Extension of measures (Lectures 10, 11, 12 and 13)

In these exercises, $\mu : \mathcal{A} :\to [0, +\infty]$ is a measure on the algebra $\mathcal{A}$.

(3.1) Show that $\mu^*(E)$ is well-defined.

(3.2) The set function $\mu^*(E)$ can take the value $+\infty$ for some sets E.

(3.3) Show that $\mu^*(E)$ can also be defined as

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \,\middle|\, A_i \in \mathcal{A}, A_i \cap A_j = \emptyset \text{ for } i \neq j \text{ and } \bigcup_{i=1}^{\infty} A_i \supseteq E \right\}.$$

(3.4) Let X be any nonempty set and let $\mathcal{A}$ be any algebra of subsets of X. Let $x_0 \in X$ be fixed. For $A \in \mathcal{A}$, define

$$\mu(A) := \begin{cases} 0 & \text{if } x_0 \notin A, \\ 1 & \text{if } x_0 \in A. \end{cases}$$

Show that μ is countably additive. Let μ^* be the outer measure induced by μ. Show that $\mu^*(A)$ is either 0 or 1 for every $A \subseteq X$, and $\mu^*(A) = 1$ if $x_0 \in A$. Can you conclude that $\mu^*(A) = 1$ implies $x_0 \in A$? Show that this is possible if $\{x_0\} \in \mathcal{A}$.

3.1. Choosing nice sets: Measurable sets

(3.5) Identify the collection of μ^*-measurable sets for μ as in example 3.7.6 in the text book.

(3.6) Let $X = [a,b]$ and let $\mathcal{S}$ be the σ-algebra of subsets of X generated by all subintervals of $[a,b]$. Let μ, ν be finite measures on $\mathcal{S}$ such that

$$\mu([a,c]) = \nu([a,c]), \;\; \forall\, c \in [a,b].$$

Show that $\mu(E) = \nu(E) \;\; \forall\, E \in \mathcal{S}$.

(3.7) Let μ_F be the measure on $\mathcal{A}(\tilde{\mathcal{I}})$, the algebra generated by left open right closed intervals. Let μ_F itself denote the unique extension of μ_F to $\mathcal{L}_F$, the σ-algebra of μ_F^*-measurable sets. Show that

(i) $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}_F$.

(ii) $\mu_F(\{x\}) = F(x) - \lim_{y \uparrow x} F(y)$. Deduce that the function F is continuous at x iff $\mu_F(\{x\}) = 0$.

(iii) Let F be differentiable with bounded derivative. If $A \subseteq \mathbb{R}$ is a Lebesgue null set, then $\mu_F^*(A) = 0$.

The measure μ_F is called the **Lebesgue-Stieltjes measure** induced by the distribution function F.

(3.8) *Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ be subsets of X. Then*

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu^*(E_n).$$

3.2. Completion of a measure space

(3.9) Let $\mathcal{N} := \{E \subseteq X \mid \mu^*(E) = 0\}$. Show that $\mathcal{N}$ is closed under countable unions.

(In fact and

$$\mathcal{S}^* = \mathcal{S}(\mathcal{A}) \cup \mathcal{N} := \{E \cup N \mid E \in \mathcal{S}(\mathcal{A}), N \in \mathcal{N}\},$$

where $\mathcal{S}^*$ is the σ-algebra of μ^*-measurable sets. Further, $\forall\, A \in \mathcal{S}^*$

$$\mu^*(A) = \mu^*(E), \text{ if } A = E \cup N, \text{ with } E \in \mathcal{S}(\mathcal{A}) \text{ and } N \in \mathcal{N}.$$

)

(3.10) Let $E \subseteq X$. A set G is called a measurable cover of E if $E \subset G$ and $\mu^*(G \setminus E) = 0$. If G_1, G_2 be two measurable covers of E, Show that $\mu^*(G_1 \Delta G_2) = 0$.

(3.11) Let $E \subseteq X$. A set K is called a measurable kernal of E if $E \supset GK$ and $\mu^*(E \setminus K) = 0$. Let K_1, K_2 be two measurable kernels of E. Show that $\mu^*(K_1 \Delta K_2) = 0$.

