4.5 Eulerian graphs

In this section, the graphs can have loops and multiple edges. Such graphs are called multi-graphs. Let $X = (V, E)$ be a graph. Then recall that a trail in X is a walk in which each edge is distinct. A graph is said to be an Eulerian graph or a closed Eulerian trail (in short Eulerian) if there is a closed trail that traverses each edge of X exactly once. Note that this is equivalent to saying that a graph X is Eulerian, if one can find a walk that traverses every edge of X exactly once and finishes at the starting vertex. A non-Eulerian graph is called an Eulerian trail if there is a walk that traverses every edge of X exactly once. The graphs that have a closed trail traversing each edge exactly once have been name "Eulerian graphs" due to the solution of Königsberg bridge problem by Euler in 1736. The problem is as follows: The city Königsberg (the present day Kaliningrad) is divided into 4 land masses by the river Pregel. These land masses are joined by 7 bridges (see Figure 4.11). The question required one to answer "is there a way to start from a land mass that passes through all the seven bridges in Figure 4.11 and returns back to the starting land mass"? Euler, rephrased the problem along the following lines: Let the four land masses be denoted by the vertices A, B, C and D of a graph and let the 7 bridges correspond to 7 edges of the graph. Then he asked "does this graph has a closed trail that traverses each edge exactly once"? He gave a necessary and sufficient condition for a graph to have such a closed trail and thus giving a negative answer to Königsberg bridge problem.

Observe that the definition implies that either the graph X is a connected multi-graph or in more generality, X may have isolated vertices but it has exactly one component that contains all the edges of X. So, let us assume that the multi-graphs in this section are connected. One can also relate this with the problem of drawing a given figure with pencil such that neither the pencil is lifted from the paper nor a line is repeated.

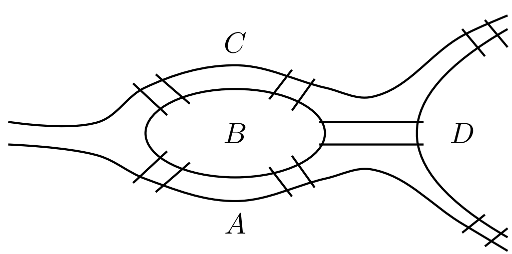

Figure 4.11: Königsberg bridge problem

To state and prove the result of Euler that states that "a connected graph $X = (V, E)$ is Eulerian if and only if $\deg(v)$ is even, for each $v \in V$, we need the following result.

Lemma 4.5.1. *Let $X = (V, E)$ be a connected multi-graph such that $\deg(v) \geq 2$, for each $v \in V$. Then X contains a circuit.*

Proof. The result is clearly true if X has either a loop or a multiple edge. So, let us assume that X is a simple graph. The proof is constructive in nature. Let us start with a vertex $v_0 \in V$.

As X is connected, there exists a vertex $v_1 \in V$ that is adjacent to v_0. Since X is a simple graph and $\deg(v) \geq 2$, for each $v \in V$, there exists a vertex $v_2 \in V$ adjacent to v_1 with $v_2 \neq v_0$. Similarly, there exists a vertex $v_3 \in V$ adjacent to v_2 with $v_3 \neq v_1$. Note that either $v_3 = v_0$, in which case, one has a circuit $[v_0 v_1 v_2 v_0]$ or else one can proceed as above to get a vertex $v_4 \in V$ and so on.

As the number of vertices is finite, the process of getting a new vertex will finally end with a vertex v_i being adjacent to a vertex v_k, for some i, $0 \leq i \leq k-2$. Hence, $[v_i v_{i+1} v_{i+2} \ldots v_k v_i]$ forms a circuit. Thus, the proof of the lemma is complete. ■

Let us now prove the following theorem.

Theorem 4.5.2 (Euler 1736)**.** *Let $X = (V, E)$ be a connected graph. Then X is an Eulerian graph if and only if each vertex of X has even degree.*

Proof. Let $X = (V, E)$ be an Eulerian graph with a closed Eulerian trail $T \equiv [v_0 v_1 \ldots v_{k-1} v_k = v_0]$. By the very nature of the trail, for each $v \in V$, the trail T enters v through an edge and departs v from another edge of X. Thus, at each stage, the process of coming in and going out, contributes 2 to degree of v. Also, the trail T passes through each edge of X exactly once and hence each vertex must be of even degree.

Conversely, let us assume that each vertex of X has even degree. We need to show that X is Eulerian. We prove the result by induction on the number of edges of X. As each vertex has even degree and X is connected, by Lemma 4.5.1 X contains a circuit, say C. If C contains every edge of X then C gives rise to a closed Eulerian trail and we are done. So, let us assume that C is a proper subset of E. Now, consider the graph X' that is obtained from X by removing all the edges in C. Then, X' may be a disconnected graph but each vertex of X' still has even degree. Hence, we can use induction to each component to X' to get a closed Eulerian trail for each component of X'.

As each component of X' has at least one vertex in common with C, we use the following method to construct the required closed Eulerian trail: start with a vertex, say v_0 of C. If there is a component of X' having v_0 as a vertex, then traverse this component and come back to v_0. This is possible as each component is Eulerian. Now, proceed along the edges of C until we get another component of X', say at v_1. Traverse the new component of X' starting with v_1 and again come back to v_1. This process will come to an end as soon as we return back to the vertex v_0 of C. Thus, we have obtained the required closed Eulerian trail. ■

We state two consequences of Theorem 4.5.2 to end this section. The proofs are omitted as they can be easily obtained using the arguments used in the proof of Theorem 4.5.2.

Corollary 4.5.3. *Let $X = (V, E)$ be a connected graph. Then X has an Eulerian trail if and only if X has exactly two vertices of odd degree.*

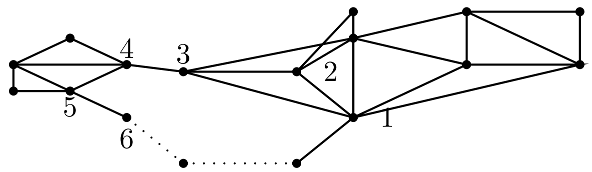

Figure 4.12: Constructing a closed Eulerian trail

Corollary 4.5.4. *Let $X = (V, E)$ be a connected graph. Then X is an Eulerian graph if and only if the edge set of X can be partitioned into cycles.*