Vectors Spaces

In this course we work with 'scalars' and 'vectors'. 'Scalars' are going to be elements of a chosen (associative) ring with (identity) $\mathbb{K}$ and 'vectors' are going to be in general non commutative and will be equipped with a chosen involution but many significant results are best obtained in a setting when further restrictions are imposed on $\mathbb{K}$. We provide here a collection of definitions and examples [of the kind of scalars that we are going to need or think the user may need a bit later], remarks, and other tit-bits, on our $\mathbb{K}$.

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ with the usual addition and multiplication are commutative rings. With the exception of $\mathbb{Z}$, they are fields,

2. fix $n > 0$ in $\mathbb{Z}$ and write $a \sim b$ iff n divides $a - b$.then

   (i) [n divides 0] $a \sim a$ so this is reflexive ,

   (ii) [$\because$ if n divides a-b then n divides $b-a$] $a \sim b$ ensures $b \sim a$ so this is a symmetric,and

   (iii) [$\because$ n divides $a-b$ and n divides $b-c$ then n divides $a-c$] $a \sim b, b \sim c$ ensure $a \sim c$, so this is transitive.

   Thus we have an equivalence relation on $\mathbb{Z}$; we read $a \sim b$ as 'a is equivalent to b modulo n'. Write $[a] := \{b \in \mathbb{Z} | a \sim b\}$ and define $\mathbb{Z}_n := \{[a] | a \in \mathbb{Z}\}$,$[a]+[b] := [a+b], [a][b] := [ab]$.

   (i) If $a' \in [a]$, $b' \in [b]$ then n divides $a - a'$ as well as $b - b'$; therefore n divides $(a + b) - (a' - b')$. Thus $[a + a'] = [b + b']$ if $[a] = [a']$ and $[b] = [b']$. This means addition above is well defined. Similarly, if $a' \in [a]$, say $a = nq + a', b' \in [b]$, say $b = np + b'$ then $ab = n^2pq + nqb' + npa' + a'b'$ i.e.$a' - b' = n[npq + qb' + a'p]$ i.e.

$a'b' \in [ab]$. then $[ab] = [a'b']$ if $[a] = [a']$ and $[b] = [b']$ which means multiplication is well defined.

(ii) Clearly $[a] + [b] = [a + b] = [b + a] = [b] + [a]$ and $0 = \{nq|q \in \mathbb{Z}\}$ so that if $a' \in [a]$, say $a = np + a'$, and $nq \in [0]$, we have $a' + nq = a - np + nq = a + n(q - p) \in [a]$ then $[a] + [0] = [a] = [0] + [a]$. Further $-a = -np - a'$ i.e. $[a] + [-a] = [0]$ which means $[-a] = -[a]$. Therefore, under addition as defined, $\mathbb{Z}_n$ an abelian group [it is clear that $[a] + ([b] + [c]) = [(a + b) + c] = ([a + b]) + [c] = ([a] + [b]) + [c]]$.

(iii) If $[a] = [1]$ then $a = nq + 1$ so that $ba = bnq + b$ hence $[ba] = [b]$ and $ab = nqb + b$ hence $[ab] = [b]$. Thus $[a] = [a][1] = [1][a]$ for each $[a] \in \mathbb{Z}_n$. Clearly $[a]([b][c]) = [a][bc] = [a(bc)] = [(ab)c] = [ab][c] = ([a][b])[c]$ for all $[a], [b], [c] \in \mathbb{Z}_n$. Therefore under multiplication as defined, $\mathbb{Z}_n$ is a monoid with [1] as identity.

(iv) We have $[a]([b] + [c]) = [a][b + c] = [a(b + c)] = [ab + ac] = [ab] + [ac] = [a][b] + [a][c]$ and $([a] + [b])[c] = [a + b][c] = [(a + b)c] = [ac + bc] = [ac] + [bc] = [a][c] + [b][c]$.Thus the distributive law hold.

(v) These calculations show that $\mathbb{Z}_n$ is a ring under the defined operations.

(vi) Since $[a][b] = [ab] = [ba]$ for each $[a], [b] \in \mathbb{Z}_n$, we see that $\mathbb{Z}_n$ is a commutative ring.

