Measure (Lectures 5, 6, 7, 8 and 9)

2.1. Set functions

(2.1) Let X be any countably infinite set and let

$$\mathcal{C} = \{\{x\} \mid x \in X\}.$$

Show that the algebra generated by $\mathcal{C}$ is

$$\mathcal{F}(\mathcal{C}) := \{A \subseteq X \mid A \text{ or } A^c \text{ is finite}\}.$$

Let $\mu : \mathcal{F}(\mathcal{C}) \longrightarrow [0, \infty)$ be defined by

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is finite,} \\ 1 & \text{if } A^c \text{is finite.} \end{cases}$$

Show that μ is finitely additive but not countably additive. If X is an uncountable set, show that μ is also countably additive.

(2.2) Let $X = \mathbb{N}$, the set of natural numbers. For every finite set $A \subseteq X$, let $\#A$ denote the number of elements in A. Define for $A \subseteq X$,

$$\mu_n(A) := \frac{\#\{m : 1 \leq m \leq n, m \in A\}}{n}.$$

Show that μ_n is countably additive for every n on $\mathcal{P}(X)$. In a sense, μ_n is the proportion of integers between 1 to n which are in A. Let

$$\mathcal{C} = \{A \subseteq X \mid \lim_{n\to\infty} \mu_n(A) \text{ exists}\}.$$

Show that $\mathcal{C}$ is closed under taking complements, finite disjoint unions and proper differences.

(2.3) Let $\mu : \tilde{\mathcal{I}} \cap (0,1] \longrightarrow [0,\infty]$ be defined by

$$\mu(a,b] := \begin{cases} b-a & \text{if } a \neq 0, 0 < a < b \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

(Recall that $\tilde{\mathcal{I}} \cap (0,1]$ is the class of all left-open right-closed intervals in $(0,1]$.)

Show that μ is finitely additive. Is μ countably additive also?

(2.4) Let $\mathcal{A}$ be an algebra of subsets of a set X.

(i) Let μ_1, μ_2 be measures on $\mathcal{A}$, and let α and β be nonnegative real numbers. Show that $\alpha\mu_1 + \beta\mu_2$ is also a measure on $\mathcal{A}$.

(ii) For any two measures μ_1, μ_2 on $\mathcal{A}$, we say

$$\mu_1 \leq \mu_2 \;\text{ if }\; \mu_1(E) \leq \mu_2(E), \;\forall\, E \in \mathcal{A}.$$

Let $\{\mu_n\}_{n\geq 1}$ be a sequence of measures on $\mathcal{A}$ such that

$$\mu_n \leq \mu_{n+1}, \;\forall\, n \geq 1.$$

Define $\forall\, E \in \mathcal{A}$,

$$\mu(E) := \lim_{n\to\infty} \mu_n(E).$$

Show that μ is also a measure on $\mathcal{A}$ and $\forall\, E \in \mathcal{B}$,

$$\mu(E) = \;\sup\{\mu_n(E) \mid n \geq 1\}.$$

(2.5) Let X be a compact topological space and $\mathcal{A}$ be the collection of all those subsets of X which are both open and closed. Show that $\mathcal{A}$ is an algebra of subsets of X. Further, every finitely additive set function on $\mathcal{A}$ is also countably additive.

Optional Exercises

(2.6) Let X be a nonempty set.

(a) Let $\mu : \mathcal{P}(X) \longrightarrow [0,\infty)$ be a finitely additive set function such that $\mu(A) = 0$ or 1 for every $A \in \mathcal{P}(X)$. Let

$$\mathcal{U} = \{A \in \mathcal{P}(X) \mid \mu(A) = 1\}.$$

Show that $\mathcal{U}$ has the following properties:

(i) $\emptyset \notin \mathcal{U}$.

(ii) If $A \in X$ and $B \supseteq A$, then $B \in \mathcal{U}$.

(iii) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

(iv) For every $A \in \mathcal{P}(X)$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

(Any $\mathcal{U} \subseteq \mathcal{P}(X)$ satisfying (i) to (iv) is called an **ultrafilter** in X.)

