Self Evaluation Test

1. Let T be a L.O. on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of linear operator T of R^2 such that $T^2 = 0$, but $T \neq 0$.

Solution.

$$\begin{aligned} T^2 = 0 &\Rightarrow T^2(v) = 0\ \forall v \in V \\ \Rightarrow T(T(V)) &= 0\ \forall v \in V \\ \Rightarrow T(V) &\in \text{Ker}T\ \forall v \in V \\ \text{i.e. Range}T &\subseteq \text{Ker}T \end{aligned}$$

Define $T : R^2 \to R^2$ such that,

$$T(x_1, x_2) = (x_2, 0)$$

T is Linear Operator. Since $T(2, 2) = (2, 0) \neq (0, 0) \Rightarrow T \neq 0$. But

$$\begin{aligned} T^2(x_1, x_2) &= T(T(x_1, x_2)) \\ &= T(x_2, 0) \\ &= (0, 0) \\ \Rightarrow T^2 &= 0 \end{aligned}$$

2. Let T be a linear operator on V and let RankT^2=Rank$T$, show that

$$\text{Range } T \cap \text{ Ker } T = \{0\}$$

Solution. $T : V \to V$, $T^2 : V \to V$ are Linear Transformations.

Rank T^2 =dim$V$-dim KerT^2 ($\because$ Rank T^2=Rank T)

$\Rightarrow$ dim Ker T=dim Ker T^2

Let $x \in$ Ker$T$ $\Rightarrow Tx = 0$ $\Rightarrow T^2(x) = T(0) = 0$

$\Rightarrow$ $x \in$ KerT^2 $\Rightarrow$ Ker$T \subseteq$ KerT^2

$\Rightarrow$ Ker$T$ = KerT^2 (as they have the same dim).

Now $x \in$ Range $T \cap$ Ker$T$

$$\begin{aligned}
\Rightarrow\ T(x) &= 0 \text{ if } x = T(y) \text{ for some } y \in V \\
\Rightarrow\ T(Ty) &= 0 \text{ i.e. } T^2(y) = 0 \\
\Rightarrow\ y \in \ \ker T^2 &= \text{Ker}T \\
\text{i.e. } T(y) &= 0 \\
\Rightarrow\ x &= 0 \\
\Rightarrow\ \text{Ker}T \cap \text{Range}T &= \{0\}
\end{aligned}$$

3. Let T be a linear operator on R^2 defined by $T(x_1, x_2) = (-x_2, x_1)$.

Let $\beta = \{e_1 = (1,0), e_2 = (0,1)\}$ and $\beta' = \{\alpha_1 = (1,2), \alpha_2 = (1,-1)\}$ be ordered basis for R^2.

Find a matrix P such that $[T]_{\beta'} = P^{-1}[T]_\beta P$.

Solution. $T(1,0) = (0,1)$; $T(0,1) = (-1,0)$

$T(1,0) = 0.e_1 + 1.e_2$; $T(0,1) = -1.e_1 + 0.e_2$

$\therefore\ [T]_\beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$T(\alpha_1) = T(1,2) = (-2,1) = \frac{-1}{3}\alpha_1 - \frac{5}{3}\alpha_2$

$T(\alpha_2) = T(1,-1) = (1,1) = \frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2$

$\therefore\ [T]'_\beta = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{bmatrix}$

Define $S : R^2 \to R^2$ such that

$$S(e_i) = \alpha_i;\ i = 1,2$$

Now $\alpha_1 = (1,2) = 1.e_1 + 2.e_2$ and $\alpha_2 = (1,-1) = 1.e_1 + (-1)e_2$

$S(\alpha_1) = 1.\alpha_1 + 2.\alpha_2$; $S(\alpha_1) = 1.\alpha_1 + (-1).\alpha_2$

$[S]_{\beta'} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$$\begin{aligned}
P &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \\
\text{and}\ \ P^{-1} &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \\
P^{-1}[T]_\beta P &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{bmatrix}$$

$$= [T]_{\beta'}$$

4. Let f_1, f_2, f_3 be three linear functionals on R^4 defined as follows:

$f_1(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$

$f_2(x_1, x_2, x_3, x_4) = 2x_2 + x_4$

$f_3(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4$

Determine the subspace W of R^4 such that $f_i(w) = 0 \; \forall \; w \in W; \; i = 1, 2, 3$

Solution Let $(x_1, x_2, x_3, x_4) \in W$. Then $f_i(x_1, x_2, x_3, x_4) = 0 \; \forall \; i = 1, 2, 3$

$$\begin{aligned} \Rightarrow \; x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ 2x_2 - x_4 &= 0 \\ -2x_1 - 4x_3 + 3x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

By elementary row transformations we get:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$\Rightarrow \; x_1 + 2x_3 = 0; \; x_2 = 0; \; x_4 = 0$

$$\begin{aligned} \therefore \; (x_1, x_2, x_3, x_4) &= (-2x_3, 0, x_3, 0) \\ &= x_3(-2, 0, 1, 0) \end{aligned}$$

$\therefore$ W is spanned by (-2,0,1,0).

5. Let $f, g \in V'$ such that $f(v) = 0 \; \Rightarrow \; g(v) = 0$, prove that $g = cf$ for some $c \in F$,(V' is dual space of V).

Solution. If $f = 0$, then $g = 0 = cf$; for any $c \in F$.

Let $f \neq 0$, then there exist $v \neq 0$ in V such that $f(v) \neq 0$.

let $c = \frac{g(v)}{f(v)}$, $h = g - cf$ and $x \in V$ and $\alpha = \frac{f(x)}{f(v)}$.

$$\begin{aligned}
\text{Then } f(x - \alpha v) &= f(x) - \alpha f(v) = 0 \\
\Rightarrow\ x - \alpha v &\in \text{Ker} f \\
\Rightarrow\ x - \alpha v &= y \in \text{Ker} f \\
\Rightarrow\ x &= y + \alpha v \\
h(x) &= g(x) - cf(x) \\
&= g(y) + \alpha g(v) - cf(y) - c\alpha f(v)\ \cdot \\
&= \alpha g(v) - c\alpha f(v) \text{ as} \\
y \in \text{Ker} f &\Rightarrow y \in \text{Ker} g \\
&= \frac{f(x)}{f(v)} g(v) - \frac{g(v)}{f(v)} . f(x) \\
&= 0\ \forall\ x \in V \\
\therefore\ h = 0 &\Rightarrow g = cf
\end{aligned}$$

Hence the result follows.