(vii) $[a][b] = 1 \; iff ab = nq + 1$. Conversely, if there exist integers c, d with $ac = nd + 1$, let $c = nq + r$,$0 \le r < n$ so that $nd + 1 = ac = anq + ar$, i.e. $ar = n(d - aq) + 1$ and thus $[a][r] = [ar] = [1]$. Thus $[a] \in \mathbb{Z}_n$ is invertible iff a is relatively prime to n i.e. $gcd(a, n) = 1, i.e.$ there exist integers c, d with $ac - nd = 1$; the multiplicative inverse of $[a]$ is then given by $[c]$ and we have $[c] = [a]^{-1}$ iff there exists an integer d such that $ac - nd = 1$

(viii) Therefore $\mathbb{Z}_n$ is a field iff n is prime.

3. The ring $\mathbb{Z}$ occupies a very special position: $\mathbb{Z} \to^f \mathbb{K}$ defined by $f(n) := n1 = 1+\ldots+1[1 \in \mathbb{K}]$ is the single-morphism for any ring $\mathbb{K}$ from $\mathbb{Z}$. [Indeed, $f(1) = 1_{\mathbb{K}}$ is must and so is $f(n) = f(1+\ldots+1) = f(1) + f(1) + \ldots f(1) = 1_{\mathbb{K}} + \ldots 1_{\mathbb{K}} = n1_{\mathbb{K}}$ so $f(n) := n1_{\mathbb{K}}$ is the only possible formula if f is a ring-morphism and it is certainly clear that this formula supplies a ring -morphism $\mathbb{Z} \to \mathbb{K}$ for any ring $\mathbb{K}$; here we wrote the identity of $\mathbb{K}$ as $1_{\mathbb{K}}$ for extra clarity because $1 \in \mathbb{Z}$ is also there in the calculation but one usually writes $1_{\mathbb{K}}$simply as 1].

We say "$\mathbb{Z}$ is an initial object in the category of rings" but this language will not be used in this course.

(i) The least integer $k \geq 0$ such that $f(k) = 0$ is called the characteristic of $\mathbb{K}$; clearly $f = \{km | m \in \mathbb{Z}\}$.

(ii) (a) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ have characteristic 0.

(b) $\mathbb{Z}_n$ has characteristic n. [$\because$ $[0] = \{nm | m \in \mathbb{Z}\}$]

(c) $\mathbb{Z}_{\ltimes} \times \mathbb{Z}_{\gtrdot}$ has characteristic $lcm(m, n)$.

(d) $\mathbb{K}$ has characteristic 1 iff $f(1) = 0$ i.e. 1=0 in $\mathbb{K}$. Since we have agreed that we have at least two distinct scalars $0 \neq 1$, no ring of scalars for this course will have characteristic 1 but it should be noted that not every textbook on linear algebra takes this position.

4. (i) In connection with 3(ii)(d) above, it is perhaps here we should note that some textbooks on linear algebra accept rings without identity can be enlarged to a ring with identity so in the good old days one always did it. with modern development ,there are situations [like 'K-theory'] in which rings,or rather algebras,without identity play an essential role.But we accept rings only with identity.

(ii) On the other hand, many text books on linear algebra take only commutative rings as scalars and most of them in fact take only fields. Some accept only division rings. The basic truble with these in as follows : Even if one works only with fields as scalars, one has to works with field as scalars, one has to work with 'block matrices' in which are noncommutative so linear algebra with noncommutative rings also comes into picture.

(iii) Working with only division rings or fields, one can avoid the term 'modula' and use only 'vector spaces'. This is an economy which is only apparent : practically all the work has to be done anyway. In particular, which one crucial fact : a linear operator $V \to^T V$ [where v is a vector space over field $\mathbb{F}$] is in fact just a modulo over the polynomial ring $\mathbb{F}[\theta]$, is never mentioned, all the computational work has still to be done. So all one gains is that a concept has not been given its proper name.

(iv) We note [but will not prove] that (i) Every module over a division ring a free module (ii)If every module over $\mathbb{K}$-modulo, $\mathbb{K}$ must be a division ring. Thus the best result are certainly developed in the context of vector spaces [which are just modules over division rings] and if one uses the very convenient characterization in terms of determinant [of matrices] then it is almost essential to work with "finite-dimensional vector spaces over fields".However,it becomes an unnecessary journey into surprizes later when one fields that many things do not work in more matrices are not invertible,that $AB = 0$ dose not mean that $A = 0$ or$B = 0$,that $AB = BA$ may not be true,and so on;but it dose not mean that one should never move out of the world of real numbers.

