	3	0

Since $z_j - c_j \geq 0 \forall j = 1..4$ -which shows that optimality is achieved.

And the optimal solution is given by:

$x_1 = 0, x_2 = 1$ optimal objective value $z = 3$

Problem 7. Solve the following Linear Programming Problem:

$$\text{Maximize } z = x_1 + x_2 + 3x_3$$

Subject to

$$3x_1 + 2x_2 + x_3 \leq 3$$

$$2x_1 + x_2 + 2x_3 \leq 2$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Let us introduce slack variables $x_4 \geq 0, x_5 \geq 0$ since both of the constraints are \leq type.

Standard form of given LPP becomes:

$$\text{Maximize } z = x_1 + x_2 + 3x_3 + 0x_4 + 0x_5$$

Subject to

$$3x_1 + 2x_2 + x_3 + x_4 = 3$$

$$2x_1 + x_2 + 2x_3 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We obtain following initial tableau:

			c_j	1	1	3	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{ij}}$
0	a_4	x_4	3	3	2	1	1	0	3
0	a_5	x_5	2	2	1	2	0	1	$1 \rightarrow$
$z_j - c_j$				-1	-1	$-3 \uparrow$	0	0	

Since $z_j - c_j$ is most negative for a_3 the variable x_3 becomes entering variable in the next basis.

Also since $\frac{x_{Br}}{y_{rj}}$ is minimum for second row, the corresponding variable x_5 is the leaving variable.

The pivot element becomes 2.

			c_j	1	1	3	0	0	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
0	a_4	x_4	2	2	3/2	0	1	-1/2	
3	a_3	x_3	1	1	1/2	1	0	1/2	
$z_j - c_j$			2	1/2	0	0	3/2		

Since $z_j - c_j \geq 0 \quad \forall j = 1..5$ -which shows that optimality is achieved.

The optimal solution is unique since $z_j - c_j > 0$ for non-basic variable and $z_j - c_j = 0$ for the basic variables.

And the optimal solution is given by: $x_1 = 0, x_2 = 0, x_3 = 1$ optimal objective value $z = 3$

Problem 8:

Solve the following Linear Programming Problem by Two Phase method:

Maximize $z = x_1 + 5x_2$

Subject to

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solution:

Let us introduce slack variables $x_3 \geq 0$, for the first constraint which is of \leq type

We also include surplus variable $x_4 \geq 0$ and artificial variable $x_5 \geq 0$ for the second constraint; since it is \geq type.

Standard form of given LPP becomes:

Maximize $z = x_1 + 5x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$

Subject to

$3x_1 + 4x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = 6$

$x_1 + 3x_2 + 0 \cdot x_3 - 1 \cdot x_4 + 1 \cdot x_5 = 3$

where $x_1, x_2, x_3, x_4, x_5 \geq 0$

PHASE-1

The auxiliary objective function for phase-1 becomes:

$z_A = -1 \cdot x_5$

We obtain following initial tableau:

			c_j	0	0	0	0	-1	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
-1	a_5	x_5	3	1	3	0	-1	1	1 \rightarrow
0	a_3	x_3	6	3	4	1	0	0	1.25
$z_j - c_j$				-1	-3 \uparrow	0	1	0	

Since $z_j - c_j < 0 \forall j=1,2$

This means the solution is not optimal. a_5 leaves the basis and a_2 enters the basis.

We proceed to the next tableau.

			c_j	0	0	0	0	-1	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
0	a_2	x_2	1	1/3	1	0	-1/3	1/3	
0	a_3	x_3	2	5/3	0	1	4/3	-4/3	
$z_j - c_j$				0	0	0	0	1	

Since $z_j - c_j \geq 0 \forall j=1..5$ -and artificial variable is in zero level; the optimal solution of the original problem exists(Phase –I is completed). Now we move to the next level; Phase –II.

PHASE-II

We obtain following initial tableau for Phase –II :

			c_j	1	5	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	$\frac{x_{Br}}{y_{rj}}$
5	a_2	x_2	1	1/3	1	0	-1/3	—
0	a_3	x_3	2	5/3	0	1	4/3	1.5

$z_j - c_j$	2/3	0	0	-5/3	
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Since $z_j - c_j$ is negative for a_4 the variable x_4 becomes entering variable in the next basis. Also since

$\frac{x_{Br}}{y_{rj}}$ is defined only for second row, the corresponding variable x_3 is the leaving variable.

The pivot element becomes **4/3**.

			c_j	1	5	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	$\frac{x_{Br}}{y_{rj}}$
5	a_2	x_2	3/2	1	1	1/4	0	
0	a_4	x_4	3/2	5/4	0	3/4	1	
$z_j - c_j$				4	0	5/4	0	

Since $z_j - c_j \geq 0 \forall j = 1..5$ -This shows that optimality is achieved.

The optimal solution is unique since $z_j - c_j > 0$ for non-basic variable and $z_j - c_j = 0$ for the basic variables.

And the optimal solution is given by: $x_1 = 0, x_2 = \frac{3}{2}$ optimal objective value $z_{\max} = \frac{15}{2}$.

