Integration (Lectures 17, 18, 19, 20 and 21)

5.1. Integral of nonnegative simple functions

(5.1) Show that for $s \in \mathbb{L}_0^+$, $\int s(x)d\mu(x)$ is well-defined by proving the following: let

$$s = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{j=1}^{m} b_j \chi_{B_j},$$

where $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_m\}$ are partitions of X by elements of $\mathcal{S}$, then

(ii)

$$s = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{A_i \cap B_j} = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \chi_{A_i \cap B_j}.$$

(ii)

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j).$$

(iii) $\displaystyle\int s(x)d\mu(x)$ is independent of the representation of the function $s(x) = \sum_{i=1}^{n} a_i \chi_{A_i}$.

(5.2) Let $s_1, s_2 \in \mathbb{L}_0^+$. Prove the following:

(i) If $s_1 \geq s_2$, then $\int s_1 d\mu \geq \int s_2 d\mu$.

(ii) Let $\forall\, x \in X$,

$(s_1 \vee s_2)(x) := \max\{s_1(x), s_2(x)\}$ and $(s_1 \wedge s_2)(x) := \min\{s_1(x), s_2(x)\}$.

Then $s_1 \wedge s_2$ and $s_1 \vee s_2 \in \mathbb{L}_0^+$ with

$$\int (s_1 \wedge s_2) d\mu \;\leq \int s_i d\mu \;\leq\; \int (s_1 \vee s_2) d\mu,\; i = 1, 2.$$

(5.3) For $s_1, s_2 \in \mathbb{L}_0^+$ compute $\{x \,|\, s_1(x) \geq s_2(x)\}$ and show that it belongs to $\mathcal{S}$. Can you say that the sets

$$\{x \in X \,|\, s_1(x) > s_2(x)\}, \{x \in X \,|\, s_1(x) \leq s_2(x)\}, \{x \in X \,|\, s_1(x) = s_2(x)\}$$

are also elements of $\mathcal{S}$?

(5.4) Let $s_1, s_2 \in \mathbb{L}_0^+$ be real valued and $s_1 \geq s_2$. Let $\phi = s_1 - s_2$. Show that $\phi \in \mathbb{L}_0^+$. Can you say that

$$\int \phi d\mu = \int s_1 d\mu - \int s_2 d\mu?$$

(5.5) Let $\{s_n\}_{n\geq 1}$ and $\{s_n^{'}\}_{n\geq 1}$ be sequences in $\mathbb{L}_0^+$ such that for each $x \in X$, both $\{s_n(x)\}_{n\geq 1}$ and $\{s_n^{'}(x)\}_{n\geq 1}$ are increasing and

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} s_n^{'}(x).$$

Show that

$$\lim_{n\to\infty} \int s_n d\mu = \lim_{n\to\infty} \int s_n^{'} d\mu.$$

(Hint: Consider sequence $\{s_n \wedge s_m^{'}\}_n$ for all fixed m to deduce that $\int s_m^{'} d\mu \leq \lim_{n\to\infty} \int s_n d\mu$.)

5.2. Integral of nonnegative measurable functions

(5.6) Let $f \in \mathbb{L}^+$.

(i) If $\displaystyle\int f d\mu = 0$, how that $f(x) = 0$, a.e.(x).

(ii) If $\displaystyle\int f d\mu < \infty$, Show that $f(x) < \infty$, a.e.(x).

(5.7) Let $f \in \mathbb{L}^+$ and let $\{s_n\}_{n\geq 1}$ be in $\mathbb{L}_0^+$ and such that $\{s_n(x)\}_{n\geq 1}$ is decreasing and $\forall\, x \in X$, $\lim\limits_{n\to\infty} s_n(x) = f(x)$. Can you conclude that

$$\int f d\mu = \lim_{n\to\infty} \int s_n d\mu?$$

(5.8) Let $s \in \mathbb{L}_0^+$

(i) Show that $\int s d\mu$ is same whether we teart it as an element of $\mathbb{L}_0^+$ or of $\mathbb{L}^+$.

