Problem Sheet

Q1. Let Y be Bernoulli distributed random variable with parameter p. The process $\{Z_n, n = 0, 1, \ldots\}$ starts at time 0 with one ancestor. $Z_0 = 1$. At time $n = 1$ this ancestor dies producing a random number of descendants $Z_1 = Y_1^{(0)}$ where $Y_1^{(0)}$ have the same distribution as Y. Each descendant behaves independently of the others living only until $n = 2$ and being then replaced by his own descendants. This process continues at $n = 3, 4, \ldots$. The random variables $Y_j^{(n)}$ have the same distribution as Y. Thus Z_{n+1} is the number of descendants in the $(n + 1)$th generation produced by Z_n individuals of generation n. Find the probability of eventual extinction for this branching process.

Q2. Consider the discrete branching process $\{Z_n, n = 0, 1, \ldots\}$. Let $\mu = E(Z_1)$ and $\sigma^2 = Var(Z_1)$. Prove that,

$$Var(Z_n) = \begin{cases} \frac{\mu^{n-1}(\mu^n-1)\sigma^2}{\mu-1}, & \mu \neq 1 \\ n\sigma^2 & \mu = 1 \end{cases}$$

Q3. Consider the Galton - Watson process $\{Z_n, n = 0, 1, 2, \ldots\}$ with offspring distribution $\{p_k\}$ where $p_k = bp_{k-1}, k = 1, 2, \ldots$. Let $H_n(s)$ be the probability generating function of Z_n. Prove that,

$$H_n(s) = \frac{np - (np + p - 1)s}{1 - p + np - nps}.$$

Q4. Let $Z(t)$ be number of particles at time t. Each particle lives random time which is exponential distributed and then splits into two particles. i.e., it follows Yule process. Thus $\{Z(t), t \geq 0\}$ is a Markov branching process. Find the probability generating function for $P_{1j}(t)$, $\phi(t; s)$.

Q5. Let $Z(t)$ be number of particles at time t. In the above problem, assume that simple birth death process with rates $n\lambda$ and $n\mu$ for birth and death respectively, when the process is in state n. i.e., it follows birth and death process. Thus $\{Z(t), t \geq 0\}$ is a Markov branching process. Find the probability generating function for $P_{1j}(t)$, $\phi(t; s)$.

Answers to Problem Sheet

Ans 1: Since Y is Bernoulli distributed random variable, we have the probability generating function of Y, denoted by $H(t)$ is

$$H(t) = pt + (1 - p).$$

We know that the probability generating function of the random variables Z_ns satisfy the following recurrence relations

$$H_1(t) = H(t); \quad H_{n+1}(t) = H(H_n(t)), n = 1, 2, \ldots$$

Hence, we have

$$d_{n+1} = H_{n+1}(0) = H(H_n(0)) = H(d_n).$$

Here,

$$\begin{aligned} d_1 &= H(d_0) = H(0) = 1 - p \\ d_2 &= H(d_1) = H(1 - p) = 1p^2 \\ d_3 &= H(d_2) = H(1 - p^2) = 1p^3 \end{aligned}$$

One can show using induction,

$$d_n = 1p^n, \quad n = 0, 1, \ldots$$

Taking the limit as n tends to infinity, we get the probability of eventual extinction

$$\lim_{n \to \infty} d_n = 1.$$

Ans 2: We know that

$$H_n(s) = H_{n-1}(H(s))$$

Differentiating the above equation with respect to s gives

$$H^{'}(s) = H^{'}_{n-1}(H(s))H^{'}(s). \tag{1}$$

Putting $s = 0$, and writing the mean of the nth generation as μ_n, yields

$$\mu_n = \mu_{n-1}\mu.$$

Hence, applying the above equation recursively, we get

$$\mu_n = \mu^n.$$

If we differentiate (1) with respect to s, we obtain

$$H^{''}(s) = H^{''}_{n-1}(H(s))(H^{'}(s))^2 + H^{'}_{n-1}(H(s))H^{''}(s)$$

Putting $s = 0$, and using σ_n^2 for the variance at the nth generation, we get

$$\sigma_n^2 = \mu^2\sigma_{n-1}^2 + \mu_{n-1}\sigma^2 = \mu^2\sigma_{n-1}^2 + \mu^{n-1}\sigma^2.$$

