Notations:

IVP: Initial Value Problem

BVP: Boundary Value Problem

Ordinary Differential Equations

(1) Find the region absolute stability of the finite difference method $2y_{n+2} = y_{n+1} - \frac{h}{3}(2f_{n+1} - f_n)$

Ans :

$\rho(z) = 2z^2 - z$

$\sigma(z) = -\frac{1}{3}(2z-1)$

Characteristic equation : $2z^2 - z + \frac{\bar{h}}{3}(2z-1) = 0$

where $\bar{h} = \lambda h$

Roots: $z = \frac{1}{2}, -\frac{\bar{h}}{3}$

region of absolute stability $|\bar{h}| < 3$

(2) Consider the IVP $y^1 = 2y - 2x^2 - 3$

$y(0) = 2$

Use Picard's method to obtain $y(0.2)$ upto 3 decimal places.

Ans :

$y^{(3)}(x) = 2 + x + x^2 - \frac{x^4}{3} - \frac{x^5}{15} + ...$

$y_{(1)}(0.2) = 2.24$

$y_{(2)}(0.2) = 2.2416$

$y_{(3)}(0.2) = 2.24192$

(3) Show that the 4th order Adams- Bash forth method

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

is strongly stable.

Ans:

Characteristic equation: $\xi^4 - \xi^3 = 0$

Roots: $\xi_i = 1, 0, 0, 0$

$\therefore$ Root condition is satisfied and hence strongly stable.

(4) Given $y^1 = \left(1 + x^2\right) y^2$ and $y(0) = 1$, use Milne-Simpson's P-C method to obtain y at $x = 0.4$ using h = 0.1. The required past values are given by

$$y(0.1) = 1.06,\ y(0.2) = 1.12,\ y(0.3) = 1.21$$

Ans:

$$y_{n+1}^{(p)} = y_{n-3} + \frac{4h}{3}\left(2y_{n-1}^1 - y_{n-2}^1 + 2y_n^1\right)$$

$$y_{(0.4)}^{(p)} = 1.5543$$

$$y_4^{(1)} = 1.4697$$

(5) Solve the initial value problem $y^1 = -3xy^2, y(0) = 1$, with $h = 0.2$ in $[0, 0.4]$ using the method

$$y_{n+1} = y_n + hy_{n+1}^1$$

Ans :

$$y(0.2) \simeq 0.9023$$

$$y(0.4) \simeq 0.7971$$

(6) Solve the system of equations

$$y' = -3y + 2x,\ y(0) = 0.5$$

$$x' = 3y - 4x,\ x(0) = 0$$

With $h = 0.2$ as the interval $(0, 0.4)$ using Euler - Cauchy method

Ans :

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$
$$k_1 = hf\left(x_n, y_n\right)$$
$$k_2 = hf\left(x_n + h, y_n + k_1\right)$$
$$x(0.2) \simeq 0.96,\ y(0.2) \simeq 0.35$$
$$x(0.4) \simeq 0.6774,\ y(0.4) \simeq 0.3602$$

(7) Use Adams-Bash forth third order method to use the IVP $y' = -2xy^2, y(0) = 0.5$ on $[0,1]$ with $h = 0.2$

Ans :

$$y_{n+1} = y_n + \frac{h}{12}(23 f_n - 16 f_{n-1} + 5 f_{n-2})$$
$$y(0.2) \simeq 0.49,\ y(0.4) = 0.4624$$
$$y(0.6) \simeq 0.4224,\ y(0.8) = 0.3779$$
$$y(1) \simeq 0.3332$$

(8) Consider the heat conduction equation

$$u_t = \frac{1}{2} u_{xx}, 0 \le x \le 1$$
$$u(x,0) = 20 + 40x$$
$$u(0,t) = 20e^{-t}; u(1,t) = 60e^{-2t}$$

Obtain the solution $\mu(x,t)$ at first time level using Grant-Nicolson method. Choose $h = 1/4, k = 1/10$ **Ans :**

$$u_{1,1} = 29.42144$$
$$u_{2,1} = 39.29975$$
$$u_{3,1} = 47.42746$$

(9) Consider a steel rod that is subjected to a temperature of $50^0 C$ on the left end and $25^0 C$ on the right. If the length of the rod is 0.05m, use implicit method to find the distribution in the rod from $t = 0$ to $t = 6$ seconds. Use $\delta x = h = 0.01m;\ \delta t = k = 3s$. Given thermal conductivity $k = 54$; density $\rho = 7800$, specific heat $c = 490$. Choose initial temperature as $20^0 C$.