(ii) Show that$f \in \mathbb{L}^+$ iff there exists a sequence $\{s_n\}_{n\geq 1}$ in $\mathbb{L}^+$ such that

$$f(x) = \lim_{n\to\infty} s_n(x), \ \forall x \in X.$$

(iii) For $f \in \mathbb{L}^+$,

$$\int f d\mu = \sup\left\{\int s d\mu \middle| 0 \leq s(x) \leq f(x) \text{ for a.e. } x(\mu), s \in \mathbb{L}_0^+\right\}.$$

(5.9) Let $\{f_n\}_{n\geq 1}$ be an increasing sequence of functions in $\mathbb{L}^+$ such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists for a.e. $x(\mu)$. Show that $f \in \mathbb{L}^+$ and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu,$$

where $f(x)$ is defined as an arbitrary constant for all those x for which $\lim_{n\to\infty} f_n(x)$ does not converge.

(5.10) Let f, $f_n \in \mathbb{L}^+$, $n = 1, 2, \ldots$, be such that $0 \leq f_n \leq f$. If $\lim_{n\to\infty} f_n(x) = f(x)$, can you deduce that

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu?$$

(5.11) Let $f \in \mathbb{L}$. For $x \in X$ and $n \geq 1$, define

$$f_n(x) := \begin{cases} f(x) & \text{if} \quad |f(x)| \leq n, \\ n & \text{if} \quad f(x) > n, \\ -n & \text{if} \quad f(x) < -n. \end{cases}$$

Prove the following:

(i) $f_n \in \mathbb{L}$ and $|f_n(x)| \leq n \ \ \forall\, n$ and $\forall\, x \in X$.

(ii) $\lim_{n\to\infty} f_n(x) = f(x) \ \ \forall\, x \in X$.

(iii) $|f_n(x)| := \min\{|f_n(x)|, n\} := (|f| \wedge n)(x)$ is an element of $\mathbb{L}^+$ and

$$\lim_{n\to\infty} \int |f_n| d\mu = \int |f| d\mu.$$

Note:
For $f \in \mathbb{L}$, the sequence $\{f_n\}_{n\geq 1}$ as defined in exercise 4.27 is called the **truncation sequence** of f. The truncation sequence is useful in proving results about functions in the class $\mathbb{L}$.

(5.12) Let $f \in \mathbb{L}$ and

$$\nu(E) := \mu\{x \in X \,|\, f(x) \in E\}, \ E \in \mathcal{B}_{\mathbb{R}}.$$

Show that ν is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Further, if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is any nonnegative $\mathcal{B}_{\mathbb{R}}$-measurable function, i.e., $g^{-1}(A) \in \mathcal{B}_{\mathbb{R}} \ \ \forall\, A \in \mathcal{B}_{\mathbb{R}}$, then $g \circ f \in \mathbb{L}$ and

$$\int g \, d\nu = \int (g \circ f) \, d\mu.$$

The measure ν is usually denoted by $\mu f^{-1}$and is called the **distribution** of the measurable function f.

(5.13) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $(X, \bar{\mathcal{S}}, \overline{\mu})$ be its completion. Let $\overline{f} : X \longrightarrow \mathbb{R}$ be an $\bar{\mathcal{S}}$-measurable function. Show that there exists an $\mathcal{S}$-measurable function $f : X \longrightarrow \mathbb{R}$ such that $\overline{f}(x) = f(x)$ for a.e. $x(\mu)$.

(5.14) Let $(X, \mathcal{S}, \mu)$ be a complete measure space and $\{f_n\}_{n\geq 1}$ be a sequence of $\mathcal{S}$-measurable functions on X. Let f be a function on X such that $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for a.e. $x(\mu)$. Show that f is $\mathcal{S}$-measurable.

