8.7 Lecture 7

Preamble: We will show that the best approximation to a quadratic irrational by a rational sequence is given by the convergents of the infinite continued fraction of the irrational.

Keywords: best approximation to a quadratic irrational

8.7.1 Best Rational Approximation to an Irrational

THEOREM 8.25. *Let x be an irrational number, and $\frac{A_n}{B_n}$ be a convergent of its continued fraction. Then $\frac{A_n}{B_n}$ is the closest rational number to x amongst all rational numbers $\frac{a}{b}$ with denominators $b \le B_n$, i.e.,*

$$\mid x - \frac{A_n}{B_n} \mid \le \mid x - \frac{a}{b} \mid \qquad a,\ b \in \mathbb{Z},\ \ 1 \le b \le B_n.$$

We will first prove the following lemma:

LEMMA 8.26. *Let $\frac{A_i}{B_i}$ denotes the i-th convergent of an irrational number x. Then for any $a,\ b \in \mathbb{Z}$ satisfying $1 \le b < B_{n+1}$, we have*

$$\mid B_n x - A_n \mid \ \le\ \mid bx - a \mid .$$

Proof: Note that x lies between $\frac{A_n}{B_n}$ and $\frac{A_{n+1}}{B_{n+1}}$, so that $B_n x - A_n$ and $B_{n+1}x - A_{n+1}$ have opposite signs. If we can express

$$\begin{aligned} b &= B_n \alpha + B_{n+1}\beta \\ a &= A_n \alpha + A_{n+1}\beta, \end{aligned} \tag{8.7}$$

where $\alpha \neq 0$ and β are integers of opposite signs (except when $\beta = 0$), then we can write

$$\begin{aligned} \mid bx - a \mid &= \mid \alpha(B_n x - A_n) + \beta(B_{n+1}x - A_{n+1}) \mid \\ &= \mid \alpha(B_n x - A_n) \mid + \mid \beta(B_{n+1}x - A_{n+1}) \mid \\ &= \mid \alpha \mid \ \mid (B_n x - A_n) \mid + \mid \beta \mid \ \mid (B_{n+1}x - A_{n+1}) \mid \\ &\ge \mid \alpha \mid \ \mid (B_n x - A_n) \mid \\ &\ge \mid (B_n x - A_n) \mid . \end{aligned}$$

The coefficients in (8.7) satisfy $A_{n+1}B_n - A_nB_{n+1} = (-1)^n$, hence we can solve the system and obtain,

$$\alpha = (-1)^n(A_{n+1}b - B_{n+1}a), \quad \beta = (-1)^n(A_nb - B_na).$$

If $\alpha = 0$, then $b = B_{n+1}\beta$, hence $\beta > 0$ and $b \geq B_{n+1}$, which is a contradiction. Now, $\beta = 0$ implies $b = B_n\alpha$, and $a = A_n\alpha$, and the inequality in the lemma follows easily. Suppose $\beta > 0$. Then $B_n\alpha + B_{n+1}\beta = b < B_{n+1}$ implies $B_n\alpha < 0$ hence $\alpha < 0$. If $\beta < 0$, then $B_n\alpha + B_{n+1}\beta = b \geq 1$ implies $B_n\alpha > 0$, i.e., $\alpha > 0$. Thus, either $\beta = 0$ or α and β have opposite signs, and the inequality in the lemma follows. □

Proof of the theorem: Consider a rational number $\frac{a}{b}$ where $1 \leq b \leq B_n(< B_{n+1})$ and a is any integer. By the previous lemma, we have

$$\begin{array}{rrcl} & \mid B_nx - A_n \mid & \leq & \mid bx - a \mid \\ \Longrightarrow & \mid B_n \mid \; \mid x - \frac{A_n}{B_n} \mid & \leq & \mid b \mid \; \mid x - \dfrac{a}{b} \mid \\ \Longrightarrow & \mid x - \frac{A_n}{B_n} \mid & \leq & \mid \dfrac{b}{B_n} \mid \; \mid x - \dfrac{a}{b} \mid \\ \Longrightarrow & \mid x - \frac{A_n}{B_n} \mid & \leq & \mid x - \dfrac{a}{b} \mid, \end{array}$$

as $\frac{b}{B_n} \leq 1$. □

8.7.2 A Sufficiently Close Rational is a Convergent

Let x be an irrational number, and let $\frac{a}{b}$ be a rational number. If $\frac{a}{b}$ is sufficiently close to x, we can show that $\frac{a}{b}$ has to be a convergent of the continued fraction of x.

Proposition 8.27. *Let $\frac{a}{b}$ be a rational number with $1 \leq b$, $gcd(a, b) = 1$ and*

$$\mid \frac{a}{b} - x \mid \; < \; \frac{1}{2b^2}.$$

Then, $\frac{a}{b}$ is a convergent of the continued fraction of x.

Proof: As the denominator B_i of convergents $\frac{A_i}{B_i}$ forms an increasing sequence of integers, we have $B_n \leq b < B_{n+1}$ for some n. Suppose, $\frac{a}{b} \neq \frac{A_n}{B_n}$, i.e., $\mid aB_n - bA_n \mid \geq 1$. As

$b < B_{n+1}$, by the preceding lemma we have

$$\begin{array}{rccl}
 & |\, B_n x - A_n \,| & \leq & |\, bx - a \,| \;<\; b \cdot \dfrac{1}{2b^2} \;=\; \dfrac{1}{2b} \\
\Longrightarrow & |\, x - \frac{A_n}{B_n} \,| & \leq & \dfrac{1}{2bB_n} \\
\Longrightarrow & |\, \frac{a}{b} - x \,| + |\, x - \frac{A_n}{B_n} \,| & < & \dfrac{1}{2b^2} + \dfrac{1}{2bB_n} \\
\Longrightarrow & |\, \frac{a}{b} - \frac{A_n}{B_n} \,| & \leq & |\, \dfrac{a}{b} - x \,| + |\, x - \dfrac{A_n}{B_n} \,| \;<\; \dfrac{1}{2b^2} + \dfrac{1}{2bB_n} \\
\Longrightarrow & \frac{|aB_n - bA_n|}{|bB_n|} & < & \dfrac{1}{2b^2} + \dfrac{1}{2bB_n} \\
\Longrightarrow & \frac{1}{bB_n} & < & |\, \dfrac{aB_n - bA_n}{bB_n} \,| \;<\; \dfrac{1}{2b^2} + \dfrac{1}{2bB_n} \\
\Longrightarrow & b & < & B_n,
\end{array}$$

which is a contradiction. $\square$