Applying the above equation recursively, we obtain

$$\sigma_n^2 = (\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}\sigma^2) = \frac{\mu^{n-1}(\mu^n - 1)}{\mu - 1}\sigma^2, \quad \mu \neq 1.$$

If $\mu = 1$, we obtain

$$\sigma_n^2 = n\sigma^2, \quad \mu = 1.$$

Ans 3: Since

$$p_k = bp_{k-1}, k = 1, 2, \ldots.$$

$$p_0 = 1 - (P_1 + p_2 + \ldots) = \frac{1 - b - p}{1 - p}.$$

We get

$$H(s) = 1 - \frac{b}{1-p} + \frac{bs}{1-ps} \quad \text{and} \quad \mu = \frac{b}{(1-p)^2}.$$

For any two points u, v, we have

$$\frac{H(s) - H(u)}{H(s) - H(v)} = \frac{s-u}{s-v}\frac{1-pv}{1-pu}. \tag{2}$$

The equation $H(s) = s$ has roots s_0 and 1. If $\mu > 1$, then $s_0 < 1$; if $\mu = 1$, $s_0 = 1$; if $\mu < 1$, $s_0 > 1$.

If we take $u = s_0$ and $v = 1$ then for $\mu \neq 1$ the above equation becomes

$$\frac{1-p}{1-ps_0} = \lim_{s\to 1} \frac{\left(\frac{H(s)-s_0}{s-s_0}\right)}{\left(\frac{H(s)-1}{s-1}\right)} = \frac{1}{\mu}.$$

Hence equation (2) becomes

$$\frac{H(s)-s_0}{H(s)-1} = \frac{s-s_0}{\mu(s-1)}.$$

Iterating the above equation, we obtain

$$\frac{H_n(s)-s_0}{H_n(s)-1} = \frac{s-s_0}{\mu^n(s-1)}$$

which can be solved explicitly for $H_n(s)$. After simplifications, we get

$$H_n(s) = 1 - \mu^n \left(\frac{1-s_0}{\mu^n - s_0}\right) + \frac{\mu^n \left(\frac{1-s_0}{\mu^n-s_0}\right)^2 s}{1-\left(\frac{\mu^n-1}{\mu^n-s_0}\right) s}, \quad \mu \neq 1.$$

If $\mu = 1$, then $b = (1-p)^2$ and $s_0 = 1$. Then

$$H(s) = \frac{p-(2p-1)s}{1-ps}$$

which can be iterated to yield

$$H_n(s) = \frac{np-(np+p-1)s}{1-p+np-nps}.$$

Ans 4: Given $a_2 = 1$, define

$$f(s) = \sum_k a_k s^k = s^2$$

Now, define $u(s) = a(f(s) - s)$. From Theorem 5, we get

$$\frac{\partial \phi(t;s)}{\partial t} = a(\phi^2(t;s) - \phi(t;s)) \text{ and } \phi(0;s) = s.$$

After solving the above partial differential equation, we obtain

$$\phi(t;s) = \frac{se^{-at}}{1-(1-e^{-at})s}.$$

If instead of splitting into two particles each particle splits into exactly k particles ($k > 2$), then the generating function is given by

$$\phi(t;s) = \frac{se^{-at}}{\left[1-(1-e^{-a(k-1)t})s^{k-1}\right]^{1/(k-1)}}.$$

Ans 5: Given $a = \lambda + \mu$ and $f(s) = \frac{\mu + \lambda s^2}{a}$. Hence,

$$\frac{\partial \phi(t;s)}{\partial t} = \lambda \phi^2(t;s) - (\lambda + \mu)\phi(t;s) + \mu \quad \text{and} \quad \phi(0;s) = s.$$

After solving the above partial differential equation, we obtain

$$\phi(t;s) = \frac{\mu(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}{\lambda(s-1) - e^{(\mu-\lambda)t}(\lambda s - \mu)}.$$