Self Evaluation Test

1. If $A = \; = \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix}$ then compute $A^{@}$

Solution. In Example 1.(Module 7, Generalized Inverse) we have

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & -\iota \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1+\iota}{2} & -1 \\ \frac{1+\iota}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1+\iota}{2} & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{bmatrix}$$

$$\text{Then } \; D^{@} = \left[\begin{array}{cc|c} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$A^{@} = VD^{@}U^{*}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-\iota}{\sqrt{3}} \\ \frac{1+\iota}{\sqrt{3}} & \frac{-\iota}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1-\iota}{2} & \frac{1}{2} \\ \frac{-1-\iota}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-1-\iota}{2\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-\iota}{\sqrt{3}} \\ \frac{1+\iota}{\sqrt{3}} & \frac{-\iota}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{4\sqrt{3}} & \frac{1-\iota}{4\sqrt{3}} & \frac{1}{4\sqrt{3}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{12} & \frac{1-\iota}{12} & \frac{1}{12} \\ \frac{1+\iota}{12} & \frac{2}{12} & \frac{1+\iota}{12} \end{bmatrix}$$

$$A^{@} = \frac{1}{12} \begin{bmatrix} 1 & 1-\iota & 1 \\ 1+\iota & 2 & 1+\iota \end{bmatrix}$$

$$AA^{@}A = A \text{ and } A^{@}AA^{@} = A^{@}$$

$$
\begin{aligned}
AA^{@}A &= \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix} \frac{1}{12} \begin{bmatrix} 1 & 1-\iota & 1 \\ 1+\iota & 2 & 1+\iota \end{bmatrix} \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix} \frac{1}{12} \begin{bmatrix} 4 & 41-\iota \\ 41+\iota & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1-\iota}{3} \\ \frac{1+\iota}{3} & \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1-\iota \\ 1+\iota & 2 \\ 1 & 1-\iota \end{bmatrix}
\end{aligned}
$$

Similarly $A^{@}AA^{@} = A^{@}$.

2. Compute $A^{@}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Solution. $A^*A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which has eigen values $\lambda_1^2 = 2, \lambda_2 = 0$ and corresponding orthogonal set of eigen vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

After normalization choose the unitary matrix V with these as columns.

$$
\begin{aligned}
V &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
AV &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\mu_1, & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mu_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
x_1 &= \mu_1 \\
x_2 &= e_2 - \frac{\langle e_2, x_1 \rangle}{\|x_1\|^2} x_1 \\
&= (0,1) - \frac{\langle (0,1),\ (1,0) \rangle}{1} (1,0) \\
&= (0,1)
\end{aligned}
$$

Therefore $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$
\begin{aligned}
\text{Then} \quad U^*AV &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \\
&= D \\
D^{@} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \\
A^{@} &= VD^{@}U^* \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \\
AA^{@}A &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\
&= A
\end{aligned}
$$

Similarly $A^{@}AA^{@} = A^{@}$.

3. Solve the $x + 2y = 1$ by Generalised inverse method.

Solution.

$$
\begin{aligned}
x + 2y &= 1 \\
\begin{bmatrix} 1, & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= 1 \\
\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1, & 2 \end{bmatrix}^{@} [1]
\end{aligned}
$$

where $\begin{bmatrix} 1, & 2 \end{bmatrix}^{@}$ is generalised inverse of $\begin{bmatrix} 1, & 2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1, & 2 \end{bmatrix}$, $A^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A^*A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1, & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ which have eigen values$\lambda_1^2 = 5$,$\lambda_2 = 0$ and the corresponding eigen vector $(1, 2)$ and $(-2, 1)$. After normalization choose the unitary matrix V with these as columns.

$$
\begin{aligned}
V &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
AV &= \begin{bmatrix} 1, & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{5}, & 0 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \mu_1, & 0 \end{bmatrix} \\
\Rightarrow \mu_1 &= 1 \\
U &= [1] \\
D &= U^* AV \\
&= [1] \begin{bmatrix} 1, & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{5}, & 0 \end{bmatrix} \\
D^{@} &= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \\
A^{@} &= V D^{@} U^* \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} [1]
\end{aligned}
$$

$$
\begin{aligned}
A^{@} &= \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} &= A^{@}[1] \\
&= \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}
\end{aligned}
$$

$$
\Rightarrow \begin{array}{l} x = \frac{1}{5} \\ y = \frac{2}{5} \end{array}
$$