Lecture 5 Compatible Systems and Charpit's Method

In this lecture, we shall study compatible systems of first-order PDEs and the Charpit's method for solving nonlinear PDEs. Let's begin with the following definition.

Definition 1. ***(Compatible systems of first-order PDEs)*** *A system of two first-order PDEs*

$$f(x, y, z, p, q) = 0 \tag{1}$$

and

$$g(x, y, z, p, q) = 0 \tag{2}$$

are said to be compatible if they have a common solution.

Theorem 2. *The equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ are compatible on a domain D if*

(i) $J = \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0$ *on D.*

(ii) p and q can be explicitly solved from (1) and (2) as $p = \phi(x, y, z)$ and $q = \psi(x, y, z)$. Further, the equation

$$dz = \phi(x, y, z)dx + \psi(x, y, z)dy$$

is integrable.

Theorem 3. *A necessary and sufficient condition for the integrability of the equation $dz = \phi(x, y, z)dx + \psi(x, y, z)dy$ is*

$$\boxed{[f, g] \equiv \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0.} \tag{3}$$

In other words, the equations (1) and (2) are compatible iff (3) holds.

Example 4. *Show that the equations*

$$xp - yq = 0, \qquad z(xp + yq) = 2xy$$

are compatible and solve them.

Solution. Take $f \equiv xp - yq = 0, \qquad g \equiv z(xp + yq) - 2xy = 0$. Note that

$$f_x = p, \;\; f_y = -q, \;\; f_z = 0, \;\; f_p = x, \;\; f_q = -y.$$

and

$$g_x = zp - 2y, \;\; g_y = zq - 2x, \;\; g_z = xp + yq, \;\; g_p = zx, \;\; g_q = zy.$$

Compute

$$J \equiv \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} = \begin{vmatrix} x & -y \\ zx & zy \end{vmatrix} = zxy + zxy = 2zxy \neq 0$$

for $x \neq 0$, $y \neq 0$, $z \neq 0$. Further,

$$\begin{aligned}
\frac{\partial(f,g)}{\partial(x,p)} &= \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix} = zxp - x(zp-2y) = 2xy \\
\frac{\partial(f,g)}{\partial(z,p)} &= \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & zx \end{vmatrix} = 0 - x(xp+yq) = -x^2p - xyq \\
\frac{\partial(f,g)}{\partial(y,q)} &= \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -qzy + y(zq-2x) = -2xy \\
\frac{\partial(f,g)}{\partial(z,q)} &= \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = y(xp+yq) = y^2q + xyp.
\end{aligned}$$

It is an easy exercise to verify that

$$\begin{aligned}
[f,g] &\equiv \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} \\
&= 2xy - x^2p^2 - xypq - 2xy + y^2q^2 + xypq \\
&= y^2q^2 - x^2p^2 \\
&= 0.
\end{aligned}$$

So the equations are compatible.

Next step to determine p and q from the two equations $xp - yq = 0$, $z(xp+yq) = 2xy$. Using these two equations, we have

$$\begin{aligned}
zxp + zyq - 2xy = 0 &\implies xp + yq = \frac{2xy}{z} \\
&\implies 2xp = \frac{2xy}{z} \implies p = \frac{y}{z} = \phi(x,y,z).
\end{aligned}$$

and

$$\begin{aligned}
xp - yq = 0 &\implies q = \frac{xp}{y} = \frac{xy}{yz} = \frac{x}{z} \\
&\implies q = \frac{x}{z} = \psi(x,y,z).
\end{aligned}$$

Substituting p and q in $dz = pdx + qdy$, we get

$$zdz = ydx + xdy = d(xy),$$

and hence integrating, we obtain

$$z^2 = 2xy + k,$$

where k is a constant.

NOTE: For the compatibility of $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ it is not necessary that every solution of $f(x, y, z, p, q) = 0$ be a solution of $g(x, y, z, p, q) = 0$ or vice-versa as is generally believed. For instance, the equations

$$f \equiv xp - yq - x = 0 \tag{4}$$

$$g \equiv x^2p + q - xz = 0 \tag{5}$$

are compatible. They have common solutions $z = x + c(1 + xy)$, where c is an arbitrary constant. Note that $z = x(y + 1)$ is a solution of (4) but not of (5).

Charpit's Method: It is a general method for finding the complete integral of a nonlinear PDE of first-order of the form

$$f(x, y, z, p, q) = 0. \tag{6}$$

Basic Idea: The basic idea of this method is to introduce another partial differential equation of the first order

$$g(x, y, z, p, q, a) = 0 \tag{7}$$

which contains an arbitrary constant a and is such that

(i) Equations (6) and (7) can be solved for p and q to obtain

$$p = p(x, y, z, a), \qquad q = q(x, y, z, a).$$

(ii) The equation

$$dz = p(x, y, z, a)dx + q(x, y, z, a)dy \tag{8}$$

is integrable.

