Problem set 4 : Splitting Fields

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(1) Let F be a field and let K be a splitting field of a polynomial $f(x) \in F[x]$. Show that $[K : F] \leq n!$.

(2) Find degrees of splitting fields over $\mathbb{Q}$ of each of the following polynomials: (a) x^3-2 (b) x^4-1 (c) x^4+1 (d) x^6+1 (e) $(x^2+1)(x^3-1)$ and (f) x^6+x^3+1.

(3) Find a splitting field K of x^3-10 over $\mathbb{Q}(\sqrt{2})$. Find $[K : \mathbb{Q}]$.

(4) Let p be a prime number. Show that the degree of a splitting field of x^p-2 over $\mathbb{Q}$ is $p(p-1)$.

(5) Let $f(x) \in \mathbb{Q}[x]$ be a cubic polynomial and K be a splitting field of $f(x)$ over $\mathbb{Q}$. Show that $[K : \mathbb{Q}]$ is either $1, 2, 3$ or 6. Provide examples in each case.

(6) Let $\mathbb{F}_q$ denote a finite field with q elements. Show that for a prime number p, the finite field $\mathbb{F}_{p^n}$ is a splitting field over $\mathbb{F}_p$ of the polynomial $f(x) = x^{p^n} - x$. [Hint: Show that $\mathbb{F}_{p^n}$ is precisely the set of roots of $f(x)$.]

(7) Let $K \subset \mathbb{C}$ be a splitting field of $f(x) = x^3 - 2$ over $\mathbb{Q}$. Find a complex number z such that $K = \mathbb{Q}(z)$.

(8) Let F be a field of characteristic p. Let $f(x) = x^p - x - c \in F[x]$. Show that either all roots of $f(x)$ lie in F or $f(x)$ is irreducible in $F[x]$. [Hint: show that if a is a root of $f(x)$ then so is $a+1$.]

(9) Let F be a field of characteristic zero and let p be an odd prime. Let $a \in F^\times$ such that a is not a p^{th} power of any element in F. Show that $f(x) = x^p - a$ is irreducible in $F[x]$. What can you say about the degree of a splitting field of $f(x)$ over F ?

(10) Let E be a splitting field over a field F of $f(x)$. Let K be a subfield of the field extension E/F. Let $\sigma : K \to E$ be a monomorphism such that $\sigma(a) = a$ for all $a \in F$. Such a map is called an F-embedding of K into E. Show that σ can be extended to an automorphism of E.