**Exercises:**

1. Show that in a topological group, the connected component of the identity is a normal subgroup.
2. Show that the action of the group $SO(n, \mathbb{R})$ on the sphere $S^{n-1}$ given by matrix multiplication is transitive. You need to employ the Gram-Schmidt theorem to complete a given unit vector to an orthonormal basis.
3. Suppose a group $G$ acts transitively on a set $S$ and $x, y$ are a pair of points in $S$ and $y = gx$. Then the subgroups stab $x$ and stab $y$ are conjugates and $g^{-1}$(stab $y)g =$ stab $x$.

   (i) Show that the map $\overline{\phi} : G/$stab $x \longrightarrow S$ given by $\overline{\phi}(\overline{g}) = gx$ is well-defined, bijective and $\overline{\phi} \circ \eta = \phi$.

   (ii) Suppose that $S$ is a topological space, $G$ is a topological group and the action $G \times S \longrightarrow S$ is continuous. Show that the map $\overline{\phi}$ is continuous.

   (iii) Deduce that if $G$ is compact and $S$ is Hausdorff then $G/$stab $x$ and $S$ are homeomorphic.

4. Examine whether the map $\phi : SU(n) \times S^1 \longrightarrow U(n)$ given by $\phi(A, z) = zA$ is a homeomorphism.
5. Show that the group of all unitary matrices $U(n)$ is compact and connected. Regarding $U(n-1)$ as a subgroup of $U(n)$ in a natural way, recognize the quotient space as a familiar space.
6. Show that the subgroups $SU(n)$ consisting of matrices in $U(n)$ with determinant one are connected for every $n$.
7. Suppose $G$ is a topological group and $H$ is a normal subgroup, prove that $G/H$ is Hausdorff if and only if $H$ is closed.