3.3. The Lebesgue measure

(3.12) Let $\mathcal{I}_0$ denote the collection of all **open intervals** of $\mathbb{R}$. For $E \subseteq X$, show that

$$\overset{*}{\lambda}(E) = \inf\left\{ \sum_{i=1}^{\infty} \lambda(I_i) \,\middle|\, I_i \in \mathcal{I}_0 \ \forall\, i, \text{ for } i \neq j \text{ and } E \subseteq \bigcup_{i=1}^{\infty} I_i \right\}.$$

(3.13) Let $E \subseteq \mathbb{R}$ and let $\epsilon > 0$ be arbitrary. Show that there exists an open set $U_\epsilon \supseteq E$ such that $\lambda(U_\epsilon) \leq \overset{*}{\lambda}(E) + \epsilon$. Can you also conclude that $\lambda(U_\epsilon \setminus E) \leq \epsilon$?

(3.14) For $E \subseteq \mathbb{R}$, let

$$\text{diameter}(E) := \sup\{|x - y| \mid x, y \in E\}.$$

Show that $\overset{*}{\lambda}(E) \leq$ diameter(E).

(3.15) Show that for $E \subseteq \mathbb{R}, \overset{*}{\lambda}(E) = 0$ if and only if for every $\epsilon > 0$, there exist a sequence $\{I_n\}_{n\geq 1}$ of intervals such that $E \subseteq \cup_{n=1}^{\infty}$ and $\overset{*}{\lambda}(\cup_{n=1}^{\infty} \setminus E) < \epsilon$. Such sets are called **Lebesgue null** sets.Prove the following:

(i) Every singleton set $\{x\}$, $x \in \mathbb{R}$, is a null set. Also every finite set is a null set.

(ii) Any countably infinite set $S = \{x_1, x_2, x_3, \ldots\}$ is a null set.

(iii) $\mathbb{Q}$, the set of rational numbers, is a null subset of $\mathbb{R}$.

(iv) Every subset of a null set is also a null set.

(iv) Let $A_1, A_2, \ldots, A_n, \ldots$ be null sets. Then $\bigcup_{n=1}^{\infty} A_n$ is a null set.

(v) Let $E \subseteq [a, b]$ be any set which has only a finite number of limit points. Can E be uncountable? Can you say E is a null set?

(vi) Let E be a null subset of $\mathbb{R}$ and $x \in \mathbb{R}$. What can you say about the sets $E + x := \{y + x \mid y \in E\}$ and $xE := \{xy \mid y \in E\}$?

(vii) Let I be an interval having at least two distinct points. Show that I is not a null set.

(viii) If E contains an interval of positive length, show that it is not a null set. Is the converse true, i.e., if $E \subseteq \mathbb{R}$ is not a null set, then does E contain an interval of positive length?

(ix) Show that Cantor's ternary set is an uncountable null set.

(3.16) Let $E \subseteq [0, 1]$ be such that $\overset{*}{\lambda}([0, 1] \setminus E) = 0$. Show that E is dense in $[0, 1]$

(3.20) Let $E \subseteq \mathbb{R}$ be such that $\overset{*}{\lambda}(E) = 0$. Show that E has empty interior.

(3.21) Let $A \in \mathcal{L}$ and $x \in \mathbb{R}$. Using equivalent definition of measurability, show that

(i) $A + x \in \mathcal{L}$, where $A + x := \{y + x \mid y \in A\}$.

(ii) $-A \in \mathcal{L}$, where $-A := \{-y \mid y \in A\}$.

(3.22) Let $E \in \mathcal{B}_{\mathbb{R}}$. Show that $E + x \in \mathcal{B}_{\mathbb{R}}$ for every $x \in \mathbb{R}$.

(3.23) Let $E \in \mathcal{L}$ and $x \in \mathbb{R}$. Let

$$xE := \{xy \mid y \in E\} \quad \text{and} \quad -E := \{-x \mid x \in E\}.$$

Show that $-E, xE \in \mathcal{L}$ for every $x \in E$. Compute $\lambda(xE)$ and $\lambda(-E)$ in terms of $\lambda(E)$.

Optional Exercises

(3.24) Let $E \subseteq \mathbb{R}$. Show that the following statements are equivalent:

(i) $E \in \mathcal{L}$.

(ii) $\overset{*}{\lambda}(I) = \overset{*}{\lambda}(E \cap I) + \overset{*}{\lambda}(E^c \cap I)$ for every interval I.

(iii) $E \cap [n, n+1) \in \mathcal{L}$ for every $n \in \mathbb{Z}$.

(iv) $\overset{*}{\lambda}(E \cap [n, n+1)) + \overset{*}{\lambda}(E^c \cap [n, n+1)) = 1$ for every $n \in \mathbb{Z}$.