Ans :

If $\theta(x,t)$ denote temperature

$\theta(0,t)=50;\ \theta(0.05,t)=25$

$\theta_{i,0}=\theta(x_i,0)=20,\ i=1,2,3,4$

$\theta_{1,1}=268.939;\ \theta_{2,1}=204.972;\ \theta_{3,1}=204.119;\ \theta_{4,1}=209.924$

$\theta_{1,2}=2003.4539;\ \theta_{2,2}=1918.3812;\ \theta_{3,2}=1915.3948$

$\theta_{4,2}=1614.8802$

(10) In order to solve the equation $y'=f(x,y)$ the following method has been defined

$$y_{n+1}=y_n+W_1K_1+W_2K_2$$
$$K_1=hf(x_n,y_n),\ K_2=hf(x_{n+h},y_n+\beta K_1)$$

Find β, w_1, w_2 such that the order of the method is two.

Ans: $\beta=1,\ W_1=1/2,\ W_2=1/2$

(11) Using central difference for the derivatives, discretize

$$y^{11}=xy,\ y(0)+y'(0)=1,\ y(1)=1$$

with $h=1/3$ and then solve for $y(0),\ y(1/3)$ and $y(2/3)$.

Ans: $y(0)=-\frac{82}{83}, y(1/3)=-\frac{27}{83}, y(2/3)=\frac{27}{83}$

(12) For the parabolic equation,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

$u(x,0)=1$ for $0\le x\le 1$

$$\frac{\partial u(0,t)}{\partial x}=u,\ \frac{\partial u(1,t)}{\partial x}=-u$$

Use explicit method with $\Delta x=0.1$ and $\lambda=\frac{1}{4}$ where λ is the the usual grid parameter and compute

$u(0,\Delta t), u(0.1,\Delta t), u(0,2\Delta t), u(0.1,2\Delta t)$

Ans:

$u(0,\Delta t)=0.95$
$u(0.1,\Delta t)=1$
$u(0,2\Delta t)=0.9275$
$u(0.1,2\Delta t)=0.9875$

(13) If $y(x_{n+1})$ is the exact solution and

$y_{n+1}=y_n+(a+b)hf_n+(pf_n+qff_y)bh^2+O(h^3)$ is a second order approximation to the I.V.P

$y'=f(x,y), y(x_0)=y_0$

Then find the relation that a, b, p, q satisfy.

Ans: $a+b=1, bp=\frac{1}{2}, bq=\frac{1}{2}$

(14) If the error equation for a single step method to solve an IVP is given by

$E_{j+1}=\left(E(\lambda h)-e^{\lambda h}\right)y_j+E(\lambda h)\in_j$

then find the conditions so that, the method is (i) absolutely stable (ii) relatively stable.

Ans :

$(i)\ |E(\lambda h)|\le 1,\ (ii)\ |E(\lambda h)|\le e^{\lambda h}$

(15) Use the method of characteristics to derive a solution of the quasi-linear equation

$$\frac{\partial^2 u}{\partial x^2}-u^2\frac{\partial^2 u}{\partial y^2}=0$$

At the first characteristic grid point $R(x_R,0), x_R>0$

Between $x=0.2$ and $x=0.3, y>0$, where u satisfies the condition $u=0.2+5x^2, \frac{\partial u}{\partial y}=3x$ along the initial line $y=0$, for $0\le x\le 1$.

(16) Solve the boundary value problem

$u_{xx}+u_{yy}-10u(u_{xx}-u_y)=-10e^{4x}cos2y(cos2y+sin2y)0\le x,y\le 1$

Using finite difference method with $h=\frac{1}{2}$. The Dirichlet boundary conditions are obtained from the exact solution $u(x,y)=e^{2x}cos2y$

Ans: $u_{11}=0.894678$ (approx).

(17) Solve the differential equation $\Delta u = 16$ (where Δ denotes the Laplacian) for a square with side 2, with $u = 0$ on the boundary

(a) Formulate the corresponding difference equation with mesh size h in both the directions.

(b) Solve the difference equation for $h = 1$ in x-direction, and $h = 1/2$ for y-direction.

(18) Solve the partial differential equation

$u_{tt} = u_{xx}$

with $u = f(x)$ at $t = 0$

where $f(x) = x, \quad 0 \le x \le 1/2$

$= (1-x), \quad 1/2 \le x \le 1$

and $u(0.t) = 0;\ u(1,t) = 0$. Find the solution up to two time steps with $h = 0.2,\ k = 1/2$.