When such a function g is found, the solution

$$F(x, y, z, a, b) = 0$$

of (8) containing two arbitrary constants a, b will be the solution of (6).

Note: Notice that another PDE g is introduced so that the equations f and g are compatible and then common solutions of f and g are determined in the Charpit's method.

The equations (6) and (7) are compatible if

$$[f, g] \equiv \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0.$$

Expanding it, we are led to the linear PDE

$$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0. \tag{9}$$

Now solve (9) to determine g by finding the integrals of the following auxiliary equations:

$$\boxed{\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}} \tag{10}$$

These equations are known as Charpit's equations which are equivalent to the characteristics equations (10) of the previous Lecture 4.

Once an integral $g(x,y,z,p,q,a)$ of this kind has been found, the problem reduces to solving for p and q, and then integrating equation (8).

Remark 5. *1. For finding integrals, all of Charpit's equations (10) need not to be used.*
2. p or q must occur in the solution obtained from (10).

Example 6. *Find a complete integral of*

$$p^2x+q^2y=z. \tag{11}$$

Solution. To find a complete integral, we proceed as follows.

Step 1: (Computing f_x, f_y, f_z, f_p, f_q).
Set $f\equiv p^2x+q^2y-z=0$. Then

$$f_x=p^2,\quad f_y=q^2,\quad f_z=-1,\quad f_p=2px,\quad f_q=2qy.$$

$$\Longrightarrow pf_p+qf_q=2p^2x+2q^2y,\ \ -(f_x+pf_z)=-p^2+p,\ \ -(f_y+qf_z)=-q^2+q.$$

Step 2: (Writing Charpit's equations and finding a solution $g(x,y,z,p,q,a)$).

The Charpit's equations (or auxiliary) equations are:

$$\begin{aligned}&\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}\\ \Longrightarrow\quad &\frac{dx}{2px}=\frac{dy}{2qy}=\frac{dz}{2(p^2x+q^2y)}=\frac{dp}{-p^2+p}=\frac{dq}{-q^2+q}\end{aligned}$$

From which it follows that

$$\begin{aligned}&\frac{p^2dx+2pxdp}{2p^3x+2p^2x-2p^3x}=\frac{q^2dy+2qydq}{2q^3y+2q^2y-2q^3y}\\ \Longrightarrow\quad &\frac{p^2dx+2pxdp}{p^2x}=\frac{q^2dy+2qydq}{q^2y}\end{aligned}$$

On integrating, we obtain

$$\begin{aligned} & \log(p^2x) = \log(q^2y) + \log a \\ \Longrightarrow \quad & p^2x = aq^2y, \end{aligned} \tag{12}$$

where a is an arbitrary constant.

Step 3: (Solving for p and q).

Using (11) and (12), we find that

$$\begin{aligned} & p^2x + q^2y = z, \quad p^2x = aq^2y \\ \Longrightarrow \quad & (aq^2y) + q^2y = z \Longrightarrow q^2y(1+a) = z \\ \Longrightarrow \quad & q^2 = \frac{z}{(1+a)y} \Longrightarrow q = \left[\frac{z}{(1+a)y}\right]^{1/2}. \end{aligned}$$

and

$$\begin{aligned} & p^2 = aq^2\frac{y}{x} = a\frac{z}{(1+a)y}\frac{y}{x} = \frac{az}{(1+a)x} \\ \Longrightarrow \quad & p = \left[\frac{az}{(1+a)x}\right]^{1/2}. \end{aligned}$$

Step 4: (Writing $dz = p(x,y,z,a)dx + q(x,y,z,a)dy$ and finding its solution).

Writing

$$\begin{aligned} & dz = \left[\frac{az}{(1+a)x}\right]^{1/2}dx + \left[\frac{z}{(1+a)y}\right]^{1/2}dy \\ \Longrightarrow \quad & \left(\frac{1+a}{z}\right)^{1/2}dz = \left(\frac{a}{x}\right)^{1/2}dx + \left(\frac{1}{y}\right)^{1/2}dy. \end{aligned}$$

Integrate to have

$$[(1+a)z]^{1/2} = (ax)^{1/2} + (y)^{1/2} + b$$

the complete integral of the equation (11).

Practice Problems

1. Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and solve them.

2. Show that the equations $f(x,y,p,q) = 0$ and $g(x,y,p,q) = 0$ are compatible if
$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,p)} = 0.$$

3. Find complete integrals of the equations:
(i) $p = (z+qy)^2$; (ii) $(p^2+q^2)y = qz$