

Note Title

3/17/2011

Gaussian Integration : Error and Convergence

Gauss 2 points

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$$f_0(x) = 1, \quad f_1(x) = x, \quad f_2(x) = x^2, \quad x \in [-1, 1]$$

$$g_0(x) = \frac{1}{\sqrt{2}}, \quad g_1(x) = \sqrt{\frac{3}{2}}x, \quad g_2(x) = \frac{2\sqrt{2}}{3\sqrt{5}}(x^2 - \frac{1}{3})$$

$$\langle g_2, g_0 \rangle = \langle g_2, g_1 \rangle = 0$$

$$\Rightarrow \int_{-1}^1 \left(x + \frac{1}{\sqrt{3}}\right) \left(x - \frac{1}{\sqrt{3}}\right) dx = 0, \quad \int_{-1}^1 \left(x + \frac{1}{\sqrt{3}}\right) \left(x - \frac{1}{\sqrt{3}}\right) x dx = 0$$

$$\text{Let } x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}$$

$$\therefore x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}, \quad x_1 - x_0 = \frac{2}{\sqrt{3}} = -2x_0$$

$$f(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$+ f[x_0, x_1, x] (x - x_0)(x - x_1)$$

$$\int_{-1}^1 f(x) dx \simeq \int_{-1}^1 \{f(x_0) + f[x_0, x_1](x - x_0)\} dx$$

$$= 2f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} \left[\frac{(x - x_0)^2}{2} \right]_{-1}^1$$

$$= 2f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} \left\{ \frac{(1 - x_0)^2}{2} - \frac{(-1 - x_0)^2}{2} \right\}$$

$$= f(x_0) + f(x_1)$$

$$\text{error} = \int_{-1}^1 f[x_0, x_1, x] (x-x_0)(x-x_1) dx.$$

$$\text{Since } \int_{-1}^1 (x-x_0)(x-x_1) dx = 0, \quad \int_{-1}^1 (x-x_0)(x-x_1)x dx = 0$$

and

$$\begin{aligned} f[x_0, x_1, x] &= f[x_0, x_0, x_1] + f[x_0, x_0, x_1, x](x-x_0) \\ &= f[x_0, x_0, x_1] + f[x_0, x_0, x_1, x_1](x-x_0) \\ &\quad + f[x_0, x_0, x_1, x_1, x](x-x_0)(x-x_1) \end{aligned}$$

$$\text{error} = \int_{-1}^1 f[x_0, x_0, x_1, x_1, x] (x-x_0)^2 (x-x_1)^2 dx$$

$$\text{error} = \int_{-1}^1 f[x_0, x_0, x_1, x_1, x] (x-x_0)^2 (x-x_1)^2 dx$$

$$= f[x_0, x_0, x_1, x_1, c] \int_{-1}^1 (x-x_0)^2 (x-x_1)^2 dx$$

by the MVT for integrals

$$= \frac{f^{(4)}(d)}{4!} \int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \frac{f^{(4)}(d)}{4!} \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{1}{9}x \right]_{-1}^1 = \frac{f^{(4)}(d)}{4!} \left\{ \frac{8}{45} \right\} = \frac{f^{(4)}(d)}{135}$$

Gauss 2-point rule

$$\int_{-1}^1 f(x) dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$\text{Error} = \frac{f^{(4)}(d)}{135}$$

We now consider $\int_a^b g(x) dx$.

Let $g : [a, b] \rightarrow \mathbb{R}$. Define 1-1, onto, affine map

$\phi : [-1, 1] \rightarrow [a, b]$ by

$$\phi(t) = \frac{(t+1)b + (1-t)a}{2} = \frac{a+b}{2} + t\left(\frac{b-a}{2}\right)$$