Ans:

$$u_1^{(2)} = 0.19375 \quad u_2^{(2)} = 0.30625 \quad u_3^{(2)} = 0.30625$$
$$u_4^{(2)} = 0.19375 \quad \text{where the superscript (2) denotes the time step}$$

(19) Find the solution of $u_t + u_x = 0$, subject to the initial condition $u(x,0) = 0 \quad x < 0$

$= x \qquad 0 \le x < 1$

$= (3-x) \quad 1 \le x < 2$

$= 0 \qquad x \ge 2$

Using the Lax-wendroff method with $h = 1/2$ and $k = 1/2$. Compute the solution up to 1st time step.

Ans: $x: \quad 0 \quad 1/2 \quad 1 \quad 3/2 \quad 2 \quad 5/2$

$u^{(1)}: -1/16 \quad \frac{1}{4} \quad \frac{15}{16} \quad \frac{17}{16} \quad \frac{17}{8} \quad 0$

(20) If $\left.\frac{dy}{dx}\right|_{x=x_j} \simeq Ay_j + By_{j+1} + Cy_{j+2} + O(h^2)$

Then find the values of A, B, C.

Ans: $A = -3/2,\ B = 2,\ C = -1/2.$

(21) Compute an approximation to $y(1),\ y'(1), y''(1)$ with Taylor series, of order $2, h = 1$, when

$$y''' + 2y'' + y' - y = cosx$$

$$0 \le x \le 1,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2$$

Ans: $y(1) = 2,\ y'(1) = 1,\ y''(1) = 3/2$

(22) Given the Laplacian and the corresponding boundary conditions as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,0) = 0,\ u(x,10) = 0,\ u(0,y) = 0,\ u(20,y) = 100$$

Use five point formula and obtain the system of equations with $\delta x = \delta y = 5$ and find the solution.

Ans: $u(5,5) = 1.786,\ u(5,10) = 7.143, u(5,15) = 27.786$

(23) Consider the IVP $y' = x(y+x) - 2,\ y(0) = 2$. Use Euler's method with $h = 0.3,\ h = 0.2$ to compute $y(0.6)$. If the true solution of $y(0.6) = 1$, what is your conclusion?

Ans: $y(0.6) = 0.953$ with $h = 0.3,\ y(0.6) = 1.00576$ with $h = 0.1$

(24) Show that Euler's method applied to $y' = \lambda y,\ y(0) = 1, \lambda < 0$ is stable for step sizes $-2 < \lambda h < 0$

Ans: hint $|E(\lambda h)| \le 1$

(25) Find the solution at $x = 0.3$ for the differential equation $y' = x - y^2,\ y(0) = 1$ by Adams-Bash forth method of order 2 with $h = 0.1$. Determine the starting value using RK-method of second order.

Ans: $y(0.1) = 0.9145;\ y(0.2) = 0.85405;\ (0.3) = 0.81146$

(26) Given the I.V.P $y' = y - x^2,\ y(0) = 1$. Use Milne- Simpson Predictor-Corrector method with $h = 0.2$ to compute $y(0.8)$. Compute past values using any of the other methods you know.

Ans: $y^{(p)}(0.8) = 2.01461 \quad y^{(c)}(0.8) = 2.014434$

(27) Find the solution of the boundary value problem

$$y''(x) = y + x, \qquad x \in [0,1]$$
$$y(0) = 0, \qquad y(1) = 0$$

Using shooting method. Use the Runge-Kutta method of 2nd order to solve the corresponding I.V.P with step size 0.2

Ans :

$$y(0.2) \approx 0.0284, \quad y(0.4) \approx -0.050080$$
$$y(0.6) \approx -0.05776, \quad y(0.8) \approx -0.04389$$

$$y(0) = 1, \quad y(0.5) = \frac{4}{9}$$

(28) Use shooting method to find the solution of Boundary Value Problem

$$y'' = 6y^2$$

$$y(0) = 1, \; y(0.5) = \frac{4}{9}$$

Assume the initial approximation $y'(0) = -1.8$

Find the solution of the corresponding initial value problem using fourth order RK method with $h = 0.1$.

The exact solution of the problem is $y(x) = \frac{1}{(1+x)^2}$

Ans :

$$y(0.1) = 0.8468, \; y(0.2) = 0.7372, \; y(0.3) = 0.6606$$

$$y(0.4) = 0.6103 \quad y(0.5) = 0.5825$$

(29) If the error of a finite difference scheme that was used to approximate $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x^2}$ is

$$\frac{k}{2}\frac{\partial^2 \mu}{\partial t^2} - \frac{h^2}{12}\frac{\partial^3 \mu}{\partial t \partial x^2} + \frac{h^4}{360}\frac{\partial^6 \mu}{\partial x^6} + \frac{\mu \partial^3 \mu}{\partial t^3}$$

Then find the order of the truncation error.