(v) Some textbooks deal with only complex numbers as scalars; this is to prepare the student for "Applicable functional analysis " and hence to many rich and exciting subjects like quantum mechanics and Wavelets differential equations and so(classical) computation,one frequently meets with $\mathbb{Z}_n$ and since there is hardly any saving in learning efforts by declaring "by a scalar we mean a real number or a complex number" we have avoided this declaration.

(vi) Clearly, the 'choice of reasonable scalars' is a subjective choice . We hope our option of working with a noncommutative but associative ring with identity with identity equipped with a preferred conjugation' prepares the users for a number of situations that may arise in subsequent mathematical education.

(vii) In all this, let us note that in $\mathbb{Z}_n$, we have $[n] = [0]$ so that, for instance, $\lambda \in \mathbb{Z}_2, \ \ \mu \in \mathbb{Z}_2$ dose not ensure that $\frac{\lambda+\mu}{2} \in \mathbb{Z}_2$. Many results will need "let $\mathbb{F}$ be a field of characteristic not equal to 2". [There are fields other than $\mathbb{Z}_2$ which have characteristic 2 ; see 7(vii) below page 10]

5. Let us note that the collection of all vectors in $\mathbb{R}^3$ is a ring with cross product as multiplication since $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times b + \underline{a} \times \underline{c}$. This not associative , [$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c}$ fails] and dose not commutative, [$\underline{a} \times \underline{b} = \underline{b} \times \underline{a}$ fails] and dose not have an identity. User for this course can treat it as an object of curiosity in the beginning but at the advanced level of linear algebra, non associative multiplication enter into picture [when one deals with 'Lie Algebra' with which we will not work].

6. If $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ is a quaternion then for any $r \in \mathbb{R}$ we clearly have $ra = ar$. Then $\mathbb{R} \subseteq cen\mathbb{H}$ [$\mathbb{H}$ the division ring of quaternions]. Conversely, suppose $q = (q_0 + q_1e_1 + q_2e_2 + q_3e_3) \in cen\mathbb{H}$ so that $qa = aq$ for each $a \in \mathbb{H}$. Then we have $qe_1 = e_1q$ in particular. Now $qe_1 = q_0e_1 + q_1e_1^2 + q_2e_2e_1 + q_3e_3e_1$
$= -q_1 + q_0e_1 + q_3e_2 - q_2e_3$ [look at modules, page 2], while
$e_1q = e_1q_0 + e_1q_1e_1 + e_1q_2e_2 + e_1q_3e_3$
$= q_0e_1 + q_1e_1e_1 + q_2e_1e_2 + q_3e_1e_3 [q_i \in \mathbb{R} \subseteq cen\mathbb{H}]$

$= -q_1 + q_0e_1 - q_3e_2 + q_2e_3$

which provides $q_3 = 0 = q_2$ [$\because \sum_{i=0}^{3} q_ie_i = \sum_{i=0}^{3} q_i'e_i$ iff $q_i = q_i'$ for each i]

Thus $q = q_0 + q_1e_1$. We also have $qe_2 = e_2q$, *i.e.* $q_0e_2 + q_1e_1e_2 = e_2q_0 + e_2q_1e_1 = q_0e_2 + q_1e_2e_1$ i.e. $q_0e_2 + q_1e_3 = q_0e_2 - q_1e_3$ which means $q_1 = 0$. Thus We have $q = q_0 \in \mathbb{R}$ i.e. $cen(\mathbb{H}) \subseteq \mathbb{R}$. To sum up : $cen\mathbb{H} = \mathbb{R}$.

[We wrote a quaternion as $\sum_{i=0}^{3} q_ie_i$ rather than $\sum e_iq_i$ but this dose not matter since $\mathbb{H}$ is an algebra over $\mathbb{R}$ and $\mathbb{R}$ is commutative; see 1.4.1, modules, page 9].

7. Our rings are rings are rings with a given involutions . We took $\overline{\lambda} = \lambda_0 - e_1\ \lambda_1 - e_2\ \lambda_2 - e_3\lambda_3$ for $\lambda = \lambda_0 + e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3 \in \mathbb{H}$ as the standard involution but $\lambda^* = \lambda_0 + e_1\ \lambda_1 - e_2\ \lambda_2 - e_3\lambda_3$ is also an involution and there could be others [such as ?]. We do not attempt an explanation of what makes an involution 'student'. But let us look at some situations.

   (i) (a) If$\mathbb{Z} \rightarrow^f \mathbb{Z}$ is a ring morphism, we have $f(1) = 1$ and thus $f(m) = f(m.1) = mf(1)$ for all $m \in \mathbb{Z}$.