ϕ : 1-1, onto, affine

$$\phi'(t) = \frac{b-a}{2}, \quad \phi''(t) = 0$$

$$\phi: [-1, 1] \rightarrow [a, b] \quad \phi(t) = \frac{a+b}{2} + t \left(\frac{b-a}{2} \right)$$

$$\int_a^b g(x) dx = \int_{-1}^1 g(\phi(t)) \phi'(t) dt = \frac{b-a}{2} \int_{-1}^1 f(t) dt$$

$$\int_{-1}^1 f(t) dt \simeq f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right), \text{ error} = \frac{f^{(4)}(d)}{135}$$

$$\int_a^b g(x) dx \simeq \frac{b-a}{2} \left\{ g\left(\phi\left(-\frac{1}{\sqrt{3}}\right)\right) + g\left(\phi\left(\frac{1}{\sqrt{3}}\right)\right) \right\}$$

$$\text{error} = \frac{b-a}{2} \frac{f^{(4)}(d)}{135}$$

$$\text{error} = \frac{f^{(4)}(d)}{135} \left(\frac{b-a}{2}\right)$$

Since $f(t) = g(\phi(t))$, by the Chain rule,

$$f'(t) = g'(\phi(t)) \phi'(t) = g'(\phi(t)) \left(\frac{b-a}{2}\right)$$

$$f''(t) = g''(\phi(t)) \phi'(t)^2 = g''(\phi(t)) \left(\frac{b-a}{2}\right)^2$$

$$f'''(t) = g'''(\phi(t)) \left(\frac{b-a}{2}\right)^3, \quad f^{(4)}(t) = g^{(4)}(\phi(t)) \left(\frac{b-a}{2}\right)^4$$

$$\text{error} = \frac{g^{(4)}(\phi(d))}{135} \left(\frac{b-a}{2}\right)^5$$

Gauss 2 point rule: $\int_a^b f(x) dx \simeq \frac{b-a}{2} (f(x_0) + f(x_1))$

$$x_0 = \frac{a+b}{2} - \frac{1}{\sqrt{3}} \left(\frac{b-a}{2} \right), \quad x_1 = \frac{a+b}{2} + \frac{1}{\sqrt{3}} \left(\frac{b-a}{2} \right)$$

$$\text{error} = \frac{f^{(4)}(d)}{135} \left(\frac{b-a}{2} \right)^5$$

Composite Gauss 2-point rule

$$a = t_0 < t_1 < \dots < t_n = b, \quad h = \frac{b-a}{n}.$$

$$\phi_i: [-1, 1] \rightarrow [t_i, t_{i+1}] \quad \phi_i(t) = \frac{t_i + t_{i+1}}{2} + t \frac{(t_{i+1} - t_i)}{2}$$

Gauss 2 points in $[t_i, t_{i+1}]$:

$$u_{2i+1} = \phi_i\left(-\frac{1}{\sqrt{3}}\right), \quad u_{2i+2} = \phi_i\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{t_i + t_{i+1}}{2} - \frac{h}{2\sqrt{3}}, \quad = \frac{t_i + t_{i+1}}{2} + \frac{h}{2\sqrt{3}},$$

$$i = 0, 1, \dots, n-1$$

$$\int_a^b f(x) dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x) dx$$

$$\approx \sum_{i=0}^{n-1} \frac{h}{2} (f(u_{2i+1}) + f(u_{2i+2}))$$

$$\text{error} = \sum_{i=0}^{n-1} \frac{f^{(4)}(d_i)}{135} \left(\frac{h}{2}\right)^5$$

$$= \frac{(b-a)f^{(4)}(n)}{270} \left(\frac{h}{2}\right)^4$$

Legendre Polynomials

$$X = C[a, b], \quad \langle f, g \rangle = \int_a^b f(x) g(x) dx, \quad \|f\|_2 = \sqrt{\langle f, f \rangle}$$
$$f_0(x) = 1, \quad f_1(x) = x, \quad \dots, \quad f_k(x) = x^k, \quad \dots$$

Gram-Schmidt Orthonormalization

$$g_0(x) = \frac{f_0}{\|f_0\|}$$

for $k = 1, 2, \dots$

$$r_k = f_k - \sum_{j=0}^{k-1} \langle f_k, g_j \rangle g_j, \quad g_k = \frac{r_k}{\|r_k\|_2}$$

$g_0, g_1, g_2, \dots, g_k, \dots$: Legendre Polynomials.

g_k : polynomial of degree k ,

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

g_k is perpendicular to any polynomial of degree $\leq k-1$,

g_k has k distinct zeros, say, x_0, x_1, \dots, x_{k-1}

Gauss Points