5.3. Integrable functions

(5.15) For $f \in \mathbb{L}$, prove the following:

(i) $f \in L_1(\mu)$ iff $|f| \in L_1(\mu)$. Further, in either case

$$\left| \int f d\mu \right| \;\leq\; \int |f| d\mu.$$

(ii) If $f \in L_1(\mu)$, then $|f(x)| < +\infty$ for a.e. $x(\mu)$.

(5.16) Let $\mu(X) < +\infty$ and let $f \in \mathbb{L}$ be such that $|f(x)| \leq M$ for a.e. $x(\mu)$ and for some M. Show that $f \in L_1(\mu)$.

(5.17) Let $f \in L_1(\mu)$ and $E \in \mathcal{S}$. Show that $\chi_E f \in L_1(\mu)$, where

$$\int_E f d\mu \;:=\; \int \chi_E f d\mu.$$

Further, if $E, F \in \mathcal{S}$ are disjoint sets, show that

$$\int_{E\cup F} f d\mu \;=\; \int_E f d\mu + \int_F f d\mu.$$

(5.18) Let $f \in L_1(\mu)$ and $E_i \in \mathcal{S}, i \geq 1$, be such that $E_i \cap E_j = \emptyset$ for $i \neq j$. Show that the series $\sum_{i=1}^{\infty} \int_{E_i} f d\mu$ is absolutely convergent, and if $E := \bigcup_{i=1}^{\infty} E_i$, then

$$\sum_{i=1}^{\infty} \int_{E_i} f d\mu \;=\; \int_E f d\mu.$$

(5.19) (i) For every $\epsilon > 0$ and $f \in L_1(\mu)$, show that

$$\mu\{x \in X \mid |f(x)| \geq \epsilon\} \;\leq\; \frac{1}{\epsilon} \int |f| d\mu \;<\; \infty.$$

This is called **Chebyshev's inequality**.

(ii) Let $f \in L_1(\mu)$, and let there exist $M > 0$ such that

$$\left| \frac{1}{\mu(E)} \int_E f d\mu \right| \;\leq\; M$$

for every $E \in \mathcal{S}$ with $0 < \mu(E) < \infty$. Show that $|f(x)| \leq M$ for a.e. $x(\mu)$.

(5.20) Let λ be the Lebesgue measure on $\mathbb{R}$, and let $f \in L_1(\mathbb{R}, \mathcal{L}, \lambda)$ be such that

$$\int_{(-\infty,x)} f(t)d\lambda(t) = 0, \quad \forall\, x \in \mathbb{R}.$$

Show that $f(x) = 0$ for a.e. $(\lambda)x \in \mathbb{R}$.

(5.21) Let $(X, \mathcal{S}, \mu)$ be a finite measure space and $\{f_n\}_{n\geq 1}$ be a sequence in $L_1(\mu)$ such that $f_n \to f$ uniformly. Show that $f \in L_1(\mu)$ and

$$\lim_{n\to\infty} \int |f_n - f\,|\, d\mu \;=\; 0.$$

Can the condition of $\mu(X) < +\infty$ be dropped?

(5.22) Let $\{f_n\}_{n\geq 1}$ and $\{g_n\}_{n\geq 1}$ be sequences of measurable functions such that $|f_n| \leq g_n \quad \forall\, n$. Let f and g be measurable functions such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for a.e. $x(\mu)$ and $\lim\limits_{n\to\infty} g_n(x) = g(x)$ for a.e. $x(\mu)$. If

$$\lim_{n\to\infty} \int g_n\, d\mu \;=\; \int g\, d\mu \;<\; +\infty,$$

show that

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f d\mu.$$

(Hint: Apply Fatou's lemma to $(g_n - f_n)$ and $(g_n + f_n)$.)

(5.23) Let $\{f_n\}_{n\geq 0}$ be a sequence in $L_1(X, \mathcal{S}, \mu)$. Show that $\left\{\int |f_n| d\mu\right\}_{n\geq 1}$ converges to $\int |f_0| d\mu$ iff $\left\{\int |f_n - f_0| d\mu\right\}_{n\geq 1}$ converges to zero.