   (b) If $Z_n \rightarrow^f Z_n$ is a ring morphism, we have have $f[a] = f(a[1]) = af[1] = a[1] = [a]$.

   (c) If $\mathbb{Q} \rightarrow^f \mathbb{Q}$ is a ring morphism, we have $nf(\frac{m}{n}) = f(n.\frac{m}{n}) = f(m) = f(m.1) = mf(1) = m = n.\frac{m}{n}$ and hence $f(\frac{m}{n}) = f(\frac{m}{n})$ for any $r \in \mathbb{Q},\ r = \frac{m}{n},\ m,\ n \in \mathbb{Z},\ n > 0$

   (d) Suppose $\mathbb{K}$ and $\mathbb{L}$ are rings, $\mathbb{K} \rightarrow^f \mathbb{L}$ a ring morphism. Write $ker\ f := \{\lambda \in \mathbb{K}/f(\lambda) = 0\}$ Then if $\lambda \in kerf$ and $a \in \mathbb{K}$, we have $f(\lambda a) = f(\lambda)f(a) = 0f(a) = 0$ i.e. $\lambda a \in$ ker f ; similarly $a\lambda \in kerf$ [This makes ker f an ideal of $\mathbb{K}$; for the definition of 'ideal' see below] Now if $\mathbb{K}$ and $\mathbb{L}$ are division rings , $0 \neq \lambda \in kerf$, then $1_{\mathbb{K}} = \lambda^{-1}\lambda \in kerf$ so that $f(1_{\mathbb{K}}) = 1_{\mathbb{L}}$ and $1 \neq 0$ in any (division)ring. Therefore $kerf = 0$. Hence f is injective [$f(\lambda) = f(\mu)$ means $f(\lambda - \mu) = f(\lambda) - f(\mu) = 0$ which ensures $\lambda - \mu \ \epsilon\ kerf$ i.e. $\lambda - \mu = 0$ i.e. $\lambda = \mu$

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To sum up: If $\mathbb{K}$ and $\mathbb{L}$ are division rings (and in particular, if they are fields), any ring morphism $\mathbb{K} \to^f \mathbb{L}$ is necessarily injective.

(ii) If $\mathbb{R} \to^f \mathbb{R}$ is a ring morphism, then by above it is injective and hence [by (i)c on preceding pase] we have $f(r) = r$ for $r \in \mathbb{Q}$. If $x \in \mathbb{R}, x > 0$ then $f(x) = f(\sqrt{x}\sqrt{x}) = (f(\sqrt{x}))^2$ so that if $a > b$ and thus $a - b > 0$, we get $f(a-b) = f(a) - f(b) > 0$ i.e.$f(a) > f(b)$. This means f must be order-preserving. Now let $x \in \mathbb{R}$ and choose $r_n, s_n \in \mathbb{Q}$ such that $r_n < x < s_n$ with $\bigcap_{n=1}^{\infty}[r_n, s_n] = \{x\}$

[This is possible because of the properties of $\mathbb{R}$]

Then we have $f(r_n) = r_n < f(x) < f(s_n) = s_n$ which means $f(x) \in \bigcap_{n=1}^{\infty}[r_n, s_n] = \{x\}$ and proves that $f(x) = x$.

Together with (i),(iii) shows that :

None of the rings $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$ admits any ring morphism from itself to itself other than the identity. Since each of them is commutative, morphisms are the same as anti-morphisms. Therefore

No nontrivial involution, isomorphisms are the same as anti-isomorphism. If $\mathbb{C} \to^f \mathbb{C}$ is an isomorphism such that $f(x) = x$ for $x \in \mathbb{R}$, then $f(x+iy) = f(x) + f(i)f(y)$ with $(f(i))^2 = f(i)f(i) = f(i^2) = f(-1) = -1$ so that $f(i) = \pm i$.To sum up:

The only nontrivial involution on $\mathbb{C}$ with $f(x) = x$ for $x \in \mathbb{R}$ is $x + iy \mapsto x - iy$

(iii) If $d \neq 0.1$ is an integer which is square free [in the sense that its prime factorization has no square], we write $\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} | a,\ b \in \mathbb{Z}\}$ and $\mathbb{Q}[\sqrt{d}] := \{a + b\sqrt{d} | a,\ b \in \mathbb{Q}\}$

Then under obvious multiplication and addition, there are commutative rings and $Q[\sqrt{d}]$ is actually a field ; both carry the nontrivial involution $a + b\sqrt{d} \mapsto a - b\sqrt{d}$. The user is requested to verify this.