Ans. Order of truncation error = $O(k + h^2)$

(30) Given the initial value problem

$$y' = x^3 + 2\mu^2, \; y(0) = 1$$

Determine first four non zero terms in the Taylor's series for $y(x)$ and hence obtain the value of $y(1)$.

Also determine x when the error in $y(x)$ obtained from first 3-non zero terms is to be less than 10^{-3}.

Ans :

$y(1) = 13$ (approx), $\quad x < \frac{1}{20}$

$$y(x) = 1 + 2x + 4x^2 + 6x^3 + 0(x^4)$$

(31) Given the initial value problem

$y' = -2x^3 y,\ y(0) = 1$

Estimate $y(0.4)$ using modified Euler-Cauchy's method and then compare the result with the exact solution.

Ans: 0.9942

Error= $|0.9942 - Exact| = 0.00691$

(32) Solve the initial value problem

$u' = 2t^4 u,\ u(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$. Use the fourth order classical Runge - Kutta method. Compare with the exact solution.

Ans: Exact Solution = 0.99591

$u(2) = 0.9961$

Error $= |0.9961 - 0.9959| \simeq 0.0019$

33. Consider the discretized equation

$$u_{i,j+1} = Au_{i,j} + Bu_{i-1,j} + Cu_{i+1,j} - Du_{i,j-1}$$

That approximates a particular PDE (which need not be known for now) whose solution is $u(x,t)$, A,B,C being known constants. If $u_t(x,0) = g(x)$,where t=0 represents initial time, obtain the corresponding approximation for u_i^1 which is explicit, however, independent of any "fictitious values".

Ans: $u_{i,1}(1+D) = Au_{i,0} + Bu_{i-1,0} + Cu_{i+1,0} + 2KDg_i$

34. Consider the hyperbolic equation $u_{tt} = u_{xx}$ subject to the boundary conditions $u(0,t) = 0, u(1,t) = 0, t > 0$ and the initial conditions $u_t(x,0) = 0, u(x,0) = 1 - x^2, 0 \le x \le 1$. Use the explicit method to obtain solution at the grid points generated by $h = 1/4, k = 0.2$ for first time level.

Ans: $u_{1,0} = 0.9375;\ u_{2,0} = 0.75;\ u_{3,0} = 0.4375;$

$u_{1,1} = 0.8975;\ u_{2,1} = 0.71;\ u_{3,1} = 0.3975$

35. If $u(r,\theta)$ denote the solution of the Laplace equation in (r,θ) polar coordinates given by

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0,$$

obtain the corresponding finite difference scheme that is implicit. Consider central difference approximation for both first and second derivatives. Given the data on the boundary as $u(r,\theta)=1, u\left(r,\frac{\pi}{2}\right)=1$ and $u(3,\theta)=-1$, solve for $u(r,\theta)$ along $\theta=\frac{\pi}{6},\frac{\pi}{3}$ at $r=1,2$. Assume the step size as $h=\delta r=1$ for radial coordinate and $k=\delta\theta=\frac{\pi}{6}$ for angular coordinate.

$$\text{Ans:}\quad u\left(1,\frac{\pi}{6}\right)=0.7541;\ u\left(1,\frac{\pi}{3}\right)=0.7535;$$
$$u\left(2,\frac{\pi}{6}\right)=0.0756;\ u\left(2,\frac{\pi}{3}\right)=0.0753$$

36. Consider the second order PDE given by

$$au_{xx}+bu_{xt}+cu_{tt}+e=0,$$

wherea,b,c are constants and $e=e(x,t)$. Assuming that the given PDE is hyperbolic, consider the two characteristics f and g intersecting at a point $R(x_R,t_R)$. Derive the corresponding discretized equation for the characteristics and hence obtain the solution for $u_{xx}-uu_{tt}+(1-x^2)=0, u(x,0)=x(1-x), u_t(x,0)=0, u(0,t)=0, u(1,t)=0$. If $P(0,2,0)$ and $Q(0,4,0)$ are the points on the initial datum, obtain u at the point $R(x_R,t_R)$.

Ans: $f_P = 0.4,\ f_Q = 0.490, g_P = -0.4, g_Q = -0.490$

$(x_R, t_R) = (0.310, 0.044)$

$p_R = 0.399, q_R = -0.246$

u_R along PR $= 0.2095$

u_R along QR $= 0.2076$

37. Find the interval of absolute stability for the method

$$u_{j+1} = u_j + \frac{h}{2}\left(u'_{j+1} + u'_j\right) + \frac{h^2}{12}\left(u''_j - u''_{j+1}\right)$$

used for solving the IVP $u' = f(x,u), u(x_0) = u_0$.

Ans: $$\sum = \frac{1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{12}}{1 - \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{12}}$$

$$\lambda h \in (-\infty, 0)$$