(b) Let $\mathcal{U}$ be any ultrafilter in X. Define $\mu : \mathcal{P}(X) \longrightarrow [0,\infty)$ by

$$\mu(A) := \begin{cases} 1 & \text{if } A \in \mathcal{U}, \\ 0 & \text{if } A \notin \mathcal{U}. \end{cases}$$

Show that μ is finitely additive.

2.2. Countably additive set functions on intervals

(2.7) Let $F(x) = [x]$, the integral part of $x, x \in \mathbb{R}$. Describe the set function μ_F induced by F on the class $\tilde{\mathcal{I}}$ of all left-open right-closed intervals.

(2.8) Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function and $\alpha \in \mathbb{R}$. Show that $F_1 := F + \alpha$ is also a distribution function and $\mu_F = \mu_{F_1}$. Is the converse true?

(2.9) (i) Let $\mathcal{C}$ be a collection of subsets of a set X and $\mu : \mathcal{C} \to [0, \infty]$ be a set function. If μ is a measure on $\mathcal{C}$, show that μ is finitely additive. Is μ monotone? Countably subadditive?

(ii) If $\mathcal{C}$ be a semi-algebra, then μ is countably subadditive iff $\forall\ A \in \mathcal{C}$ with $A \subseteq \bigcup_{i=1}^{\infty} A_i,\ A_i \in \mathcal{C}$ implies

$$\mu(A) \ \leq\ \sum_{i=1}^{\infty} \mu(A_i).$$

2.3. Set functions on algebras

(2.10) Let $\mathcal{A}$ be an algebra of subsets of a set X and $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive set function.

(i) Show that in general, for a decreasing sequence $\{A_k\}_{k\geq 1}$ in $\mathcal{A}$ with $\bigcap_{k=1}^{\infty} A_k = A \in \mathcal{A}$ need not imply that $\mu(A) = \lim_{n\to\infty} \mu(A_n)$, even if μ is countably additive.

(ii) If $\mu(X) < +\infty$, show that the following statements are equivalent:

(a) $\lim\limits_{k\to\infty} \mu(A_k) = 0$, whenever $\{A_k\}_{k\geq 1}$ is a sequence in $\mathcal{A}$ with $A_k \supseteq A_{k+1}\ \forall\ k$, and $\bigcap_{k=1}^{\infty} A_k = \emptyset$.

(b) μ is countably additive.

(2.11) Let $\mathcal{A}$ be a σ-algebra and $\mu : \mathcal{A} \to [0, \infty]$ be a measure. For any sequence $\{E_n\}_{n\geq 1}$ in $\mathcal{A}$, show that

(i) $\mu\left(\liminf_{n\to\infty} E_n\right) \ \leq\ \liminf_{n\to\infty} \mu(E_n)$.

(ii) $\mu\left(\limsup_{n\to\infty} E_n\right) \ \geq\ \limsup_{n\to\infty} \mu(E_n)$.

(Hint: For a sequence $\{E_n\}_{n\geq 1}$ of subsets of a set X,

$$\liminf_{n\to\infty} E_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k \ \subseteq\ \limsup_{n\to\infty} E_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.\,)$$

Optional Exercise

(2.12) Let $\mathcal{A}$ be a **semi- algebra** of subsets of a set X and $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive set function. Show that the following statements are equivalent:

(a) $\lim\limits_{k\to\infty} \mu(A_k) = 0$, whenever $\{A_k\}_{k\geq 1}$ is an increasing sequence in $\mathcal{A}$ with $\bigcup_{k=1}^{\infty} A_k = A \in \mathcal{A}$.

(b) μ is countably additive.

(Hint: Extend μ to the algebra generated by $\mathcal{A}$.)

2.4. Uniqueness problem for measures

(2.13) Let $\mathcal{A}$ be an algebra of subsets of a set X. Let μ_1 and μ_2 be σ-finite measures on a σ-algebra $\mathcal{S}(\mathcal{A})$ such that $\mu_1(A) = \mu_2(A)\ \ \forall\, A \in \mathcal{A}$. Then, $\mu_1(A) = \mu_2(A)\ \forall A \in \mathcal{S}(\mathcal{A})$.

(2.14) Show that a measure μ defined on an algebra $\mathcal{A}$ of subsets of a set X is finite if and only if $\mu(X) < +\infty$.