(iv) For a prime number p, $\mathbb{Z}_p$ is a finite field with exactly p element ; this has been introduced in 2(viii) pase 2 above. We supply the following facts without proof; the proof are not exactly obvious and are properly speaking part of an algebra course which pays some attention to finite fields.

(a) A finite field has exactly p^r elements, p a prime, $r > 0$ an integer.

(b) There is exactly one finite field with p^r elements

(c) For the finite field with p^r elements, the characteristic is p. Further, $\lambda \mapsto \lambda^p$ is a ring morphism and $(\lambda + \mu)^p = \lambda^p + \mu^p$ holds.

(d) Thus on the field $\mathbb{F}$ with p^2 elements, $x \mapsto \overline{x} := x^p$ is an involution.

(e) Let us examine the field $\mathbb{F}$ with p^2 elements for $p = 2$. Quite clearly, this cannot be $\mathbb{Z}_4$ [in fact the field with p^r elements, $r > 1$, will never be $\mathbb{Z}_q, q = p^r$] because $\mathbb{Z}_n$ is a field iff p is a prime [2(viii), page 2 above]. verify that it carries addition and multiplication given by the tables

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

and

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	o	a	b	1
b	0	b	1	a

Its multiplicative fragment {1, a, b} must be an abelian group since it is a field. Verify that this is same as the additive fragment of $\mathbb{Z}_3$. There are no finite division rings other than finite fields.

8. Consider the collection $\mathbb{K} := \{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} | \alpha, \beta \in \mathbb{C}\}$.

Since $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} + \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix} = \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ -\overline{\beta}-\overline{\delta} & \overline{\alpha}+\overline{\gamma} \end{pmatrix} = \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ -\overline{\beta+\delta} & \overline{\alpha+\gamma} \end{pmatrix}$,

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix} = \begin{pmatrix} \alpha\gamma - \beta\overline{\delta} & \alpha\delta + \beta\overline{\gamma} \\ -\overline{\beta}\gamma - \overline{\alpha}\overline{\delta} & -\overline{\beta}\delta + \overline{\alpha}\,\overline{\gamma} \end{pmatrix} = \begin{pmatrix} \alpha\gamma - \beta\overline{\delta} & \alpha\delta + \beta\overline{\gamma} \\ -(\overline{\alpha\delta + \beta\overline{\gamma}}) & \overline{\alpha\gamma - \beta\overline{\delta}} \end{pmatrix},$$

We find that $\mathbb{K}$ is closed under the usual matrix addition and multiplication and thus obviously a ring [—the distributive law flows from the usual matrix situation $Mat_2(\mathbb{C})$] which is a sub ring of Mat_2 ($\mathbb{C}$) because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1d_2 \in \mathbb{K}$.

(i)We write $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,$K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and note

$$JK = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = I,$$

$$KJ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} - I,$$

$$KI = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

$$IK = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -J$$

$$IJ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = K$$

$$JI = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -K$$

$$J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -Id_2$$

$K^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -ID_2$

$I^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -Id_2$ If $\alpha = a + bi, \beta = c + di, a, b, c, d \in \mathbb{R}$ we can write a quaternion uniquely as $q = aId_2 + bI + cJ + dK$ and conversely a matrix of the sort $a1d_2 + bI + cJ + dK$ would stand for the quaternion This can be done by correspondence $e_0 \leftrightarrow Id_2, e_1 \leftrightarrow I, e_2 \leftrightarrow J, e_3 \leftrightarrow K$ The calculation above show that the multiplication table for quaternions [on page 2 Modules]corresponds to the matrix multiplication performed above;since the addition obviously dose the mapping $f(e_0) = 1d_2, f(e_1) = I, f(e_2) = J, f(e_3) = K$establishes an isomorphism of the division algebra of the quaternions onto the matrix algebra introduced just now. So the two are the same. Further,$f(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} = a1d_2 + bI$ for $\alpha = a + bi\epsilon\mathbb{C}$ embeds the field $\mathbb{C}$ into $\mathbb{H}$ via $a \mapsto a + bi = \alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$

Similarly, with $a \in \mathbb{R}$ being identified to $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ we can show that the matrices $a + b_j = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ form a field under multiplication and addition which is isomorphic to $\mathbb{C}$ via $a + b_i \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.