Problem set 3 : Symmetric Polynomials

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(1) Write the symmetric polynomials $f(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$ and $g(x, y, z) = x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3$ in terms of elementary symmetric polynomials.

(2) Let $(f(x))$ denote the discriminant of a polynomial $f(x)$. Show that
   (a) $(x^3 + px + q) = -(4p^3 + 27q^2)$.
   (b) $(x^4 + px^2 + r) = 16r(p^2 - 4r)^2$.
   (c) $(x^4 + qx + r) = -27q^4 + 256r^3$.

(3) Find the sum of $7th$ powers of the roots of $x^3 + px + q$.

(4) Let $g(x) = x^3 + px + q$ where $q \neq 0$. Determine the monic polynomial whose roots are inverses of the squares of the roots of $g(x)$.

(5) (a) Show that $(x^n - 1) = (-1)^{\binom{n}{2}+n-1}n^n$.
   (b) Let $g(x)$ and $h(x)$ be monic polynomials and $g(x) = (x-a)h(x)$. Show that $(g(x)) = h(a)^2 dis(h(x))$.
   (c) Show that $(x^{n-1} + x^{n-2} + \cdots + 1) = (-1)^{(n-1)(n+2)/2}n^{n-2}$.

(6) Show that a polynomial $f \in S := R[x_1, x_2, \ldots, x_n]$ where R is a commutative ring is fixed under all the automorphisms of S induced by even permutations in S_n if and only if $f = g + \delta h$ where g and h are symmetric polynomials and $\delta = \prod_{i \leq j}(x_i - x_j)$.

(7) Let $f(x) = \prod_{i=1}^{n}(x - r_i)$. Show that $dis(f(x)) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} f'(r_i)$. Use this formula to show that $(\Phi_p(x)) = (-1)^{\binom{p}{2}}p^{p-2}$. Here $\Phi_p(x) =$ irr $(\zeta_p, \mathbb{Q})$ for a prime number p.