Problem 9: Solve the following Linear Programming Problem by Two Phase method:

$$\text{Maximize } z = x_1 + 5x_2$$

Subject to

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Problem 10 : Solve the following Linear Programming Problem by Big-M method:

$$\text{Maximize } z = x_1 + 5x_2$$

Subject to

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solution:

Let us introduce slack variables $x_3 \geq 0$, for the first constraint which is of \leq type

We also include surplus variable $x_4 \geq 0$ and artificial variable $x_5 \geq 0$ for the second constraint; since it is \geq type.

Standard form of given LPP becomes:

$$\text{Maximize } z = x_1 + 5x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$$

Subject to

$$3x_1 + 4x_2 + 1 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = 6$$

$$x_1 + 3x_2 + 0 \cdot x_3 - 1 \cdot x_4 + 1 \cdot x_5 = 3$$

$$\text{where } x_1, x_2, x_3, x_4, x_5 \geq 0$$

The auxiliary objective function for Big-M becomes:

$$z_A = x_1 + 5x_2 + 0 \cdot x_3 + 0 \cdot x_4 - M \cdot x_5$$

We obtain following initial tableau:

			c_j	1	5	0	0	-M	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
-M	a_5	x_5	3	1	3	0	-1	1	1 →
0	a_3	x_3	6	3	4	1	0	0	1.25
$z_j - c_j$			-M-1	-3M-5 ↑	0	M	0	0	

Since $z_j - c_j < 0 \forall j=1,2$

This means the solution is not optimal. a_5 leaves the basis and a_2 enters the basis.

We proceed to the next tableau.

			c_j	1	5	0	0	-M	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
5	a_2	x_2	1	1/3	1	0	-1/3	1/3	—
0	a_3	x_3	2	5/3	0	1	4/3	-4/3	3/2
$z_j - c_j$			2/3	0	0	0	-5/3 ↑	5/3+M	

Since $z_3 - c_3 < 0$

This means the solution is not optimal. a_3 leaves the basis and a_4 enters the basis.

We proceed to the next tableau.

			c_j	1	5	0	0	-M	
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	$\frac{x_{Br}}{y_{rj}}$
5	a_2	x_2	3/2	1	1	1/4	0	5	
0	a_4	x_4	3/2	5/4	0	3/4	1	0	
$z_j - c_j$				4	0	5/4	0	25+M	

Since $z_j - c_j \geq 0 \forall j=1..5$ -This shows that optimality is achieved.

The optimal solution is unique since $z_j - c_j > 0$ for non-basic variable and $z_j - c_j = 0$ for the basic variables.

And the optimal solution is given by: $x_1 = 0, x_2 = \frac{3}{2}$ optimal objective value $z_{\max} = \frac{15}{2}$.

Problem 11. Solve the following Linear Programming Problem by Two Phase method:

$$\text{Maximize } z = x_1 + 5x_2$$

Subject to

$$3x_1 + 4x_2 \leq 6$$

$$x_1 + 3x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Problem 12. Convert the following Linear Programming Problem to its dual formulation:

$$\text{Maximize } z = x_1 - x_2 + 3x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 10$$

$$x_1 - x_3 \leq 2$$

$$2x_1 - 2x_2 + 3x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Since the problem is already stated in Canonical form.

Primal LP problem is maximization \Rightarrow Dual LP problem is minimization.

Primal LP problem has 3 variables \Rightarrow Dual LP problem will have 3 constraints.

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Using dual variables $v_1, v_2, v_3 \geq 0$ corresponding to first, second and third constraints respectively.

We now write the dual formulation of the given LPP as:

$$\text{Minimize } w = v_1 + 5v_2 + 6v_3$$

Subject to

$$v_1 + 2v_2 + 2v_3 \geq 1$$

$$-v_1 + 2v_3 \leq 1$$

$$v_1 - v_2 + 3v_3 \geq 3$$

$$\text{where } v_1, v_2, v_3 \geq 0$$

Problem 13. Convert the following Linear Programming Problem to its dual formulation:

$$\text{Maximize } z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$$

Subject to

$$4x_1 - 5x_2 - 9x_3 + x_4 - 2x_5 \leq 6$$

$$2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 \leq 9$$

$$x_1 + x_2 - 5x_3 - 7x_4 + 11x_5 \leq 10$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution:

The problem is already stated in Canonical form.

Primal LP problem is maximization \Rightarrow Dual LP problem is minimization.

Primal LP problem has 5 variables \Rightarrow Dual LP problem will have 5 constraints.

Primal LP problem has 3 constraints \Rightarrow Dual LP problem will have 3 variables.

Using dual variables $v_1, v_2, v_3 \geq 0$ corresponding to first, second and third constraints respectively.

We write the dual formulation of the given LPP as:

$$\text{Minimize } w = v_1 + 5v_2 + 6v_3$$

Subject to

$$4v_1 + 2v_2 + 3v_3 \geq 3$$

$$-5v_1 + 3v_2 + v_3 \geq 1$$

$$-9v_1 + 4v_2 - 5v_3 \geq 4$$

$$v_1 - 5v_2 - 7v_3 \geq 1$$

$$-2v_1 + v_2 + 11v_3 \geq 9$$

$$\text{where } v_1, v_2, v_3 \geq 0$$