(5.24) Let $(X, \mathcal{S})$ be a measurable space and $f : X \longrightarrow \mathbb{R}$ be $\mathcal{S}$-measurable. Prove the following:

(i) $\mathcal{S}_0 := \{f^{-1}(E) \,|\, E \in \mathcal{B}_{\mathbb{R}}\}$ is the σ-algebra of subsets of X, and $\mathcal{S}_0 \subseteq \mathcal{S}$.

(ii) If $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is Borel measurable, i.e., $\phi^{-1}(E) \in \mathcal{B}_{\mathbb{R}}\ \forall\, E \in \mathcal{B}_{\mathbb{R}}$, then $\phi \circ f$ is an $\mathcal{S}_0$-measurable function on X.

(iii) If $\psi : X \longrightarrow \mathbb{R}$ is any $\mathcal{S}_0$-measurable function, then there exists a Borel measurable function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\psi = \phi \circ f$.
(Hint: Use the simple function technique and note that if ψ is a simple $\mathcal{S}_0$-measurable function, then $\psi = \sum\limits_{i=1}^{n} a_n \chi_{f^{-1}(E_i)}$ for some positive integer n, $a_i \in \mathbb{R}$ for each i, and $E_i \in \mathcal{B}_{\mathbb{R}}$, then $\psi = (\sum\limits_{i=1}^{n} a_i \chi_{E_i}) \circ f$.

(5.25) Let μ, ν be as in proposition 4.4.9. Let $\mathcal{S}_\nu$ denote the σ-algebra of all ν^*-measurable subsets of X. Prove the following:

(i) $\mathcal{S} \subseteq \mathcal{S}_\nu$.

(ii) There exist examples such that $\mathcal{S}$ is a proper subclass of $\mathcal{S}_\nu$. Show that $\mathcal{S} = \mathcal{S}_\nu$ if $\mu^*\{x \in X \,|\, f(x) = 0\} = 0$.

5.4. The Lebesgue integral and its relation with the Riemann integral

(5.26) Let $f : [0,1] \longrightarrow [0,\infty)$ be Riemann integrable on $[\epsilon, 1]$ for all $\epsilon > 0$. Show that $f \in L_1[0,1]$ iff $\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx$ exists, and in that case

$$\int f(x)d\lambda(x) = \lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx.$$

(5.27) Let $f(x) = 1/x^p$ if $0 < x \leq 1$, and $f(0) = 0$. Find necessary and sufficient condition on p such that $f \in L_1[0,1]$. Compute $\int_0^1 f(x)d\lambda(x)$ in that case.
(Hint: Use exercise 4.46.)

(5.28) (**Mean value property):**Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function and let $E \subseteq [a,b], E \in \mathcal{L}$, be such that $\lambda(E) > 0$. Show that there exists a real number α such that

$$\int_E f(x)\lambda(x) = \alpha\lambda(E).$$

(5.29) Let $f \in L_1(\mathbb{R})$, and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function such that $\alpha \leq g(x) \leq \beta$ for a.e. $x(\lambda)$. Show that $fg \in L_1(\mathbb{R})$ and there exists $\gamma \in [\alpha, \beta]$ such that

$$\int |f|gd\lambda = \gamma\int |f|d\lambda.$$

(5.30) Let $f \in L_1(\mathbb{R}, \mathcal{L}, \lambda)$ and let $a \in \mathbb{R}$ be fixed. Define

$$F(x) := \begin{cases} \int_{[a,x]} f(t)\, d\lambda(t) & \text{for } x \geq a, \\ \int_{[x,a]} f(t) d\lambda(t) & \text{for } x \geq a. \end{cases}$$

Show that F is continuous.
(Hint: Without loss of generality take $f \geq 0$ and show that F is continuous from the left and right. In fact, F is actually uniformly continuous.)